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*APPLETONS' MATHEMATICAL SERIES*

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NUMBERS UNIVERSALIZED  
AN  
ADVANCED ALGEBRA

BY

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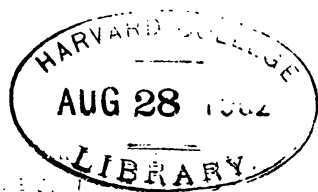
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## PREFACE.

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**NUMBERS UNIVERSALIZED** is believed to embrace all algebraic subjects usually taught in the preparatory and scientific schools and colleges of this country. For convenience, it is divided into two parts, which are bound separately and together, to accommodate all kinds and grades of schools sufficiently advanced to adopt its use.

Part First is intended as an advanced elementary algebra, and is especially adapted to schools preparing students for college, and to advanced classes in high and normal schools.

In treatment, the great simplicity noticeable in the author's book for beginners, "**Numbers Symbolized**," has been avoided; yet care has been exercised to everywhere keep intact the logical sequence of thought, and to thus prevent the discussions from becoming unnecessarily abstruse and difficult. Definitions are logically arranged and expressed in simple, concise, and exact language. Illustrations are resorted to only when concepts described are not sufficiently clear without them. Fundamental principles and propositions are carefully formulated, and, whenever practicable, rigidly proved. Examples have been selected with special reference to variety in combination and methods of reduction. Unusual prominence is given to the subjects of factoring, radicals, and quadratic equations, as the author has learned by experience that few pupils entering classes in the higher mathematics are well enough drilled in these departments of work to follow with rapidity discussions involving them. The introduction into this part of the book of so much of indeterminate equations, loga-

rithms, higher series, interest and annuities, and permutations and combinations, as have a practical interest to students, it is believed, will be acceptable to many teachers, and may be omitted by others who desire a shorter course without doing violence to the logic of other parts. The doctrine of limits is introduced before the treatment of proportion, geometrical progressions, and annuities, to enable the author to discuss the limiting cases under these subjects, and to thus complete a treatment that would otherwise be necessarily imperfect or partial. The use of the term *ultimum*, to denote the final state of a variable as it approaches its limit, has given rise to simpler demonstrations of the more difficult propositions under this head than are usually found.

Part Second is treated in five chapters, as follows : One embracing serial functions, including development of functions into series, convergency and divergency of infinite series, the binomial formula, the binomial theorem, the exponential and logarithmic series, summation of series, reversion of series, recurring series, and decomposition of rational fractional functions ; one treating of complex numbers, graphically and analytically, including fundamental operations with complex numbers, general principles of moduli and norms, and the development and representation of sine, cosine, and tangent ; one embodying a discussion on the theory of functions, including graphical representations of the meaning of the terms independent and dependent variables, continuous and discontinuous functions, increasing and decreasing functions, and turning values and limits of functions, and also a treatment of differentials and derivatives, and maxima and minima values of functions ; one treating of the theory of equations, including a discussion of the properties of the roots, real and imaginary, of an equation, methods of determining the commensurable roots of a numerical equation, Sturm's theorem for detecting the number and situation of real roots, Horner's method of root extension, Cardan's for-

mula for solving cubic equations, and a short treatment of reciprocal and binomial equations; one treating of determinants and probabilities, so far as these subjects are of interest and value to the general student. The volume closes with a supplementary discussion of continued fractions and theory of numbers.

The aim of the author in preparing this part of his work has not been so much to give completeness to the various subjects treated as to lead the student to a comprehension of the fundamentals of a wider range of subjects, and to cultivate in him a taste for mathematical investigation. It is believed that the plan adopted will give the general student a broader and more practical knowledge of algebra, and will lead to better results in a preparatory course of study for the university than would a completer treatment of fewer subjects requiring an equal amount of space in their development and more time in their mastery. While a sufficient number of examples have been placed under each head to offer opportunity for the application of the principles and laws developed, there will not be found an unnecessary multiplicity of them to retard the progress of the pupil in his onward course.

In conclusion, the author desires to acknowledge his indebtedness to the English authors, Hall and Knight, Chrystal, Aldis, Whitworth, and C. S. Smith, whose works he frequently consulted, and from which he obtained many new and valuable ideas.

DAVID M. SENSENIG.

NORMAL SCHOOL, WEST CHESTER, PA., }  
December 2, 1889.



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# NUMBERS UNIVERSALIZED.

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## INTRODUCTION.

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### GENERAL DEFINITIONS.

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1. Anything that may be conceived to be made up of equal definite amounts is *Quantity*; as, length, superficies, volume, weight, value, momentum, etc.

2. A quantity may be *increased* or *diminished*, and, if not indefinitely large or small, *measured*.

3. To measure a quantity of any kind is to assume a definite amount of the same kind of quantity as a standard of measure, and to determine *how many times* the quantity contains this standard.

4. Any definite amount of quantity assumed as a standard of measure is a *unit* of quantity; as, a foot, a pound, a dollar, etc.

5. That which denotes how many units a quantity contains, how many objects are in a group, or how many times anything is taken or done, is a *Number*. *One* is the *unit* of number.

6. A *number* is one, or a collection of ones.

7. The measure of a quantity in units is the *magnitude* of the quantity.

Thus, weight is *quantity*; twenty-five pounds is the *magnitude* of a weight; and twenty-five is the *number* of units in the weight, or in the magnitude of the weight.

8. By a figure of speech, magnitudes and numbers are also called quantities.

9. A quantity that contains a definite, or *specific*, number of units is a *Specific Quantity*; as, five feet.

10. A quantity that contains a *general* number of units, that is, a quantity that may contain any number of units, is a *General Quantity*; as, the *weight* of a ball.

11. Specific numbers are expressed by characters called *figures*, and general numbers by *letters of the alphabet*, or by a combination of figures and letters.

12. Numbers expressed by figures are sometimes called *Numerical Quantities*, and numbers expressed by letters, or a combination of figures and letters, *Literal Quantities*.

13. The result of combining two or more quantities into one is the *sum* of the quantities, and the process of finding the sum is *Addition*.

14. The symbol of addition is  $+$ , read *plus*, and, when placed before a quantity, it denotes that the quantity is to be added to what has gone before. Thus,  $8 + 5$  denotes that 5 is to be added to 8, and  $8 + 5 + 3$  that 5 is to be added to 8 and 3 to the result. Likewise,  $a + b + c$  denotes that  $b$  is to be added to  $a$  and  $c$  to the result.

15. The *difference* of two quantities is such a quantity as, added to one of them, called the *subtrahend*, will produce the other, called the *minuend*, and the process of finding the difference is *Subtraction*.

Thus, the difference of the minuend 8 and the subtrahend 5 is 3, since 3 added to 5 makes 8.

16. The symbol of subtraction is  $-$ , read *minus*, and denotes, when written before a quantity, that the quantity is to be subtracted from what has gone before. Thus,  $x - y$  denotes that  $y$  is to be subtracted from  $x$ ;  $x - y - z$ , that  $y$  is to be subtracted from  $x$  and  $z$  from the result; and  $x - y + z$ , that  $y$  is to be subtracted from  $x$  and  $z$  added to the result.

Hence, the order of operations in addition and subtraction is from left to right.

17. The *product* of two quantities is the result of taking one of them, called the *multiplicand*, as many times as is indicated by the other, called the *multiplier*, and the process of finding the product is *Multiplication*.

18. The *product*, or *continued product*, of three or more quantities is the result of multiplying the first by the second, the product by the third, and so on until all the quantities have been used, and the process of finding a continued product is *Continued Multiplication*.

19. The quantities multiplied together to produce a product are called the *Factors* of the product.

20. A quantity that is the product of two or more factors other than itself and unity is a *Composite Quantity*.

21. A quantity that consists of no other factors than itself and unity is a *Prime Quantity*.

22. The *prime factors* of a quantity are the *prime* quantities which, multiplied together, will produce the quantity.

23. The symbol of multiplication is  $\times$  or  $.$ , read *into* or *multiplied by*, and denotes, primarily, when written before a quantity, that what precedes it is to be multiplied by the quantity. Thus,  $a \times b$ , or  $a.b$ , denotes that  $a$  is to be multiplied by  $b$ , and  $a \times b \times c$ , or  $a.b.c$ , that  $a$  is to be multiplied by  $b$  and the result by  $c$ .

24. The multiplication sign is usually omitted between letters, or between a figure and a letter. Thus,  $abc$  means the same as  $a \times b \times c$ , and  $5x$  the same as  $5 \times x$ . This sign, however, can not be omitted between figures, since  $35$  is not  $3 \times 5$ .

25. When a product consists of two or more like factors, it is called a *power* of a factor. Thus,  $aa$  is the second power of  $a$ ;  $aaa$  the third power of  $a$ ;  $aaaa\dots$  to  $n$  factors the  $n$ th power of  $a$ .  $a$  is sometimes, but with doubtful propriety, called the first power of  $a$ . The second power is called the *square* and the third power the *cube*.

26. A *power* of a quantity may be defined as the result of using a quantity two or more times as a factor.

27. The most convenient method of denoting a power of a quantity is to write on the right hand, above the quantity, the number of times it is to be used as a factor. Thus, the cube of  $a$ , or  $aaa$ , may be written  $a^3$ .

28. The small figure or letter used to express how many times a quantity is used as a factor is called an *index* or *exponent*.

---

29. The *quotient* of two quantities is the number of times one of them, called the *divisor*, must be taken to produce the other, called the *dividend*, and the process of finding the quotient is *Division*.

30. The symbol of division is  $\div$ , read *divided by*, and denotes, when written between two quantities, that the first is to be divided by the second. Thus,  $a \div b$  denotes that  $a$  is to be divided by  $b$ ; that is, the number of times the divisor  $b$  must be taken to produce the dividend  $a$  is to be determined.

$a \div b \div c$  denotes that  $a$  is to be divided by  $b$  and the quotient by  $c$ .  $a \times b \div c$  denotes that  $a$  is to be multiplied by  $b$  and the product divided by  $c$ .  $a \div b \times c$  denotes that  $a$  is to be divided by  $b$  and the quotient multiplied by  $c$ .

Hence, the order of operations in multiplication and division is from left to right.

31. To indicate that one quantity is to be divided by another, the dividend is sometimes written above and the divisor below a horizontal line. Thus,  $\frac{a}{b}$  is equivalent to  $a \div b$ .

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32. A *root* of a quantity is one of the equal factors of the quantity, and the process of finding a root is *Evolution*. One of the *two* equal factors of a quantity is the *square root*, one of the *three* equal factors the *cube root*, and one of the  $n$  equal factors the  *$n$ th root*.

**33.** The symbol of root is  $\sqrt{\phantom{a}}$ , called the *radical sign*. A number, called the *index* of the root, is written in the angle of the sign to denote what root is to be extracted. Thus,  $\sqrt[3]{a}$  denotes the cube root of  $a$ , and  $\sqrt[n]{a}$  the  $n$ th root of  $a$ . When no index is expressed, the square root is denoted. Thus,  $\sqrt{a}$  is the square root of  $a$ .

**34.** Any written quantity or indicated operation is called a *mathematical expression*, or simply an *expression*; as,  $a$  or  $a + b - c$ .

**35.** The value of a mathematical expression is the quantity that will result from performing all the operations therein indicated. Thus, the value of  $8 + 4 - 3$  is 9.

**36.** The symbol  $=$  is read *equals*, or *is equal to*.

**37.** Two expressions are equal:

1. When they have the same form and value; as,  $4x = 4x$ .

This is known as *the law of identity*.

2. When they differ in form but have the same value; as,  $3x + 4x = 5x + 2x$ .

This is known as *the law of association*.

**38.** An expression denoting that two other expressions are equal is an *Equation*, and the expressions which are placed equal to each other are the *members* of the equation.

Thus,  $7 \times x = 5x + 2x$  is an equation, in which  $7 \times x$  is the *first* member, and  $5x + 2x$  the *second* member.

**39.** The symbol  $>$  is read *greater than*;  $<$  *less than*;  $\neq$  *not equal to*;  $\succ$  *not greater than*; and  $\prec$  *not less than*.

Thus,  $a > b$  denotes that  $a$  is greater than  $b$ ;  $a < b$ , that  $a$  is less than  $b$ ;  $a \neq b$ , that  $a$  is not equal to  $b$ ;  $a \succ b$ , that  $a$  is not greater than  $b$ ; and  $a \prec b$ , that  $a$  is not less than  $b$ .

**40.** An expression denoting that two other expressions are not equal is an *Inequality*, and the unequal expressions are the *members* of the inequality.

41. The symbol  $\therefore$  denotes *therefore* or *hence*.

42. The symbol  $\because$  denotes *because* or *since*.

43. To denote that an expression is to be treated as a whole, it is placed between a pair of *symbols of aggregation*. Thus,  $(a + b) + (c + d)$  denotes that the sum of  $c$  and  $d$  is to be added to the sum of  $a$  and  $b$ ;  $(a + b)(c + d)$  denotes that the sum of  $a$  and  $b$  is to be multiplied by the sum of  $c$  and  $d$ ;  $a \div (b - c)$  denotes that  $a$  is to be divided by the difference of  $b$  and  $c$ .

44. The symbols of aggregation are  $()$ , called a *parenthesis*;  $\{ \}$ , called *braces*;  $[ ]$ , called *brackets*; and  $\text{---}$ , called a *vinculum*. The vinculum is drawn over the expression that is to be considered as a whole. Thus,  $\overline{a + b} \times c = (a + b)c$ . The term *parenthesis* is often made to include all symbols of aggregation.

45. An expression of two or more parts connected by  $+$  or separated by  $-$  is called a *polynomial* or a *multinomial*, as,  $3x^2 + 2x - 7$ . The parts of a polynomial are called its *terms*.

46. A polynomial of *two* terms is a *binomial*; one of *three* terms, a *trinomial*; one of *four* terms, a *quadri-nomial*, etc. A simple quantity, or a quantity of one term, is a *monomial*.

47. Terms containing the same literal factors affected by the same exponents are *similar*; as,  $5a^2b^3c$  and  $7a^2b^3c$ .

48. The *coefficient* of a term is any factor, or group of factors, belonging to it, and which is assumed to show how many times the remaining factor, or group of factors, is to be taken. Thus, in  $3a(b)$ ,  $3a$  is the coefficient; in  $3b(a)$ ,  $3b$  is the coefficient; and in  $3(ab)$ ,  $3$  is the coefficient. The numerical factor is usually considered the coefficient unless a special reason exists for regarding some other as the coefficient. When the term has no numerical factor expressed, the factor 1 is understood to be the numerical coefficient.

49. If a term has  $n$  literal factors, it is said to be of  $n$  *dimensions*, or of the  $n$ th degree. Thus,  $3a^2b^3$  or  $3aabb^3$  is of five dimensions, or of the fifth degree.

50. When all the terms of a polynomial are of the same degree, the polynomial is said to be *homogeneous*.

51. A quantity may be in one of two diametrically opposite states. Thus, a *sum* of money may represent a gain or a loss; *distance* traveled may be in the direction of one's destination or in the opposite direction.

52. Quantities in opposite states, and hence, too, the number of units they contain, are distinguished by the terms *positive* and *negative*.

53. It is, generally speaking, indifferent which of two opposing quantities is considered positive and which negative, if consistency is maintained throughout an investigation or a discussion. But, in certain of the arts and sciences, a conventional usage has sprung up which should not be wholly disregarded in practice. Thus :

1. In business, gains and incomes are usually considered positive, and losses and outlays negative.

2. In meteorology, rise above a given temperature is considered positive, and fall below it negative.

3. In mechanics, accelerating forces are usually considered positive, and retarding forces negative.

4. In the sciences of space, distance rightward or upward from a given line is considered positive, and distance leftward or downward negative.

5. In general, quantities that tend to increase the magnitude of a given quantity when combined with it are considered positive, and quantities that tend to diminish it negative.

54. A number or an expression is marked positive by placing before it the sign  $+$  (plus), and negative by placing before it the sign  $-$  (minus).

For this reason, positive and negative expressions are sometimes called *Symbolized Numbers*.

**55.** The science which treats of numbers without regard to their character as positive or negative is *Arithmetic*. Arithmetic based on the literal notation is *Literal Arithmetic*.

**56.** The science which treats of positive and negative, or symbolized numbers, is *Algebra*.

**57.** Literal arithmetic and algebra are usually treated together as a single science under the title of *Algebra*. In this broader sense, algebra may be defined as the science of generalized and symbolized numbers, or *universalized number*.

**58.** Symbolized numbers are sometimes called *algebraic quantities*, and unsymbolized numbers *arithmetical quantities*.

**Note.**—When no sign is written before an algebraic quantity, + is understood.

**59.** An *Axiom* is a self-evident truth.

**60.** The following are the principal axioms relating to numbers :

1. Things that are equal to the same thing are equal to each other.

2. If equals are added to equals the sums are equal.

3. If equals are subtracted from equals the remainders are equal.

4. If equals are multiplied by equals the products are equal.

5. If equals are divided by equals the quotients are equal.

6. Equal powers of equal quantities are equal.

7. Equal roots of equal quantities are equal.

8. The whole is equal to the sum of all its parts.

9. If the same quantity be both added to and subtracted from a given quantity, its value will not be changed.

**61.** A *Theorem* is a truth to be proved.

**62.** A *Demonstration* is the course of reasoning employed in proving a theorem.

63. A *Problem* is a question proposed for solution.

64. A *Solution* is a description of the method employed in doing a problem.

## EXERCISE 1.

1. If  $a$  men can build a wall in  $b$  days, how long will it take  $c$  men to build it?

**Solution:** If  $a$  men can build it in  $b$  days, one man can build it in  $ab$  days, and  $c$  men can build it in one  $c$ th of  $ab$  days, or  $\frac{ab}{c}$  days.

**Note.**—Read  $\frac{ab}{c}$   $ab$  divided by  $c$ .

2. A can say  $m$  words in a minute. How long will it take him to deliver a speech of  $n$  pages, each containing  $a$  lines of  $b$  words?

3. If a man earns  $x$  dollars a year, and spends  $b$  cents a day, how much will he save in  $\frac{2}{3}$  of a year?

4. How many men can do as much work in  $a + b$  days as  $c$  men can do in  $a - b$  days?

5. A farmer exchanged  $m$  bushels of potatoes at  $n$  cents a bushel for clover-seed at  $y$  dollars a bushel. How many bushels of clover-seed did he get?

6. If a pole  $c$  feet high casts a shadow  $r$  feet long, what is the height of a pole that casts a shadow  $m + n$  feet long at the same time of day?

7. The circumference of a fore-wheel of a carriage is  $x$  feet, and of a hind-wheel  $y$  feet. How much oftener does one turn than the other in going one mile?

8. How many kegs, each containing  $a$  gal.  $a$  qt.  $a$  pt., can be filled from a hoghead containing  $m$  gal.  $m$  qt.  $m$  pt.?

9. How many square feet on the surface of a block  $a$  feet long,  $b$  feet wide, and  $c$  feet high?

10. How many cords of wood in a pile  $m$  feet long,  $n$  feet wide, and  $a$  feet high?

11. A farmer had  $a$  acres of land and sold  $x$  per cent of it. How many acres had he remaining?

12. A man bought a boat-load of coal for  $c$  dollars, and, by retailing it at  $\$5\frac{1}{2}$  a ton, gained  $m$  dollars. How many tons were in the load?

13. A clerk spends  $m$  dollars a year, which is  $y$  per cent of his salary. What is his salary?

14. A regiment went into battle with  $x$  men and came out with  $y$  men. What per cent did it lose?

15. The base of a right-angled triangle is  $a$  and the perpendicular is  $b$ ; required the hypotenuse.

16. The diagonal of a rectangle is  $x$  feet and the length  $y$  feet; required the width.

17. What is the interest of  $x$  dollars for  $y$  years at  $r$  per cent?

18. What principal will in  $t$  years at  $r$  per cent amount to  $a$  dollars?

19. What must be the face of a note which, when discounted at a bank for  $d$  days at 6 per cent per annum, will yield a proceed of  $p$  dollars?

20. If  $a$  men in  $b$  days of  $c$  hours each can dig a ditch  $d$  rods long,  $e$  feet wide, and  $f$  feet deep, how many men will be required to dig a ditch  $g$  rods long,  $h$  feet wide, and  $k$  feet deep in  $l$  days of  $m$  hours each?

If  $x$  represents one quantity and  $y$  another, write:

21. Their sum; their difference; their product; their quotient.

22. The sum of their squares; the square of their sum; the square of their difference; the difference of their squares.

23. The sum of their cubes; the cube of their sum; the product of their sum and difference; the quotient of their sum and difference.

24. Their product times their sum; their quotient times their difference; the sum of their square roots; the square root of their sum.

25. Their product divided by the sum of their square roots; their difference divided by the difference of their squares; their quotient multiplied by the sum of their cube roots.

### Numerical Values of Literal Expressions.

#### EXERCISE 2.

Find the values of the following expressions when  $a = 10$ ,  $b = 8$ ,  $c = 6$ , and  $d = 5$ :

$$1. \frac{(a+b)a}{(a-c)d} \quad \text{Explanation: } \frac{(a+b)a}{(a-c)d} = \frac{(10+8) \times 10}{(10-6) \times 5} = \frac{180}{20} = 9$$

$$2. \frac{a^2 - b^2}{c^2 - d^2} \quad 3. \frac{a+d}{b(c-d)} \quad 4. \frac{ab+cd}{ab-cd}$$

$$5. \frac{a}{b} \times \frac{c+d}{c-d} \quad 6. \frac{a+b}{a-b} \div \frac{c+d}{c-d}$$

$$7. \left( \frac{ad-bc}{abcd} \right) \left( \frac{ad+bc}{ab \div cd} \right) \quad 8. \frac{\sqrt{a+c}}{\sqrt{c+cd}} \times \frac{a-b}{b-c}$$

Find the value of the following expressions when  $p = 1$ ,  $q = 4$ ,  $r = 9$ , and  $s = 16$ :

$$9. \sqrt{\frac{rs}{r+s}} \quad 10. s + \sqrt{\frac{pr}{qs}} \quad 11. \frac{(r+s)^2}{\sqrt{r+s}}$$

$$12. \frac{rp+qs}{\sqrt{pqrs}} \quad 13. \frac{q\sqrt{s+r}}{p\sqrt{qr}} \quad 14. \frac{\sqrt{pqr}s}{pq\sqrt{4r+4s}}$$

Find the value of the following expressions when  $x = 2\frac{1}{2}$ ,  $y = \frac{3}{4}$ ,  $z = \frac{1}{2}$ , and  $u = 0$ :

$$15. \frac{x^2 - y^2}{x+y} \div \frac{y^2 - z^2}{y-z} \quad 16. \frac{ux+yz}{u y - xz}$$

$$17. \frac{(x+y+z)u}{x-y+z} \quad 18. \frac{z^2 - u^2}{x^2 + u^2} \times \frac{(x-y)^2}{(z+u)^2}$$

$$19. \frac{ux-uy}{xy-uy} \div \sqrt{z(y+z)} \quad 20. \frac{x+y}{x-y} \times \frac{y+u}{z-u}$$

## CHAPTER 1.

### INTEGRAL QUANTITIES.

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#### Addition and Subtraction.

##### Definitions and Principles.

**65.** The *arithmetical sum* or *difference* of two quantities is their sum or difference when no regard is had to their character as positive or negative.

**66.** The *algebraic sum* or *difference* of two quantities is their sum or difference when regard is had to their character as positive or negative.

**67.** If  $b$  units be combined with  $a$  units, there will be a certain sum. If, now, the  $a$  units be removed, the  $b$  units will remain. If the  $a$  units be replaced (added to the  $b$  units), there will evidently be the same sum as before. Therefore, according to the law of association,

$$a + b = b + a.$$

In like manner, it may be shown that  $a + b + c = a + c + b = b + a + c = b + c + a = c + a + b = c + b + a$ , and so on, for any number of quantities. Therefore,

**Principle 1.**—*The sum of two or more quantities is the same in whatever order they are combined.*

This is *the commutative law of addition.*

**68.** The arithmetical sum of two or more different literal quantities is denoted by placing in parenthesis the expression that they are to be added.

Thus, the sum of  $a$ ,  $b$ , and  $c$  is  $(a + b + c)$ .

69. The arithmetical difference of two different literal quantities follows the same law in its expression as the arithmetical sum.

Thus, the difference of  $a$  and  $b$  is  $(a - b)$ .

70. The parenthesis including a sum or a difference is often omitted in practice.

71. The sum of  $a$  positive units,  $b$  positive units,  $c$  positive units, etc., is evidently  $(a + b + c + \text{etc.})$  positive units. Therefore,

$$(+a) + (+b) + (+c) + \text{etc.} = +(a + b + c + \text{etc.}).$$

The sum of  $a$  negative units,  $b$  negative units,  $c$  negative units, etc., is evidently  $(a + b + c + \text{etc.})$  negative units. Therefore,

$$(-a) + (-b) + (-c) + \text{etc.} = -(a + b + c + \text{etc.}).$$

Hence,

*Prin. 2.—The algebraic sum of two or more quantities with like signs equals their arithmetical sum with the same sign.*

72. If  $a > b$  and  $b$  negative units be combined with  $a$  positive units, they will destroy  $b$  positive units, and  $(a - b)$  positive units will remain. Therefore,

$$(+a) + (-b) = +(a - b), \text{ when } a > b.$$

If  $a < b$  and  $b$  negative units be combined with  $a$  positive units, the  $a$  positive units will destroy  $a$  negative units, and  $b - a$  negative units will remain. Therefore,

$$(+a) + (-b) = -(b - a), \text{ when } a < b.$$

Hence,

*Prin. 3.—The algebraic sum of two quantities with unlike signs equals their arithmetical difference with the sign of the greater.*

73. Principles 2 and 3 together constitute the algebraic law of addition.

**74.** Since the difference of two quantities is such a quantity as added to the subtrahend will produce the minuend, the algebraic difference of two quantities, whatever their signs and relative values, may readily be found in three steps, as follows :

1. Find what quantity must be added to the subtrahend to produce *zero*. This is evidently the subtrahend with the sign changed.

2. Find what quantity must be added to zero to produce the minuend. This is evidently the minuend.

3. Find the sum of the two quantities added. This is evidently the difference. Therefore,

*Prin. 4.*—*The algebraic difference of two quantities is the algebraic sum obtained by adding to the minuend the subtrahend with the sign changed.*

This is *the algebraic law of subtraction*.

### Problems.

#### 1. To add similar monomials.

**Illustration.**—Find the sum of  $+2a^2b$ ,  $-3a^2b$ ,  $+4a^2b$ , and  $-6a^2b$ .

**Solution:** The sum of  $+2a^2b$  and  $+4a^2b$  is  $+6a^2b$ , and the sum of  $-3a^2b$  and  $-6a^2b$  is  $-9a^2b$  [P. 2]. The sum of  $-9a^2b$  and  $+6a^2b$  is  $-3a^2b$  [P. 3]. Therefore,  $(+2a^2b) + (-3a^2b) + (+4a^2b) + (-6a^2b) = -3a^2b$ .

**Form.**

$+2a^2b$	$-3a^2b$
$+4a^2b$	$-6a^2b$
$+6a^2b$	$-9a^2b$
	$+6a^2b$
	$-3a^2b$

#### 2. To add dissimilar monomials.

**75.** In the addition of dissimilar terms with unlike signs, when the relative absolute values of the terms are not known, precedence is given to the values of the positive terms, and the real sign of the sum is determined afterward, when the relative values become known.

**Illustration.**—Find the sum of  $+2a$ ,  $+3b$ , and  $-4c$ .

**Solution:** The sum of  $+2a$  and  $+3b$  is  $(2a+3b)$  [P. 2]. The sum of  $(2a+3b)$  and  $-4c$  is  $+(2a+3b-4c)$  [P. 3]. Therefore,  $(+2a) + (+3b) + (-4c) = +(2a+3b-4c) = 2a+3b-4c$  [70, 58, n.].

**76.** Since  $(+2a) + (+3b) + (-4c) + (-d) = 2a+3b-4c-d$ , by addition,  $2a+3b-4c-d$  and  $(+2a) + (+3b) + (-4c) + (-d)$  are equivalent expressions. Therefore,

**Prin. 5.**—Any polynomial may be reduced to the form of  $a+b+c$  etc., in which  $a, b, c$ , etc., may have any values, positive or negative.

### 3. To add monomials with like factors.

**Illustration.**—Find the sum of  $ax$ ,  $bx$ , and  $-cx$ .

**Solution:**  $b$  times  $x$  added to  $a$  times  $x$  is evidently  $(a+b)x$ , and  $-c$  times  $x$  added to  $a+b$  times  $x$  is evidently  $(a+b-c)x$ .

Form.	
$a$	$x$
$b$	$x$
$-c$	$x$
$(a+b-c)x$	

### 4. To subtract monomials.

**Illustration.**—

Find the difference of  $+3a^2b$  and  $-4xy$ .

**Solution:** The difference of

$$\left\{ \begin{array}{c} +3a^2b \\ \text{and} \\ -4xy \end{array} \right\} = \text{the sum of } \left\{ \begin{array}{c} +3a^2b \\ \text{and} \\ +4xy \end{array} \right\} \quad [\text{P. 4}] = 3a^2b + 4xy \quad [\text{Prob. 2}].$$

### 5. To add polynomials.

**Illustration.**—Find the sum of  $4x^2 - 3xy + 2y^2$ ,  $3x^2 + 7xy - 3y^2$ , and  $-6x^2 + 4xy - 2y^2$ .

Form.							
$4x^2 - 3xy + 2y^2 =$	$4x^2 + (-3xy) + (+2y^2)$ [P. 5]						
$3x^2 + 7xy - 3y^2 =$	$3x^2 + (+7xy) + (-3y^2)$ [P. 5]						
$-6x^2 + 4xy - 2y^2 =$	$-6x^2 + (+4xy) + (-2y^2)$ [P. 5]						
<table style="width: 100%; border-collapse: collapse;"> <tr> <td style="padding: 5px;">Sum</td> <td style="padding: 5px;"><math>=</math></td> <td style="padding: 5px;"><math>x^2 + (+8xy) + (-3y^2)</math>, or</td> </tr> <tr> <td style="padding: 5px;">Sum</td> <td style="padding: 5px;"><math>=</math></td> <td style="padding: 5px;"><math>x^2 + 8xy - 3y^2</math> [P. 5]</td> </tr> </table>		Sum	$=$	$x^2 + (+8xy) + (-3y^2)$ , or	Sum	$=$	$x^2 + 8xy - 3y^2$ [P. 5]
Sum	$=$	$x^2 + (+8xy) + (-3y^2)$ , or					
Sum	$=$	$x^2 + 8xy - 3y^2$ [P. 5]					

**Solution:** Since the sum of two or more terms is the same in whatever order they are combined [P. 1], we arrange the terms of the polynomials so that like terms stand in the same column, then add the columns separately, and combine the results according to the law of algebraic addition.

In practice, it will not be necessary to put each polynomial into the expanded form, as precisely the same result will be reached if the columns in the original polynomials be added with reference to their signs, and the results taken as the terms of the sum.

### 6. To subtract polynomials.

**Illustration.**—Find the difference of  $4x^2 - 3xy + 2y^2$  and  $3x^2 + 7xy - 3y^2$ .

$$\begin{array}{l} \text{Form.} \\ \text{Diff. of } \left\{ \begin{array}{l} 4x^2 - 3xy + 2y^2 \\ 3x^2 + 7xy - 3y^2 \end{array} \right\} = \text{sum of } \left\{ \begin{array}{l} 4x^2 - 3xy + 2y^2 \\ -3x^2 - 7xy + 3y^2 \end{array} \right\} \\ \text{which is } \underline{x^2 - 10xy + 5y^2} \end{array}$$

**Solution:** The difference of  $4x^2 - 3xy + 2y^2$  and  $3x^2 + 7xy - 3y^2$  is such a quantity as added to  $3x^2 + 7xy - 3y^2$  will produce  $4x^2 - 3xy + 2y^2$ .

If  $-3x^2 - 7xy + 3y^2$  be added to  $3x^2 + 7xy - 3y^2$ , the result will be zero; and if  $4x^2 - 3xy + 2y^2$  be added to zero, the result will be  $4x^2 - 3xy + 2y^2$ .

Therefore, if the sum of  $-3x^2 - 7xy + 3y^2$  and  $4x^2 - 3xy + 2y^2$  be added to  $3x^2 + 7xy - 3y^2$ , the result will be  $4x^2 - 3xy + 2y^2$ . Therefore, *the difference is the minuend added to the subtrahend with the signs of its terms changed.*

### EXERCISE 8.

Find the value of :

1.  $3x^2y + (-6x^2y) + (+7x^2y) + (-2x^2y)$
2.  $3x^2y^2z^2 + (5x^2y^2z^2) + (-2x^2y^2z^2) + (-3x^2y^2z^2)$
3.  $-7a^3b^2 - (+2a^3b^2)$
4.  $10x^2y^2 - (-7x^2y^2)$
5.  $+a - (+c)$
6.  $-a - (-c)$
7.  $ax + (-bx) + (+cx) + (-dx)$
8.  $5(a-b) + [-7(a-b)] + 8(a-b) + [-5(a-b)] + [-3(a-b)]$

9.  $2ab + (-3cd) + 4x + (-mn)$
10.  $a(x+y) - [-b(x+y)]$
11.  $3p^2q^2 - (-4r^2s^2)$
12. Collect  $3(x+y)^2 + 4(x+y)^2 - 5(x+y)^2 - 3(x+y)^2 + 2(x+y)^2$
13. Collect  $2mp(p-q) + 3mp(p-q) - 5mp(p-q) - mp(p-q)$
14. Collect  $3a(x^2+y^2) - 4a(x^2+y^2) - 6a(x^2+y^2) + 5a(x^2+y^2)$
15. Collect  $4(a+b) - 3(x+y) + 4(a+b) - 7(x+y) - 3(a+b)$
16. Add  $5x^3 - 3x^2 + 7x - 1$ ,  $4x^3 - 7x^2 - 3x + 6$ ,  
 $2x^3 - 5x^2 + 7x + 9$ ,  $9x - 6x^3 + 7x^2 - 5$ ,  
 and  $8 - 7x + 6x^2 - 5x^3$
17. Add  $12mnp - 10rst + 6mt$ ,  
 $3rst - 9mnp - 7mt$ ,  $5mnp - 2mt + 8rst$ ,  
 and  $2mt - 2rst - 6mnp$
18. Add  $4(a+b)^2 - 3(a+b) + 10$ ,  
 $7(a+b)^2 + 5(a+b) - 6$ ,  $-9(a+b)^2 - 4(a+b) - 3$ ,  
 and  $-2(a+b)^2 - 6(a+b) + 9$
19. Subtract  $12x^3 - 7x^2 + 5x + 6$  from  
 $9x^3 + 3x^2 - 5x + 2$
20. From  $2a^3 - 3b^3$  subtract  $5a^3 + 3a^2b - 6ab^2 + 2b^3$
21. From  $x^3 + 1$  subtract  $3x^2 + 2x - 6$
22. Subtract  $5(a+b)^2 + 3(a+b) - 5$  from  
 $3(a+b)^2 - 4(a+b)$
23. Subtract  $1 + a + a^2 + a^3 + a^4$  from  $1 + a^3$ ,  
 and add  $a - a^4$  to the result.
24. What must be added to  $3m^2 - 2b^2 + 3n^2$   
 in order that the sum may be  $bn + mn + bm$ ?

## Symbols of Aggregation.

## Principles.

**77.** The expression  $x + (y - z)$  denotes that  $y - z$  is to be added to  $x$ . Performing the operation indicated, we have  $x + y - z$ . Therefore,

**Prin. 1.**—*If a number of terms stand between a pair of symbols of aggregation that are preceded by plus, the symbols and the sign before them may be removed without changing the value of the expression.*

**78.** Since  $x + (y - z) = x + y - z$  [P. 1],  $x + y - z = x + (y - z)$ . Therefore,

**Prin. 2.**—*Any number of terms may be placed between a pair of symbols of aggregation, and preceded by plus, without changing the value of the expression.*

**79.** The expression  $a - (b - c)$  denotes that  $b - c$  is to be subtracted from  $a$ . Performing the operation indicated, we have  $a - b + c$ . Therefore,

**Prin. 3.**—*If a number of terms stand between a pair of symbols of aggregation that are preceded by minus, the symbols and the sign before them may be removed, if the sign of every term be changed.*

**80.** Since  $a - (b - c) = a - b + c$  [P. 3],  $a - b + c = a - (b - c)$ . Therefore,

**Prin. 4.**—*Any number of terms may be placed between a pair of symbols of aggregation, and preceded by minus, if the sign of every term be changed.*

These four principles constitute the law of brackets.

## Problems.

## 1. To simplify parenthetical expressions.

**Illustrations.**—1. Simplify  $3ab + (4ab - 3ab + 2ab)$ .

**Solution :**  $3ab + (4ab - 3ab + 2ab) =$

$$3ab + 4ab - 3ab + 2ab \text{ [P. 1]} = 6ab.$$

2. Simplify  $6xy - (3xy - 2xy + 4xy)$ .

**Solution:**  $6xy - (3xy - 2xy + 4xy) =$

$$6xy - 3xy + 2xy - 4xy \text{ [P. 8]} = xy.$$

3. Simplify  $5x^2 - [3x^2 + \{2x^2 - (x^2 + 3x^2) + 5x^2\} - x^2]$ .

**Solution:**  $5x^2 - [3x^2 + \{2x^2 - (x^2 + 3x^2) + 5x^2\} - x^2]$

$$= 5x^2 - [3x^2 + \{2x^2 - x^2 - 3x^2 + 5x^2\} - x^2],$$

removing the parenthesis,

$$= 5x^2 - [3x^2 + 2x^2 - x^2 - 3x^2 + 5x^2 - x^2],$$

removing the braces,

$$= 5x^2 - 3x^2 - 2x^2 + x^2 + 3x^2 - 5x^2 + x^2 = 0,$$

removing brackets and uniting.

**Note.**—The operation may often be simplified by collecting the terms as the symbols are removed.

Thus  $5x^2 - [3x^2 + \{2x^2 - (x^2 + 3x^2) + 5x^2\} - x^2] =$

$$5x^2 - [3x^2 + \{2x^2 - 4x^2 + 5x^2\} - x^2] =$$

$$5x^2 - [3x^2 + 3x^2 - x^2] = 0.$$

#### EXERCISE 4.

Simplify :

1.  $2x^3 - (3x^3 - 4x^3 + 2x^3)$

2.  $4m^2n + (3m^2n - 6m^2n + 2m^2n)$

3.  $3yz - \{2yz + (yz - 2yz)\}$

4.  $4x^2y^2 + \{(-3x^2y^2 + 2x^2y^2) - x^2y^2\}$

5.  $-(x^2y^3 + 3x^2y^3) - (4x^2y^3 - 6x^2y^3)$

6.  $- \{(-ab + 2ab) - (3ab - 2ab)\}$

7.  $- \{1 - (-1) - 1\} - \{-1 + (-1) - 1\}$

8.  $- \{4 - 3 - 2 - (-2 - 3 - 4) - 4 - 3\}$

9.  $(x - y) - (x + y) - (y - x) - (x - y)$

10.  $-7abc - [-3abc - \{2abc - (4abc - abc)\}]$

11.  $(a + b - c) - (2a + b - c) - (a - \overline{b - c})$

12.  $(3a + 7b + 3c) - (4a + 2b + 2c) - (c + 5b)$

13.  $(4a - 7a) - (6a - 4a) - (5a - 4a) - (3a - 9a)$

14.  $6x - (3z - 2y) - (2x - \overline{3y - 4z}) - (z - \overline{7x - 5y})$

15.  $3a - [3b + \{3a - (3b + 3a)\}]$
16.  $x^2 - \{-(x^2 - x^2 - x^2) - (x^2 - x^2) - x^2\} - (2x^2 + x^2)$
17.  $5 - [5 - \{5 - (5 - 5) - 5\} - 5] - \{5 + 5 - (5 + 5)\}$
18.  $4z - (12x - 2y) - \{2z - (10x + 4y) - (2x - 6y)\}$
19.  $3ax^2 - \{2ax^2 + (5ax^2 - 3ax^2) - (4ax^2 - 2ax^2) + 3ax^2\}$
20.  $pq - \{pq - (pq - \overline{pq - pq}) - (pq - \overline{pq - pq})\}$

## 2. To affect terms by symbols of aggregation.

**Rule.**—Prefix the pair of symbols by the sign of the first term to be affected; if this be *plus*, do not make any change in the terms [P. 2], but if it be *minus*, change the sign of every term affected [P. 4].

## EXERCISE 5.

Express in binomial and in trinomial terms :

1.  $3x + 4b - 2c + 4d - 5x - 3b$
2.  $4y - 2z + 3x - 5m - 6n + 2z$
3.  $7a - 3b - c - 2d - e - 3f$
4.  $-5m + 2n - 3mn + 4m^2 - 6n^2 + 3am$
5.  $x^2 - 3xy - 2y^2 - 7x^2y + 7xy^2 - y^3$
6.  $-3a^3 - 2ab - 3b^2 + 4a - 2b + 3c$

Express  $a + b - c - d + e - f + g - h - k + l - m + n$

7. In binomial terms.
8. In trinomial terms.
9. In quadrinomial terms.
10. In trinomial terms having the last two terms of each affected by a vinculum.
11. In quadrinomial terms having the last two terms of each affected by a parenthesis.
12. In quadrinomial terms having the last three terms of each affected by a parenthesis, and the last two terms of each again affected by a vinculum.

## Multiplication.

## Definitions and Principles.

81. The *arithmetical product* of two or more quantities is their product when no regard is had to their character as positive or negative, and the *algebraic product* their product when regard is had to their character as positive or negative.

82. In algebraic notation the multiplier denotes how many times and in what manner, whether additively or subtractively, the multiplicand is to be taken. The algebraic multiplication of two quantities may, therefore, be defined as the process of taking one algebraic quantity as many times and in such a manner as is indicated by another.

Thus,  $(+b) \times (+a)$  denotes that  $b$  positive units are to be taken  $a$  times *additively*. That is,

$$(+b) \times (+a) = (+b) + (+b) + (+b) + \dots$$

to  $a$  terms.

83. If  $b$  units be placed in a row and  $a$  such rows be formed, the whole number of units will evidently be  $a$  times  $b$ . If, now, one unit be taken from each row,  $a$  units will be taken; and if this be done  $b$  times,  $b$  times  $a$  units will be taken, and none will remain. Therefore, according to the law of association,

$$b \times a = a \times b.$$

By a continuation of this process of reasoning, it may be shown that

$$a \times b \times c = a \times c \times b = b \times a \times c =$$

$$b \times c \times a = c \times a \times b = c \times b \times a,$$

and so on for any number of factors. Therefore,

**Prin. 1.**—*The product of two or more factors is the same, in whatever order they are taken.*

This is *the commutative law of multiplication.*

According to this law, the literal factors of a product may be arranged in alphabetical order.

**84.**  $xyz \times n = xy \times nz = xz \times ny = yz \times nx$ , according to the commutative law. Therefore,

**Prin. 2.**—*Multiplying one factor of a quantity multiplies the quantity.*

This is *the factorial law of multiplication.*

**85.**  $(a + b + c) \times d$ , when  $d$  is a positive integer, and  $a$ ,  $b$ , and  $c$  have any values, positive or negative, equals

$$\begin{aligned} & (a + b + c) + (a + b + c) + (a + b + c) \dots \text{to } d \text{ terms} \\ &= a + b + c + a + b + c + a + b + c + \dots [77, P. 1] \\ &= a + a + a + \dots \text{to } d \text{ terms} + b + b + b + \dots \text{to } d \text{ terms} \\ &+ c + c + c + \dots \text{to } d \text{ terms} [67, P. 1] \\ &= ad + bd + cd. \end{aligned}$$

When  $d$  is a negative integer,

$$\begin{aligned} & (a + b + c) \times (-d) = -(a + b + c) - (a + b + c) - \\ & (a + b + c) - \dots \text{to } d \text{ terms} \\ &= -a - b - c - a - b - c - a - b - c - \text{etc.} [79, P. 3] \\ &= -a - a - a \dots \text{to } d \text{ terms} - b - b - b - \dots \text{to } \\ & d \text{ terms} - c - c - c - \dots \text{to } d \text{ terms} \\ &= -ad - bd - cd. \text{ Therefore,} \end{aligned}$$

**Prin. 3.**—*Multiplying every term of a quantity multiplies the quantity.*

This is *the distributive law of multiplication.*

**86.**  $a^m = a \times a \times a \times \dots$  to  $m$  factors

$a^n = a \times a \times a \times a \times \dots$  to  $n$  factors

$\therefore a^m \times a^n = a \times a \times a \times a \dots$  to  $m + n$  factors  $= a^{m+n}$ .

Therefore,

**Prin. 4.**—*The exponent of a factor in the product equals the sum of the exponents of the same factor in the multiplicand and the multiplier.*

This is the exponential law of multiplication, and will, hereafter, be proved true for any rational values of  $m$  and  $n$ .

**87. 1.**  $+a$  times  $+b = b$  positive units taken  $a$  times additively  $= ab$  positive units taken additively  $= +(+ab) = +ab$  [77, P. 1].

**2.**  $-a$  times  $-b = b$  negative units taken  $a$  times subtractively  $= ab$  negative units taken subtractively  $= -(-ab) = +ab$  [79, P. 3].

**3.**  $+a$  times  $-b = b$  negative units taken  $a$  times additively  $= ab$  negative units taken additively  $= +(-ab) = -ab$  [77, P. 1].

**4.**  $-a$  times  $+b = b$  positive units taken  $a$  times subtractively  $= ab$  positive units taken subtractively  $= -(+ab) = -ab$  [79, P. 3]. Therefore,

**Prin. 5.**—*The product of two quantities with like signs is positive, and with unlike signs is negative.*

This is the law of signs in multiplication, and is in no way controlled by the numerical values of the quantities multiplied together.

### Problems.

#### 1. To multiply a monomial by a monomial.

**Illustration.**—Multiply  $-4a^3b^2c$  by  $+3a^2bd$ .

**Solution:** Since multiplying one factor of a quantity multiplies the quantity [P. 2],  $-4 \times a^3 \times b^2 \times c$  is multiplied by  $+3 \times a^2 \times b \times d$ , if  $-4$  is multiplied by  $+3$ ,  $a^3$  by  $a^2$ ,  $b^2$  by  $b$ , and  $c$  by  $d$ .  $+3$  times  $-4$  is  $-12$  [P. 5];  $a^3$  times  $a^2$  is  $a^5$ , and  $b^2$  times  $b$  is  $b^3$  [P. 4], and  $d$  times  $c$  is  $cd$ ; hence, the product is  $-12a^5b^3cd$ .

	Form.
$-4a^3b^2c$	$-4a^3b^2c$
$+3a^2bd$	$+3a^2bd$
$-12a^5b^3cd$	$-12a^5b^3cd$

### 2. To multiply a polynomial by a monomial.

**Illustration.**—Multiply  $2a^2 - 3ab + 4b^2$  by  $-4ab$ .

**Solution :** Since multiplying every term of a quantity multiplies the quantity [P. 3], we multiply every term of  $2a^2 - 3ab + 4b^2$  by  $-4ab$ , and obtain  $-8a^3b + 12a^2b^2 - 16ab^3$ .

Form.
$  \begin{array}{r}  2a^2 - 3ab + 4b^2 \\  -4ab \\  \hline  -8a^3b + 12a^2b^2 - 16ab^3  \end{array}  $

### 3. To multiply a polynomial by a polynomial.

**Illustration.**—Multiply  $x^2 + xy + y^2$  by  $x^2 - xy + y^2$ .

**Solution :**  $x^2 - xy + y^2 = x^2 + (-xy) + (+y^2)$ ; hence,  $x^2 - xy + y^2$  times  $x^2 + xy + y^2$  equals the sum of

1.  $x^2$  times  $x^2 + xy + y^2 =$

2.  $-xy$  times  $x^2 + xy + y^2 =$

3.  $+y^2$  times  $x^2 + xy + y^2 =$

which =

Form.
$  \begin{array}{r}  x^2 + xy + y^2 \\  x^2 - xy + y^2 \\  \hline  x^4 + x^3y + x^2y^2 \\  -x^3y - x^2y^2 - xy^3 \\  \hline  \phantom{x^4 +} + x^2y^2 + xy^3 + y^4 \\  \hline  x^4 + x^2y^2 + y^4  \end{array}  $

**88.** For convenience of multiplication, the terms of both multiplicand and multiplier should be arranged according to the ascending or descending powers of some letter assumed as the leading letter. Thus,

$$(x^3 + ax^2 + bx + c)(x^3 - ax^2 - bx + c)$$

is a better form than

$$(x^3 + bx + c + ax^2)(c - ax^2 + x^3 - bx).$$

**89.** If the terms of both multiplier and multiplicand are arranged according to the ascending or descending powers of a leading letter, much of the work of multiplication may be avoided by using in the operation only the coefficients, and supplying the letters afterward in the product. Thus,

$$\begin{array}{r}
 3x^2 + 2x + 5 \\
 2x^2 - 4x + 3 \\
 \hline
 6x^4 + 4x^3 + 10x^2 \\
 -12x^3 - 8x^2 - 20x \\
 + 9x^2 + 6x + 15 \\
 \hline
 6x^4 - 8x^3 + 11x^2 - 14x + 15
 \end{array}
 \qquad
 \begin{array}{r}
 3 + 2 + 5 \\
 2 - 4 + 3 \\
 \hline
 6 + 4 + 10 \\
 -12 - 8 - 20 \\
 + 9 + 6 + 15 \\
 \hline
 6 - 8 + 11 - 14 + 15
 \end{array}$$

The highest power of  $x$  in the product is  $(x^2 \times x^2)$ , or  $x^4$ , and the rest follow in order, and may therefore be supplied without carrying them through the partial products.

90. When some of the powers of the leading letter are wanting in the polynomials, their places must be supplied by 0's. Thus,

$$\begin{array}{r}
 3x^3 + 2x + 4 \\
 2x^2 + 3x + 2 \\
 \hline
 6x^6 + 4x^4 + 8x^3 \\
 + 9x^4 + 6x^3 + 12x \\
 + 6x^3 + 4x + 8 \\
 \hline
 6x^6 + 13x^4 + 14x^3 + 6x^2 + 16x + 8
 \end{array}$$
  

$$\begin{array}{r}
 3 \quad + 0 \quad + 2 \quad + 4 \\
 2 \quad + 0 \quad + 3 \quad + 2 \\
 \hline
 6 \quad + 0 \quad + 4 \quad + 8 \\
 \qquad \qquad \qquad 9 \quad + 0 \quad + 6 \quad + 12 \\
 \qquad \qquad \qquad \qquad \qquad 6 \quad + 0 \quad + 4 \quad + 8 \\
 \hline
 6 \quad + 0 \quad + 13 \quad + 14 \quad + 6 \quad + 16 \quad + 8
 \end{array}$$

or,  $6x^6 + 0x^5 + 13x^4 + 14x^3 + 6x^2 + 16x + 8$

This process is known as multiplication by *detached coefficients*.

#### EXERCISE 6.

1. Show that the product of any even number of factors with like signs is positive.

2. Show that the product of any odd number of factors with like signs has the same sign as the factors.

3. Show that if the signs of an even number of factors be changed, the sign of their product will remain unchanged.

4. Show that if the signs of an odd number of factors be changed, the sign of their product will be changed.

5. Show that changing the sign of every term of a polynomial changes the sign of the polynomial.

6. Show that  $a^m \times a^n \times a^p = a^{m+n+p}$

7. Tell which of the following statements are true and which false, and why :

1.  $(m - n) = -(n - m)$

2.  $(-x) \times (-y) \times (-z) = x \times y \times z$

3.  $(y - x)(x - y) = -(x - y)^2$

4.  $(x - z)(z - y) = (z - x)(y - z)$

5.  $(m - y)(y - m)(m - y) = (y - m)^3$

6.  $(x - y)^2(x - z)^2(z - x)^2(y - x)^2 = (x - y)^4(z - x)^4$

Find the product of :

8.  $(-3x^2y^2z)$  and  $2x^3y^2z$

9.  $-4(m + n)^2(x - y)^3$  and  $5(m + n)(x - y)$

10.  $2a^m(x + y)^n$  and  $3a^n(x + y)^m$

11.  $(x + y)^{m+n}$  and  $(x + y)^{n-m}$

12. Multiply  $3mnp - 5m^2p^2 - 6n^2q + 7$  by  $-5mq$

13. Multiply  $2a^2(x + y)^3 + 3ab(x + y)^2 - 6b^2(x + y) + 2$  by  $-3ab(x + y)^3$

14.  $(x^m + 2xy + y^n) \times 3x^ny^n = \text{what?}$

15.  $(a^3 + a^2b + ab^2 + b^3)(a - b) = \text{what?}$

16.  $(3a^{2n} - 4a^n b^n + 2b^{2n})(2a^{2n} - 3a^n b^n - b^{2n}) = \text{what?}$

17.  $(4x^{3p} - 3x^{2p}y^p + 6x^py^{2p} + 2y^{3p})(2x^p + 3y^p) = \text{what?}$

18.  $(16a^{4n} - 24a^{3n}b^n + 36a^{2n}b^{2n} - 54a^nb^{3n} + 81b^{4n})(2a^n + 3b^n) = \text{what?}$

19. Multiply by detached coefficients  $x^4 + 3x^3 - 2x + 5$   
and  $x^3 - 2x^2 + 7x - 3$

20. Multiply by detached coefficients  $x^5 + 4x^3 - 3$  by  
 $3x^4 - 2x^2 + 4$

### Special Principles of Multiplication.

91. The sum of  $a$  and  $b$  multiplied by their difference,  
or  $(a + b)(a - b) = a^2 - b^2$ . Therefore,

*Prin. 1.—The product of the sum and difference of two quantities equals the square of the first minus the square of the second.*

$$\begin{aligned} 92. \quad (x + a)(x + b) &= x^2 + (a + b)x + ab \\ (x + a)(x - b) &= x^2 + (a - b)x - ab \\ (x - a)(x - b) &= x^2 - (a + b)x + ab \end{aligned}$$

Notice—1. That in each of the above examples we have found the product of two binomials having a common term  $x$ , and two unlike terms,  $a$  and  $b$ , both positive in the first, one positive and one negative in the second, and both negative in the third.

2. That the first term of each product is the square of the common term  $x$ , the second term is the algebraic sum of the unlike terms  $(+a$  and  $+b$ ,  $+a$  and  $-b$ ,  $-a$  and  $-b)$  into the common term  $x$ , and the third term is the algebraic product of the unlike terms  $(+a \times +b)$ ,  $(a \times -b)$ ,  $(-a \times -b)$ . Therefore,

*Prin. 2.—The product of two binomials having a common term equals the square of the common term and the algebraic sum of the unlike terms into the common term, and the algebraic product of the unlike terms.*

$$\begin{aligned} 93. \quad (a + b)(x + y) &= ax + (ay + bx) + by \\ (a - b)(x + y) &= ax + (ay - bx) - by \\ (a - b)(x - y) &= ax - (ay + bx) + by \end{aligned}$$

Notice—1. That in each of these examples we have found the product of two binomials.

2. That the first term of each product is the product of the first terms of the binomials.

3. That the second term of each product is the algebraic sum of the products obtained by a cross-multiplication of the first and second terms.

4. That the third term of each product is the algebraic product of the second terms. Therefore,

**Prin. 3.**—*The product of any two binomials equals the product of their first terms, and the algebraic sum of the products obtained by a cross-multiplication of the first and second terms, and the algebraic product of the second terms.*

**94.** The following convenient formulas may be readily proved by actual multiplication :

$$1. (x^2 - xy + y^2)(x + y) = x^3 + y^3$$

$$2. (x^2 + xy + y^2)(x - y) = x^3 - y^3$$

$$3. (x^{2n} + x^n y^n + y^{2n})(x^{2n} - x^n y^n + y^{2n}) = x^{4n} + x^{2n} y^{2n} + y^{4n}$$

$$4. (x^2 + y^2 + z^2 - xy - xz - yz)(x + y + z) = x^3 + y^3 + z^3 - 3xyz$$

#### EXERCISE 7.

Expand :

1.  $(m^2 - n^2)(m^2 + n^2)$
2.  $(x^n + y^n)(x^n - y^n)$
3.  $(x + 3y)(x + y)$
4.  $(x - 3a)(x - 2a)$
5.  $(x^m + 6)(x^m - 5)$
6.  $(a + 3)(2a + 1)$
7.  $(3x^2 + 4)(3x^2 - 5)$
8.  $(5x^4 + 2)(3x^4 - 5)$
9.  $(x^m + y^n)(x^m + y^n)$
10.  $(ax + by)(ax + cy)$
11.  $(x^2 + xy + y^2)(x^2 - xy + y^2)$
12.  $(4x^2 - 6xy + 9y^2)(4x^2 + 6xy + 9y^2)$
13.  $(4x^2 - 6xy + 9y^2)(2x + 3y)$
14.  $(4x^2 + 6xy + 9y^2)(2x - 3y)$
15.  $(\overline{a + b + 2c})(\overline{a + b - 2c})$
16.  $(\overline{a + x - y})(\overline{a - x - y})$
17.  $(x + y)(x - y)(x^2 + y^2)(x^4 + y^4)$
18.  $(m + 6)(m - 6)(m^2 + 9)$
19.  $(\overline{1 - x} + 7)(\overline{1 - x} - 6)$
20.  $(m^4 + n^4 + p^4 - m^2 n^2 - m^2 p^2 - n^2 p^2)(m^2 + n^2 + p^2)$

## Division.

## Definitions and Principles.

**95.** The *arithmetical quotient* of two quantities is their quotient when no regard is had to their character as positive or negative; and the *algebraic quotient* is their quotient when regard is had to their character as positive or negative.

**96.** In algebraic division, the quotient denotes how many times and in what manner the divisor must be taken to produce the dividend.

**97.** The general problem of division is: "*Given the product of two quantities and one of them, to find the other.*"

**98.** Since  $xyz \times n = (xy) \times (nz) = (xz) \times (ny) = (yz) \times (nx) = nxyz$ , according to the factorial law of multiplication, it follows that  $nxyz \div n = (xy) \times (nz \div n) = (xz) \times (ny \div n) = (yz) \times (nx \div n) = xyz$ . Therefore,

**Prin. 1.**—*Dividing one factor of a quantity divides the quantity.*

This is *the factorial law of division*.

**99.** Since  $(a + b - c) \times d = a \times d + b \times d - c \times d = ad + bd - cd$ , according to the distributive law of multiplication, it follows that  $(ad + bd - cd) \div d = ad \div d + bd \div d - cd \div d = a + b - c$ . Therefore,

**Prin. 2.**—*Dividing every term of a quantity divides the quantity.*

This is *the distributive law of division*.

**100.** Since  $a^m \times a^n = a^{m+n}$ , according to the exponential law of multiplication, it follows that  $a^{m+n} \div a^m = a^{(m+n)-m} = a^n$ . Therefore,

**Prin. 3.**—*The exponent of a factor in the quotient equals the exponent of that factor in the dividend, minus the exponent of the same factor in the divisor.*

This is the exponential law of division, and will hereafter be proved true for any values of the exponents.

101. According to the law of signs in multiplication,

$$1. (+a) \times (+b) = +ab, \quad \therefore (+ab) \div (+a) = +b$$

$$2. (-a) \times (+b) = -ab, \quad \therefore (-ab) \div (-a) = +b$$

$$3. (+a) \times (-b) = -ab, \quad \therefore (-ab) \div (+a) = -b$$

$$4. (-a) \times (-b) = +ab, \quad \therefore (+ab) \div (-a) = -b$$

Therefore,

**Prin. 4.**—*The quotient of two quantities with like signs is positive, and with unlike signs negative.*

This is the law of signs in division.

102.  $a^0 \div a^0 = a^0$  [P. 3]; but  $a^0 \div a^0 = 1$ ; therefore,  $a^0 = 1$ ; whence we have

**Prin. 5.**—*Any quantity with an exponent of zero equals unity.*

Problems.

1. To divide a monomial by a monomial.

**Illustration.**—Divide

$$+16a^5b^3c \text{ by } -2a^2b. \quad \begin{array}{r} \text{Form.} \\ -2a^2b \overline{) 16a^5b^3c} \\ \underline{-8a^3b^2c} \end{array}$$

**Solution:** We divide  $+16$  by  $-2$ ,  $a^5$  by  $a^2$ ,  $b^3$  by  $b$ , and  $c$  by  $1$  [P. 1].  $+16$  divided by  $-2$  is  $-8$  [P. 4];  $a^5$  divided by  $a^2$  is  $a^3$ , and  $b^3$  divided by  $b$  is  $b^2$  [P. 3]; and  $c$  divided by  $1$  is  $c$ . Therefore,  $16a^5b^3c$  divided by  $-2a^2b$  is  $-8a^3b^2c$ .

2. To divide a polynomial by a monomial.

**Illustration.**—Divide

$$6a^3 - 9a^2b + 12ab^2 \text{ by } 3a. \quad \begin{array}{r} \text{Form.} \\ 3a \overline{) 6a^3 - 9a^2b + 12ab^2} \\ \underline{2a^2 - 3ab + 4b^2} \end{array}$$

**Solution:** Since dividing every term of a quantity divides the quantity [P. 2], we divide each term of  $6a^3 - 9a^2b + 12ab^2$  by  $3a$ , and obtain  $2a^2 - 3ab + 4b^2$ .

## 3. To divide a polynomial by a polynomial.

**Illustration.**—Divide  $x^4 + x^3y^2 + y^4$  by  $x^2 + xy + y^2$ .

**Form.**

$$\begin{array}{r}
 x^2 + xy + y^2 \overline{) x^4 + x^3y^2 + y^4} \quad (x^2 - xy + y^2 \\
 \underline{x^4 + x^3y + x^2y^2} \phantom{+ y^4} \\
 -x^2y + y^4 \\
 \underline{-x^2y - x^2y^2 - xy^3} \phantom{+ y^4} \\
 x^2y^2 + xy^3 + y^4 \\
 \underline{x^2y^2 + xy^3 + y^4} \\
 0
 \end{array}$$

**Solution:**  $x^2 + xy + y^2$  is contained in  $x^4 + x^3y^2 + y^4$  as many times as it can be taken out of it.  $x^2$  is contained in  $x^4$ ,  $x^2$  times; taking  $x^2$  times  $x^2 + xy + y^2$ , or  $x^4 + x^3y + x^2y^2$ , out of  $x^4 + x^3y^2 + y^4$  by subtraction, there remains  $-x^2y + y^4$ .  $x^2$  is contained in  $-x^2y$ ,  $-xy$  times; taking  $-xy$  times  $x^2 + xy + y^2$ , or  $-x^3y - x^2y^2 - xy^3$ , out of  $-x^2y + y^4$ , there remains  $x^2y^2 + xy^3 + y^4$ .  $x^2$  is contained in  $x^2y^2$ ,  $y^2$  times; taking  $y^2$  times  $x^2 + xy + y^2$ , or  $x^2y^2 + xy^3 + y^4$ , out of  $x^2y^2 + xy^3 + y^4$ , nothing remains. Therefore,  $x^2 + xy + y^2$  is contained in  $x^4 + x^3y^2 + y^4$ ,  $x^2 - xy + y^2$  times.

**Note.**—Before dividing, arrange both dividend and divisor according to the ascending or descending powers of some letter assumed as a leading letter.

**103.** When the terms of the dividend and divisor are arranged according to the ascending or descending powers of a leading letter, the method of *detached coefficients* may often be used with advantage. Thus, to divide

$x^5 + 6x^4 + 11x^3 + 11x^2 + 7x + 20$  by  $x^2 + 3x + 4$ , do as follows :

$$\begin{array}{r}
 1 + 3 + 4 \overline{) 1 + 6 + 11 + 11 + 7 + 20} \quad (1 + 3 - 2 + 5 \\
 \underline{1 + 3 + 4} \phantom{+ 11 + 11 + 7 + 20} \\
 3 + 7 + 11 \\
 \underline{3 + 9 + 12} \\
 - 2 - 1 + 7 \\
 \underline{- 2 - 6 - 8} \\
 5 + 15 + 20 \\
 \underline{5 + 15 + 20} \\
 0
 \end{array}$$

The first term of the quotient is  $x^3 + x^3 = x^6$ , and the other powers of the letter follow in order. Therefore, the quotient is  $x^3 + 3x^2 - 2x + 5$ .

As in multiplication by detached coefficients, the places of all powers that are wanting must be supplied by 0's.

**104.** When the divisor is of the form of  $x \pm a$ , the division may be still further abridged.

Let us first divide by the method of detached coefficients  $3x^4 + 4x^3 + 6x - 4$  by  $x + 2$ .

$$\begin{array}{r}
 \text{Form.} \\
 1 + 2 \overline{) 3 + 4 + 0 + 6 - 4} \quad (3 - 2 + 4 - 2 \\
 \underline{3 + 6} \\
 - 2 \\
 - 2 - 4 \\
 \hline
 + 4 \\
 + 4 + 8 \\
 \hline
 - 2 \\
 - 2 - 4 \\
 \hline
 0
 \end{array}$$

Notice—1. That the first term of the quotient is the same as the first term of the dividend, and that the remaining terms are the successive remainders.

2. That the first remainder is obtained by subtracting from the second term of the dividend the product of the first term of the dividend and the second term of the divisor.

3. That each subsequent remainder is obtained by subtracting from the next term of the dividend the product of the preceding remainder and the second term of the divisor.

Hence, the form of work may be placed as follows—

$$\begin{array}{r}
 + 2) 3 + 4 + 0 + 6 - 4 \\
 \quad + 6 - 4 + 8 - 4 \\
 \hline
 3 - 2 + 4 - 2
 \end{array}$$

and explained thus: 3 times + 2 is + 6; + 6 subtracted from + 4 is - 2; - 2 times + 2 is - 4; - 4 subtracted from 0 is + 4; + 4 times + 2 is + 8; + 8 subtracted from + 6 is - 2; - 2 times + 2 is - 4; - 4 subtracted from - 4 is 0.

**105.** The work may be put in a still more convenient form by changing the sign of the second term of the di-

visor and adding the successive products to the terms of the dividend. Thus,

$$\begin{array}{r} -2) 3 + 4 + 0 + 6 - 4 \\ \quad -6 + 4 - 8 + 4 \\ \hline 3 - 2 + 4 - 2 \end{array}$$

The quotient is  $3x^3 - 2x^2 + 4x - 2$ .

This shorter process is known as *Synthetic Division*, and will be found of great convenience in subsequent work.

## EXERCISE 8.

Divide :

1.  $-42m^{10}n^8p^3q$  by  $-7m^8n^6p$     2.  $-6x^{m+2}$  by  $2x^{m+3}$
3.  $12m^3n^2$  by  $-6mn$     4.  $-20x^{m+n-1}$  by  $-8x^{m-n+1}$
5.  $9x^{2m} + 6x^{3m} - 12x^{4m}$  by  $3x^m$
6.  $8x^2y(x+y)^2 - 12xy^2(x-y)^2 + 16x^2y^2(x^2-y^2)$   
by  $-4xy$
7.  $6(x-y)^{n-2} - 9(x-y)^{n-1} + 12(x-y)^n$   
by  $3(x-y)^{n-3}$
8.  $a^4 - 2a^2b^2 + b^4 + 4ab^2c - c^4$  by  $a^2 - 2ab + b^2 + c^2$
9.  $9x^2 + 24xy + 12y^2 + 30xz + 24yz + 9z^2$   
by  $x + 2y + 3z$
10.  $a^{2n} - b^{2n}$  by  $a^{2n} - b^{2n}$     11.  $a^{2n} - b^{2n}$  by  $a^n - b^n$
12.  $a^n - b^n$  by  $a - b$  to 5 terms
13.  $a^2x^4 + (2ac - b^2)x^2y^2 + c^2y^4$  by  $ax^2 + bxy + cy^2$
14.  $x^4 - (a+b)^2x^2 + 2abx^2 + a^2b^2$  by  $x^2 + (a+b)x + ab$
15.  $x^3 - (a^2 + ab + b^2)x - ab(a+b)$  by  $x - (a+b)$
16.  $8a^4 - 22a^3b + 43a^2b^2 - 38ab^3 + 24b^4$   
by  $2a^2 - 3ab + 4b^2$ ,  
by the method of detached coefficients.
17.  $2a^7b - 5a^6b^2 - 11a^5b^3 + 5a^4b^4 - 26a^3b^5 + 7a^2b^6$   
 $- 12ab^7$  by  $a^4 - 4a^3b + a^2b^2 - 3ab^3$ ,  
by the method of detached coefficients.

18.  $x^5 - 11x^4 + 28x^3 + 2x^2 - 16x + 32$  by  $x - 4$ ,  
by the method of synthetic division.
19.  $x^6 + 2x^5 - 3x^4 + 7x^3 - 2x + 183$  by  $x + 3$ ,  
by synthetic division.
20.  $3x^5 - 4x^3 - 2x - 60$  by  $x - 2$ , by synthetic division.

### Numerical Values of Algebraic Expressions.

#### EXERCISE 9.

Find the values of the following expressions when  $a = -2$ ,  $b = +3$ ,  $c = -3$ , and  $d = +4$ :

1.  $\frac{ab - cd}{ab + cd}$

**Solution:** Substituting the values of the letters for the letters,  

$$\frac{ab - cd}{ab + cd} = \frac{(-2) \times (+3) - (-3) \times (+4)}{(-2) \times (+3) + (-3) \times (+4)} = \frac{-6 - (-12)}{-6 + (-12)} = \frac{+6}{-18} = -\frac{1}{3}$$

2.  $\frac{ab}{cd}$

3.  $\frac{a + d}{b - c}$

4.  $\frac{ac + bd}{ac - bd}$

5.  $\frac{a}{c} \sqrt{b^2 + d^2}$

6.  $\frac{a^2 + b^2}{c^2 + d^2}$

7.  $\frac{a^2}{b^2} \times \frac{c^2}{d^2}$

8.  $-\frac{(a + b)c}{d}$

9.  $\frac{-a - b}{-cd}$

10.  $ab \sqrt{c^2 + d^2}$

Find the values of the following expressions when  $m = -1$ ,  $n = 0$ ,  $x = +\frac{5}{6}$ , and  $y = -\frac{2}{3}$ :

11.  $\frac{m + n}{x + y}$

12.  $\frac{m^2 - n^2}{x^2 - y^2}$

13.  $\frac{mx + ny}{mx - ny}$

14.  $(m + n) \sqrt{x^2 - y^2}$

15.  $\frac{n}{m} \sqrt{x^2 + y^2}$

16.  $\frac{m^2 + x^2}{n^2 - y^2}$

17.  $x \sqrt{\frac{n^2}{x^2} \times \frac{m^2}{y^2}}$

18.  $\frac{x^2 - n^2}{\sqrt{x^2 - y^2}}$

19.  $\frac{(m - n)^2}{x \sqrt{y^2 - n^2}}$

20.  $\frac{(m + n)(x + y)}{(x - n)(y - m)}$

$$21. \frac{mx^2 - ny^2}{\sqrt{m^2x^2 - n^2y^2}}$$

$$22. \frac{\sqrt{m^2y^2 + n^2x^2}}{\sqrt{x^2 - y^2}}$$

106. The value of any polynomial in  $x$ , when  $x = a$ , is the remainder left after dividing the polynomial by  $x - a$ .

**Demonstration.**—Let  $V$  stand for the value of a polynomial ( $P$ ) in  $x$ , when  $x = a$ .

Let  $q$  be the quotient and  $r$  the remainder when the polynomial is divided by  $x - a$ . Then

$$P = q(x - a) + r$$

Put  $a$  for  $x$  and  $V$  for  $P$ ,

$$V = q(a - a) + r = r. \quad \text{Q. E. D.}$$

Find the value of  $x^4 - 3x^3 + 2x^2 - 5x + 7$ , when  $x = -3$ .

**Solution:** Divide  $x^4 - 3x^3 + 2x^2 - 5x + 7$  by  $x + 3$  by synthetic division, and determine the remainder. Thus:

$$\begin{array}{r} -3 \mid 1 - 3 + 2 - 5 + 7 \\ \quad -3 + 18 - 60 + 195 \\ \hline \quad 1 - 6 + 20 - 65 + 202^* \end{array}$$

\* Therefore, the value is 202.

Find the value of:

$$23. x^5 - 3x^4 + 4x^2 - 3x + 2, \text{ when } x = 2$$

$$24. x^5 + 3x^4 + 5x^3 - 3x + 14, \text{ when } x = 5$$

$$25. 3x^4 - 2x^3 + 7x^2 - 3x - 8, \text{ when } x = 3$$

$$26. 2x^5 - 2x^4 + 3x^3 - x + 5, \text{ when } x = -2$$

$$27. 4x^6 - 3x^4 + 2x^2 - 20, \text{ when } x = -4$$

## Involution.

### Definitions and Principles.

107. An *even power* is one whose exponent is divisible by 2, and an *odd power* one whose exponent is not divisible by 2.

108. The general form of an even number is  $2n$ , and of an odd number  $2n-1$ , in which  $n$  may have any integral numerical value.

Thus, if  $n=1$ ,  $2n=2$ , and  $2n-1=1$ ; if  $n=2$ ,  $2n=4$ , and  $2n-1=3$ , etc.

109.  $(a^n)^n = a^n \times a^n \times a^n \times \dots$  to  $n$  factors  $= a^{n+n+\dots}$  to  $n$  terms, according to the exponential law of multiplication,  $= a^{n^2}$ . Therefore,

*Prin. 1.*—*Multiplying the exponent of a factor by the exponent of a power raises the factor to that power.*

This is the exponential law of involution.

110.  $(a^x b^y c^z)^n =$

$a^x b^y c^z \times a^x b^y c^z \times a^x b^y c^z \times \dots$  to  $n$  factors  $=$

$a^x \times a^x \times a^x \times \dots$  to  $n$  factors  $\times$

$b^y \times b^y \times b^y \times \dots$  to  $n$  factors  $\times$

$c^z \times c^z \times c^z \times \dots$  to  $n$  factors,

according to the commutative law of multiplication,

$= (a^x)^n \times (b^y)^n \times (c^z)^n$ . Therefore,

*Prin. 2.*—*Raising every factor of a quantity to a power raises the quantity to the power.*

This is the distributive law of involution.

111.  $(\pm a)^2 = (\pm a) \times (\pm a) = +a^2$

$(\pm a)^3 = (\pm a)^2 \times (\pm a) = (+a^2) \times (\pm a) = \pm a^3$

$(\pm a)^4 = (\pm a)^2 \times (\pm a)^2 = (+a^2) \times (+a^2) = +a^4$

$(\pm a)^5 = (\pm a)^4 \times (\pm a) = (+a^4) \times (\pm a) = \pm a^5$

$(\pm a)^6 = (\pm a)^4 \times (\pm a)^2 = (+a^4) \times (+a^2) = +a^6$ .

Therefore, generally

$(\pm a)^{2n} = +a^{2n}$ , and  $(\pm a)^{2n-1} = \pm a^{2n-1}$ .

Hence,

*Prin. 3.*—*All even powers are positive, and all odd powers have the same sign as the first power.*

## Problems.

## 1. To raise a monomial to any power.

**Illustration.**—1. Raise  $-2$  to the 10th power.

**Solution:**  $(-2)^{10} = (-1 \times 2)^{10} = (-1)^{10} \times 2^{10}$  [P. 2]  $= 1 \times 2^{10}$  [P. 8]  $= 2^{10}$ .  $2^2 = 4$ ;  $2^3 \times 2^3 = 2^4 = 4 \times 4 = 16$ ;  $2^4 \times 2^4 = 2^8 = 16 \times 16 = 256$ ;  $2^8 \times 2^2 = 2^{10} = 256 \times 4 = 1024$ .

2. Find the value of  $(-2x^3y^2z)^5$ .

**Solution:**  $(-2x^3y^2z)^5 = (-2)^5 \times (x^3)^5 \times (y^2)^5 \times (z)^5$  [P. 2].  $(-2)^5 = -32$  [P. 8];  $(x^3)^5 = x^{15}$ ,  $(y^2)^5 = y^{10}$ ,  $(z)^5 = z^5$  [P. 1].  $\therefore (-2x^3y^2z)^5 = -32x^{15}y^{10}z^5$ .

## EXERCISE 10.

Find the value of :

- |  |  |
|--|--|
| 1. $(3a^3b^2c)^3$  | 11. $\{3(a-b)^2\}^3$                       |
| 2. $(-4a^2b^2c^2)^3$   | 12. $\{-a^2(x+y)^3\}^4$                    |
| 3. $(-2x^2y^3z^3)^7$   | 13. $\{(a+b)^2(x+y)^3\}^5$                 |
| 4. $(a^m b^n)^5$   | 14. $\{-3(x^2+y^2)^2(x^3-y^3)^3\}^4$       |
| 5. $(-2a^2b^2)^6$  | 15. $\{(a+b)^m(a-b)^{n-1}\}^n$             |
| 6. $(a^m b^n c^p)^n$   | 16. $\{3xy^2(x^3-y^3)^{m+n}\}^{2n}$        |
| 7. $(-2a^5b)^{2n}$   | 17. $\{(a+b)^3(a+b)^2\}^m$                 |
| 8. $(6^m a^4 b^3)^{2n}$  | 18. $[\{(a^m)^n\}^p]^q$                    |
| 9. $(-m^2 n^4 p q^3)^{5n}$   | 19. $\{(x^m)^n\}^m \times [\{(x^n)^m\}^n]$ |
| 10. $(-\frac{2}{3}a^m b^n c^3)^3$                                    | 20. $(x^m)^p \times (x^p)^n \times x^{pn}$ |
| 21. $\{(-x)^{2n}\}^m \times \{(-x)^{3n+1}\}^{2n-1}$                  |  |
| 22. $[\{(1+1)^2+1\}^2+1]^2+2^2\{(2+2)^2+2\}^2$                       |  |
| 23. $[\{(5^2)^2+5\}^2-5] \div [\{(5)^2+5\}^2-5]$                     |  |
| 24. $\{(x+x)^2+x^2\}^3 \times \{x^3+(x^2)x\}^3$                      |  |
| 25. $\{(a+x)^2\}^{3n} \times \{(a+x)^3\}^{2n} \div \{(a+x)^2\}^{2n}$ |  |
| 26. Show that $\{(a^x)^y\}^z = \{(a^y)^x\}^z = \{(a^z)^x\}^y$        |  |

## Involution of Binomials.

## Special Principles.

112. The square of the sum of  $a$  and  $b$ , or

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2.$$

Therefore,

*Prin. 1.*—The square of the sum of two quantities equals the square of the first, plus twice the product of the two, plus the square of the second.

113. The square of the difference of  $a$  and  $b$ , or

$$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2.$$

Therefore,

*Prin. 2.*—The square of the difference of two quantities equals the square of the first, minus twice the product of the two, plus the square of the second.

114. The cube of the sum of  $a$  and  $b$ , or

$$(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3.$$

Therefore,

*Prin. 3.*—The cube of the sum of two quantities equals the cube of the first, plus three times the square of the first into the second, plus three times the first into the square of the second, plus the cube of the second.

115. The cube of the difference of  $a$  and  $b$ , or

$$(a - b)^3 = (a - b)(a - b)(a - b) = a^3 - 3a^2b + 3ab^2 - b^3.$$

Therefore,

*Prin. 4.*—The cube of the difference of two quantities equals the cube of the first, minus three times the square of the first into the second, plus three times the first into the square of the second, minus the cube of the second.

*Note.*—1. The square of a binomial equals the sum of the squares of its terms, and twice the product of the terms.

2. The cube of a binomial equals the sum of the cubes of its terms, and three times the square of each term into the other.

## 116. The Binomial Theorem.

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$$

The above results may all be obtained by actual multiplication. By a careful inspection of them, the following laws will appear :

1. *The number of terms is one greater than the exponent of the binomial.*

2. *If the binomial represents the sum of two quantities, the sign of every term in the power is plus ; if the difference of two quantities, the signs are alternately plus and minus.*

3. *The first letter occurs in every term but the last, and the second letter in every term but the first.*

4. *The exponent of the leading letter in the first term is the same as the exponent of the binomial, and decreases by unity in each succeeding term. The exponent of the second letter is unity in the second term, and increases by unity in each succeeding term.*

5. *The coefficient of the first term is unity ; of the second term, the exponent of the binomial ; and that of each succeeding term may be found by multiplying the coefficient of the preceding term by the exponent of the leading letter in that term, and dividing the product by the number of that term.*

**Note.**—The coefficients after the middle term are the same in an inverse order as those before it. When the exponent of the binomial is an odd number, there are two middle terms with like coefficients.

## EXERCISE 11.

Expand :

1.  $(x^2 + 2y^2)^2$

6.  $(x^2y - xy^2)^3$

11.  $(x - 1)^5$

2.  $\left(\frac{2}{3}x^2 - \frac{3}{4}y^2\right)^2$

7.  $(x + y)^4$

12.  $(x - y)^6$

3.  $(x^{m-2} - x^{n-2})^2$

8.  $(x - y)^4$

13.  $(x + y)^7$

4.  $(2x - 3y)^3$

9.  $(x + 1)^4$

14.  $(x - 1)^7$

5.  $(2x^2 + 4y)^3$

10.  $(x + y)^5$

15.  $(x + y)^n$

16. The fourth power of the sum of two quantities = what ?

17. The fourth power of the difference of two quantities = what ?

18. The fifth power of the sum of two quantities = what ?

19. The fifth power of the difference of two quantities = what ?

20.  $(2x^2 - 3y)^4 = \text{what ?}$

**Solution :** Let  $m = 2x^2$ , and  $n = 3y$ ; then  $(2x^2 - 3y)^4 = (m - n)^4$   
 $= m^4 - 4m^3n + 6m^2n^2 - 4mn^3 + n^4 = (2x^2)^4 - 4 \times (2x^2)^3 \times$   
 $3y + 6 \times (2x^2)^2 \times (3y)^2 - 4 \times (2x^2) \times (3y)^3 + (3y)^4 =$   
 $16x^8 - 96x^6y + 216x^4y^2 - 216x^2y^3 + 81y^4.$

21.  $(x^2 + y^2)^4 =$

24.  $(a^2x + by^2)^5$

27.  $(a^3 - 1)^7$

22.  $(2x - 3y)^4 =$

25.  $(5x^3 - 3y^2)^4$

28.  $(2a^3 + 3)^8$

23.  $(3x^2 - y^2)^5 =$

26.  $(ab - cd)^6$

29.  $(a^n + 2)^8$

30.  $(-2x - 3y)^4 = \{-1(2x + 3y)\}^4 =$

$(-1)^4 \times (2x + 3y)^4 = \text{what ?}$

31.  $\left(-\frac{1}{2}x^3 + 3y^2\right)^4 = ?$

32.  $(-2 - 5y)^6 = ?$

33.  $(-x^2 - ax)^5 = ?$

34.  $(-x - 2y)^5 = ?$

35.  $(-2x^2 + 3)^6 = ?$

36.  $(-2bd + 3ac)^5 = ?$

37.  $(a^m + b^n)^4 = ?$

38.  $(a^{n+1} - a^{n-1})^5 = ?$

39.  $(x^p - x^q)^6 = ?$

40.  $(ax^2 + by^n)^4 = ?$

## Involution of Polynomials.

$$117. (a + b - c + d)^2 =$$

$$(a + b - c + d)(a + b - c + d) = a^2 + b^2 + c^2 + d^2 + 2ab - 2ac + 2ad - 2bc + 2bd - 2cd. \text{ Therefore,}$$

**Prin. 1.**—*The square of any polynomial equals the sum of the squares of its terms, and twice the product of each term into all the following terms.*

$$118. (a + b - c + d)^3 =$$

$$\begin{aligned} &(a + b - c + d)(a + b - c + d)(a + b - c + d) = \\ &(a^3 + b^3 - c^3 + d^3) + (3a^2b - 3a^2c + 3a^2d) + \\ &(3b^2a - 3b^2c + 3b^2d) + (3c^2a + 3c^2b + 3c^2d) + \\ &(3d^2a + 3d^2b - 3d^2c) + \\ &(-6abc + 6abd - 6acd - 6bcd). \text{ Therefore,} \end{aligned}$$

**Prin. 2.**—*The cube of any polynomial equals the sum of the cubes of its terms, and three times the square of each term into all the other terms, and six times the product of every three terms.*

**Problem.** To raise a polynomial to any power.

**Illustrations.**—1. Expand  $(x^2 + xy + y^2)^2$ .

$$\begin{aligned} \text{Solution: } (x^2 + xy + y^2)^2 &= (x^2)^2 + (xy)^2 + (y^2)^2 + 2x^2(xy + y^2) + \\ &2xy \times y^2 \text{ [P. 1]} = x^4 + x^3y + y^4 + 2x^3y + 2x^2y^2 + \\ &2xy^3 = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4. \end{aligned}$$

2. Expand  $(x^2 - xy + y^2)^3$ .

$$\begin{aligned} \text{Solution: } (x^2 - xy + y^2)^3 &= (x^2)^3 + (-xy)^3 + (y^2)^3 + \\ &3(x^2)^2(-xy + y^2) + 3(-xy)^2(x^2 + y^2) + 3(y^2)^2(x^2 - xy) + \\ &6 \times x^2 \times (-xy) \times y^2 \text{ [P. 2]} = x^6 - x^3y^3 + y^6 - 3x^4y + \\ &3x^4y^2 + 3x^2y^4 + 3x^2y^4 - 3xy^5 - 6x^2y^3 = \\ &x^6 - 3x^4y + 6x^4y^2 - 7x^2y^3 + 6x^2y^4 - 3xy^5 + y^6. \end{aligned}$$

## EXERCISE 12.

Expand :

$$1. (x^2 + x + 1)^2$$

$$3. (x^2 - xy + y^2)^2$$

$$2. (x^2 + 2x - 1)^2$$

$$4. (2x - 3y + 4z)^2$$

5.  $(x - y + z)^3$

10.  $(x^2 - x + 1)^6$

6.  $(x^2 - x - 1)^3$

11.  $(x^2 + xy + y^2)^5$

7.  $(x^2 - 2x - 4)^3$

12.  $(x^m + x^{m-1} + x^{m-2})^2$

8.  $(2x^2 + 3x - 2)^3$

13.  $(a + b - c - d)^2$

9.  $(x^2 + x + 1)^4$

14.  $(x^2 - x^2 + x - 1)^3$

**Evolution.****Definition.**

**119.** Evolution is the process of finding a root of a quantity.

**Evolution of Monomials.****Principles.**

**120.** Since  $(a^n)^m = a^{nm}$  [109, P. 1],  $\sqrt[n]{a^{nm}} = a^m$ .

Therefore,

**Prin. 1.**—Dividing the exponent of a factor by the index of a root takes that root of a factor.

This is the exponential law of evolution.

**121.** Since  $(a^m b^n c^p)^r = (a^m)^r \times (b^n)^r \times (c^p)^r$  [110, P. 2] =  $a^{mr} b^{nr} c^{pr}$ ,  $\sqrt[r]{a^{mr} b^{nr} c^{pr}} = \sqrt[r]{a^{mr}} \times \sqrt[r]{b^{nr}} \times \sqrt[r]{c^{pr}} = a^m b^n c^p$ .

Therefore,

**Prin. 2.**—Taking a root of every factor of a quantity takes the root of the quantity.

This is the distributive law of evolution.

**122.** Since  $(\pm a)^{2n} = +a^{2n}$  and never  $-a^{2n}$  [111, P. 3],  $\sqrt[2n]{a^{2n}} = \pm a$ , and  $\sqrt[2n]{-a^{2n}}$  is impossible. Therefore,

**Prin. 3.**—An even root of a positive quantity may be either positive or negative, and an even root of a negative quantity is impossible.

**123.** Since  $(\pm a)^{2n-1} = \pm a^{2n-1}$  [111, P. 3],  $^{2n-1}\sqrt{+a^{2n-1}} = +a$ , and  $^{2n-1}\sqrt{-a^{2n-1}} = -a$ . Therefore,

**Prin. 4.**—Any odd root of a quantity has the same sign as the quantity.

Prins. 3 and 4 constitute the law of signs in evolution.

**124.**  $\sqrt[m]{\sqrt[n]{a}}$  = one of the  $m$  equal factors of one of the  $n$  equal factors of  $a$ , which is one of the  $mn$  equal factors of  $a$ , or  $\sqrt[mn]{a}$ . Therefore,

**Prin. 5.**— $\sqrt[m]{\sqrt[n]{\text{any quantity}}} = \text{the } \sqrt[mn]{\text{that quantity}}$ .

**Problem.** To extract any root of a monomial.

**Illustrations.**—1. Extract the cube root of 1728.

**Solution:**  $1728 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3$ . Since the cube root of a quantity is one of the three equal factors of the quantity [Def.], we take every third equal factor as a factor of the cube root. Therefore,  $\sqrt[3]{1728} = 2 \times 2 \times 3 = 12$ .

2. Find the value of  $\sqrt[5]{-32x^{15}y^{10}z^5}$ .

**Solution:**  $\sqrt[5]{-32x^{15}y^{10}z^5} = \sqrt[5]{-32} \times \sqrt[5]{x^{15}} \times \sqrt[5]{y^{10}} \times \sqrt[5]{z^5}$  [P. 2].  
 $\sqrt[5]{-32} = -2$  [P. 4].  $\sqrt[5]{x^{15}} = x^3$ ,  $\sqrt[5]{y^{10}} = y^2$ , and  $\sqrt[5]{z^5} = z$  [P. 1].  
 $\therefore \sqrt[5]{-32x^{15}y^{10}z^5} = -2x^3y^2z$ .

#### EXERCISE 18.

Find the value of :

- |                                      |                                      |                     |                     |
|--------------------------------------|--------------------------------------|---------------------|---------------------|
| 1. $\sqrt{6561}$                     | 2. $\sqrt[3]{5832}$                  | 3. $\sqrt[4]{4096}$ | 4. $\sqrt[5]{7776}$ |
| 5. $\sqrt[4]{16x^{12}y^{16}z^{20}}$  | 11. $\sqrt{x^2y^2(x+y)^4}$           |                     |                     |
| 6. $\sqrt[3]{-27a^3b^6c^9}$          | 12. $\sqrt[3]{-x^3y^6(x-y)^9}$       |                     |                     |
| 7. $\sqrt[5]{-32a^5b^{10}c^{20}}$    | 13. $\sqrt{9x^{2n}(x+a)^4}$          |                     |                     |
| 8. $\sqrt[5]{243x^{20}y^{30}z^{40}}$ | 14. $\sqrt[3]{(a+b)^3(a-b)^3}$       |                     |                     |
| 9. $\sqrt[5]{64a^5b^{12}c^{18}}$     | 15. $\sqrt{4(a+b)(a+b)^3}$           |                     |                     |
| 10. $\sqrt{144a^{2n}b^{4n}c^8}$      | 16. $\sqrt[2n]{x^{2n}(xy+y^2)^{2n}}$ |                     |                     |

17.  $\sqrt[3]{-27a^{3n}b^{6n}c^6}$

21.  $\sqrt[2n+1]{x^{2n+1}y^{4n+2}z^{6n+3}}$

18.  $\sqrt[4]{16a^{8n}b^{4n}c^4}$

22.  $\sqrt[n-1]{8^{2n-2} \times 7^{n-1} \times 3^{3n-3}}$

19.  $\sqrt[2]{a^{2n}b^{3n}c^n}$

23.  $\sqrt[2]{3^{2n} \times 4^n \times 5^{3n}}$

20.  $\sqrt[4]{\sqrt[3]{x^{12}(x+y)^{18}}}$

24.  $\sqrt{\sqrt{x^8y^{12}(x-y)^{4n}}}$

## Evolution of Polynomials.

### Problems.

#### 1. To extract a root of a polynomial by inspection.

Sometimes the terms of a polynomial may be so arranged as to conform to the laws enunciated in Art. 116, 117, or 118, in which case the root may be determined by inspection.

#### Illustrations.—

1.  $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = (a+b)^4$  [116].

$\therefore \sqrt[4]{(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)} = \pm (a+b)$  [P. 3].

2.  $a^2 + b^2 + c^2 - 2ab + 2ac - 2bc = (a-b+c)^2$

[117, P. 1].

$\therefore \sqrt{(a^2 + b^2 + c^2 - 2ab + 2ac - 2bc)} = \pm (a-b+c)$   
[P. 3].

3.  $a^3 - b^3 + c^3 - 3a^2b + 3a^2c + 3ab^2 + 3b^2c +$

$3ac^2 - 3bc^2 - 6abc = (a-b+c)^3$  [118, P. 2].

$\therefore \sqrt[3]{(a^3 - b^3 + c^3 - 3a^2b + 3a^2c + 3ab^2 + 3b^2c +$

$3ac^2 - 3bc^2 - 6abc)} = \pm (a-b+c)$  [P. 4].

125. To determine whether a polynomial is a perfect power of a binomial, arrange the terms according to the ascending or descending powers of some letter assumed as a leading letter; then extract the root of the first and last terms, and see whether the corresponding power of the sum or difference of these results will produce the polynomial.

## EXERCISE 14.

Find the value of :

1.  $\sqrt{(x^2 + 2xy + y^2)}$
2.  $\sqrt{(x^2 + 10x + 25)}$
3.  $\sqrt[3]{(x^3 + 3x^2y + 3xy^2 + y^3)}$
4.  $\sqrt[3]{(x^3 - 9x^2 + 27x - 27)}$
5.  $\sqrt{(9x^2 - 12xy + 4y^2)}$
6.  $\sqrt[4]{(a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4)}$
7.  $\sqrt[3]{(27x^3 - 54x^2y + 36xy^2 - 8y^3)}$
8.  $\sqrt[4]{(x^4 - 8x^3 + 24x^2 - 32x + 16)}$
9.  $\sqrt[5]{(x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5)}$
10.  $\sqrt{(x^2 + y^2 + z^2 - 2xy - 2xz + 2yz)}$
11.  $\sqrt[3]{(x^3 - y^3 - z^3 - 3x^2y - 3x^2z + 3xy^2 - 3yz^2 + 3xz^2 - 3yz^2 + 6xyz)}$
12.  $\sqrt{(a^2 + b^2 + c^2 + d^2 + 2ab - 2ac - 2ad - 2bc - 2bd + 2cd)}$

## 2. To extract the square root of any polynomial.

**Illustration.**—Extract the square root of

$$x^2 - 4xy^2 + 4y^4 + 6xz - 12y^2z + 9z^2.$$

$(m+n)^2 = m^2 + (2m+n)n$	<b>Form.</b>	$\begin{array}{r} m+n \\ \overline{m+n} \\ x-2y^2+3z \end{array}$
	$\begin{array}{r} x^2 - 4xy^2 + 4y^4 + 6xz - 12y^2z + 9z^2 \\ \hline x^2 \end{array}$	
	$\begin{array}{r} 2x - 2y^2 \quad \overline{-4xy^2 + 4y^4} \\ \hline \quad \quad \quad \overline{-4xy^2 + 4y^4} \end{array}$	
	$\begin{array}{r} 2x - 4y^2 + 3z \quad \overline{6xz - 12y^2z + 9z^2} \\ \hline \quad \quad \quad \overline{6xz - 12y^2z + 9z^2} \end{array}$	

**Solution:** A quantity whose square is equal to the given polynomial is required. Having arranged the terms according to the descending powers of  $x$  and the ascending powers of  $y$  and  $z$ , we will first endeavor to take from the polynomial the square of the first two terms of the root. For this purpose, let  $m$  represent the first term and  $n$  the second term of the root. Now,  $(m+n)^2 = m^2 + 2mn + n^2$ ,

or  $m^2 + (2m + n)n$ .  $m^2$  obviously equals  $x^2$ , whence  $m = x$ , the first term of the root. Subtracting  $x^2$  from the polynomial, and bringing down the next two terms, we have  $-4xy^2 + 4y^4$ . This remainder consists mainly or entirely of  $(2m + n)n$ ; hence, if we use  $2m$ , or  $2x$ , as a *trial divisor*, we shall obtain the value of  $n$ , which is  $-4xy^2 \div 2x$ , or  $-2y^2$ ; adding this value of  $n$  to  $2m$ , we have  $2x - 2y^2$ , the *complete divisor*; multiplying the value of  $2m + n$ , or  $2x - 2y^2$ , by the value of  $n$ , or  $-2y^2$ , we have  $-4xy^2 + 4y^4$ . Subtracting this product from  $-4xy^2 + 4y^4$ , and bringing down the remaining terms, we have  $6xz - 12y^2z + 9z^2$ . Having now taken from the polynomial the square of  $x - 2y^2$ , we let  $m$  represent  $x - 2y^2$ , and  $n$  the next term of the root, and proceed as before to take from the polynomial the square of  $m + n$ , or  $m^2 + (2m + n)n$ .  $m^2$ , or  $(x - 2y^2)^2$ , has already been subtracted; hence,  $6xz - 12y^2z + 9z^2$  consists mainly or wholly of  $(2m + n)n$ . Using  $2m$ , or  $2x - 4y^2$ , as a *trial divisor*, we obtain  $3z$  for  $n$ ; adding this to the value of  $2m$ , or  $2x - 4y^2$ , we have  $2x - 4y^2 + 3z$  for the *complete divisor*. Multiplying the complete divisor by  $3z$ , and subtracting, nothing remains. Therefore, the given polynomial is the square of  $x - 2y^2 + 3z$ ; whence  $x - 2y^2 + 3z$  is the square root of the polynomial.

Hence the following

**Rule.**—1. *Arrange the terms of the polynomial according to the ascending or descending powers of some letter assumed as a leading letter.*

2. *Take the square root of the first term of the polynomial for the first term of the root. Subtract the square of this term of the root from the polynomial.*

3. *Double the root already found for a trial divisor. Divide the first term of the remainder by the trial divisor for the next term of the root.*

4. *Add the last term of the root found to the trial divisor for the complete divisor. Multiply the complete divisor by the last term of the root found, and subtract the product from the remainder, and bring down such terms as are needed.*

5. *If the root has more than two terms, double the root already found for a new trial divisor, and proceed as before to obtain the next term of the root and the complete divisor. Continue this process until all the terms of the polynomial have been used.*

**Note.**—In the formula  $m^2 + (2m + n)n$ ,  $m$  represents the root already found,  $2m$  the trial divisor,  $n$  the next term of the root and also the correction, and  $2m + n$  the complete divisor.

## EXERCISE 18.

Extract the square root of :

1.  $4x^6 - 8x^5 + 16x^4 - 12x^3 + 9$

2.  $4x^4 - 12x^3y + 13x^2y^2 - 6xy^3 + y^4$

3.  $x^4 - 4x^2y^2 + 4y^4 + 6x^2z - 12y^2z + 9z^2$

4.  $16a^{12} - 24x^{12}y^2 - 7x^{12}y^2 + 12x^2y^{12} + 4y^{12}$

5.  $4a^4 - 12a^2b^2 + 9b^4 + 16a^2c^2 - 24b^2c^2 + 16c^4$

6.  $9x^6 + 52x^4y^2 + 44x^2y^4 - 24x^5y - 54x^3y^3 - 12x^2y^5 + y^6$

7.  $x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8$

8.  $a^{12} + 6a^{10}b^2 + 15a^8b^4 + 20a^6b^6 + 15a^4b^8 + 6a^2b^{10} + b^{12}$

9.  $1 - 2a^2 + 3a^4 - 4a^6 + 3a^8 - 2a^{10} + a^{12}$

## 3. To extract the cube root of any polynomial.

**Illustration.**—Extract the cube root of  $x^6 - 3x^5y + 6x^4y^2 - 7x^3y^3 + 6x^2y^4 - 3xy^5 + y^6$ .

Form.

$$(m+n)^3 = m^3 + (3m^2 + 3mn + n^2)n$$

$$\begin{array}{r} m+n \\ \overline{x^3 - xy^2 + y^3} \end{array}$$

$$\begin{array}{r} x^6 - 3x^5y + 6x^4y^2 - 7x^3y^3 + 6x^2y^4 - 3xy^5 + y^6 \\ \underline{x^6} \end{array}$$

T. D. =  $3x^4$

1st Cor. =  $-3x^2y$

2d Cor. =  $x^2y^2$

$$\begin{array}{r} -3x^5y + 6x^4y^2 - 7x^3y^3 \end{array}$$

C. D. =  $3x^4 - 3x^2y + x^2y^2$

$$\begin{array}{r} -3x^5y + 3x^4y^2 - x^2y^3 \end{array}$$

T. D. =  $3x^4 - 6x^2y + 3x^2y^2$

1st Cor. =  $3x^2y^2 - 3xy^3$

2d Cor. =  $y^4$

$$\begin{array}{r} 3x^4y^2 - 6x^3y^3 + 6x^2y^4 - 3xy^5 + y^6 \end{array}$$

C. D. =  $3x^4 - 6x^2y + 6x^2y^2 - 3xy^3 + y^4$

$$\begin{array}{r} 3x^4y^2 - 6x^3y^3 + 6x^2y^4 - 3xy^5 + y^6 \end{array}$$

**Solution:** A quantity whose cube is equal to the given polynomial is required. Having arranged the terms according to the descending powers of  $x$  and the ascending powers of  $y$ , we will first endeavor to take from it the cube of the first two terms of the root. For this purpose, we let  $m$  represent the first and  $n$  the second term of the root. Now,  $(m + n)^3 = m^3 + 3m^2n + 3mn^2 + n^3$ , or  $m^3 + (3m^2 + 3mn + n^2)n$ .  $m^3$  is obviously  $x^6$ , whence  $m = x^2$ . Subtracting  $(x^2)^3$ , or  $x^6$ , from  $x^6$ , and bringing down the next three terms, we have  $-3x^4y + 6x^2y^2 - 7x^2y^3$ . This remainder consists mainly of  $(3m^2 + 3mn + n^2)n$ ; hence, if we use  $3m^2$ , or  $3x^4$ , as a trial divisor, we will obtain the value of  $n$ , which is  $-3x^2y + 3x^4$ , or  $-xy$ . Substituting the values of  $m$  and  $n$  in  $3mn$  and  $n^2$ , we have  $-3x^3y$  for the first correction, and  $x^2y^2$  for the second correction; adding the two corrections to the trial divisor, we have  $3x^4 - 3x^3y + x^2y^2$  for  $3m^2 + 3mn + n^2$ , the complete divisor; multiplying the complete divisor by  $-xy$ , we have  $-3x^5y + 3x^4y^2 - x^3y^3$ , which we subtract from  $-3x^4y + 6x^2y^2 - 7x^2y^3$ , and bring down the remaining terms.

Having now taken from the polynomial the cube of  $x^2 - xy$ , we let  $m$  represent  $x^2 - xy$ , and  $n$  the next term of the root, and proceed as before to take from the polynomial the cube of  $m + n$ , or  $m^3 + (3m^2 + 3mn + n^2)n$ .  $m^3$ , or  $(x^2 - xy)^3$ , has already been subtracted; hence,  $3x^4y^2 - 6x^3y^3 + 6x^2y^4 - 3xy^5 + y^6$  consists of  $(3m^2 + 3mn + n^2)n$ . Using  $3m^2$ , or  $3x^4 - 6x^3y + 3x^2y^2$  as a *trial divisor*, we obtain  $y^2$  for  $n$ ; substituting the values of  $m$  and  $n$  in  $3mn$  and  $n^2$ , we have  $3x^3y^2 - 3xy^3$  for the first correction and  $y^4$  for the second correction, and  $3x^4 - 6x^3y + 6x^2y^2 - 3xy^3 + y^4$  for the complete divisor. Multiplying the complete divisor by  $y^2$ , and subtracting the product from the last remainder, nothing remains. Therefore,  $x^2 - xy + y^2$  is the cube root of the polynomial.

**Note.**—In the formula  $m^3 + (3m^2 + 3mn + n^2)n$ ,  $m$  represents the root already found,  $3m^2$  the trial divisor,  $3mn$  the first correction,  $n^2$  the second correction,  $3m^2 + 3mn + n^2$  the complete divisor, and  $n$  the next term of the root.

From an inspection of the above solution the following rule will appear :

1. *Arrange the terms of the polynomial according to the ascending or descending powers of some letter assumed as a leading letter.*

2. *Take the cube root of the first term of the polynomial for the first term of the root. Subtract the cube of this term of the root from the polynomial.*

3. Take three times the square of the root already found for a trial divisor. Divide the first term of the remainder by the first term of the trial divisor for the next term of the root.

4. Add to the trial divisor three times the last term of the root multiplied by the preceding terms, and the square of the last term, for a complete divisor. Multiply the complete divisor by the last term of the root, and subtract the result from the remainder, bringing down only the terms needed.

5. If the root has more than two terms, take three times the square of the root already found for a new trial divisor, and proceed as before to obtain the next term of the root and the new complete divisor. Continue the process until all the terms of the polynomial have been used.

## EXERCISE 16.

Find the cube root of :

1.  $8a^3 - 36a^2 + 54a - 27$
2.  $x^5 - 9x^4y^2 + 27x^2y^4 - 27y^6$
3.  $1000x^9 - 600x^5 + 120x^3 - 8$
4.  $x^{3m} + 6x^{2m} + 12x^m + 8$
5.  $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$
6.  $x^6 - 6x^5 + 6x^4 + 16x^3 - 12x^2 - 24x - 8$
7.  $8x^{6n} - 36x^{5n} + 66x^{4n} - 63x^{3n} + 33x^{2n} - 9x^n + 1$
8.  $x^{12} + 9x^{10}y + 24x^8y^2 + 9x^6y^3 - 24x^4y^4 + 9x^2y^5 - y^6$
9.  $x^3 - 6x^2y + 12xy^2 - 8y^3 - 3x^2z + 3xz^2 + 12xyz - 12y^2z - 6yz^2 - z^3$
10.  $8x^6 - 24x^4y^2 + 24x^2y^4 - 8y^6 + 24x^4z^3 + 24x^2z^4 - 48x^2y^2z^3 + 24y^4z^2 - 24y^2z^4 + 8z^6$
11.  $27a^3 + 54a^2b + 27a^2c + 36ab^2 + 36abc + 8b^3 + 9ac^2 + 12b^2c + 6bc^2 + c^3$
12.  $a^{9m} - 3a^{8m} + 6a^{7m} - 10a^{6m} + 12a^{5m} - 12a^{4m} + 10a^{3m} - 6a^{2m} + 3a^m - 1$

## 4. To extract the square root of decimal numbers.

126. Every decimal number of two or more figures may be regarded a polynomial. Thus,  $14376 = 1$  ten-thousand  $+ 43$  hundred  $+ 76$ , or  $14$  thousand  $+ 376$ .

## Principles.

127. *The square of a unit is a unit, the square of a ten is a hundred, the square of a hundred is a ten-thousand, the square of a thousand is a million, etc.*

Therefore,

128. *The square root of a unit is a unit, the square root of a hundred is a ten, the square root of a ten-thousand is a hundred, the square root of a million is a thousand, etc.*

**Note.**—The square decimal units in order are the *unit*, the *hundred*, the *ten-thousand*, the *million*, the *hundred-million*, the *ten-billion*, etc.

**Example.**—Extract the square root of 54756.

**Solution:** We point the number off into terms of two figures each to keep the unit of each term a perfect square [128, n.].  $54756 = 5$  ten-thousands  $+ 47$  hundreds  $+ 56$  units. The square root of 5 ten-thousands is 2 hundreds, the first term of the root. Squaring 2 hundreds, we have 4 ten-thousands; subtracting 4 ten-thousands from 5 ten-thousands, and bringing down the next term, we have 147 hundreds. Doubling the root already found for a trial divisor, we have 4 hundreds; dividing 14 thousand by 4 hundred, we have 3 tens for the next term of the root; adding 3 tens to the trial divisor, we have 43 tens for the complete divisor; multiplying 43 tens by 3 tens, we have 129 hundreds; subtracting 129 hundreds from 147 hundreds, and bringing down the next term, we have 1856 units. Doubling 23 tens, we have 46 tens for a trial divisor; dividing 185 tens by 46 tens, we have 4 units for the next term of the root; adding 4 units to 46 tens, we have

		<b>Form.</b>
		5'4 7'5 6   2 3 4
		4
43.	147	
	129	
464	1856	
	1856	

464 units for the complete divisor; multiplying 464 units by 4, we have 1856 units; subtracting this product from the remainder, nothing remains. Therefore, 284 is the square root of 54756.

## EXERCISE 17.

Extract the square root of :

- |          |           |            |              |
|----------|-----------|------------|--------------|
| 1. 3136  | 4. 42436  | 7. 603729  | 10. 1809025  |
| 2. 6561  | 5. 105625 | 8. 978121  | 11. 36072036 |
| 3. 15376 | 6. 258064 | 9. 1048576 | 12. 60855601 |

## 5. To extract the cube root of a decimal number.

## Principles.

129. *The cube of a unit is a unit, the cube of a ten is a thousand, the cube of a hundred is a million, etc.*

Therefore,

130. *The cube root of a unit is a unit, the cube root of a thousand is a ten, the cube root of a million is a hundred, etc.*

**Note.**—The cubic decimal units in regular order are the *unit*, the *thousand*, the *million*, the *billion*, etc.

**Example.**—Extract the cube root of 14706125.

Form.

$$\begin{array}{rcl}
 (m+n)^3 = m^3 + (3m^2 + 3mn + n^2)n & & \begin{array}{|l} m+n \\ \hline m+n \\ 245 \end{array} \\
 & & 14'706'125 \\
 m^3 = & 8 & \\
 3m^2 = 3 \times (2..)^2 = 12.... & 6706 & \\
 3mn = 3 \times 2.. \times 4. = 24... & & \\
 n^2 = (4.)^2 = 16.. & & \\
 \hline
 3m^2 + 3mn + n^2 = 1456.. & 5824 & \\
 3m^2 = 3 \times (24.)^2 = 1728.. & 882125 & \\
 3mn = 3 \times 24. \times 5 = 360. & & \\
 n^2 = 5^2 = 25 & & \\
 \hline
 3m^2 + 3mn + n^2 = 176425 & 882125 & \\
 \hline
 \end{array}$$

**Solution:** We point off the number into terms of three figures each to keep the unit of each term a perfect cube  $[180, n]$ .  $14,706,125 = 14$  million +  $706$  thousand +  $125$  units. The cube root of  $14$  million is  $2$  hundred. Cubing  $2$  hundred, we have  $8$  million; subtracting  $8$  million from  $14$  million, and bringing down the next term, we have  $6706$  thousand. Taking three times the square of the root already found ( $3m^2$ ) for a trial divisor, we have  $12$  ten-thousands; dividing  $67$  hundred-thousands by  $12$  ten-thousands, we have  $4$  tens ( $n$ ); taking  $3$  times the root previously found ( $3m$ ), and multiplying it by the last term of the root ( $n$ ), we have  $24$  thousand for the first correction, and, squaring the last term of the root ( $n^2$ ), we have  $16$  hundred for the second correction; adding the trial divisor and the two corrections, we have  $1456$  hundred for the complete divisor ( $3m^2 + 3mn + n^2$ ); multiplying the complete divisor by the last term of the root ( $n$ ), we have  $5824$  thousand; subtracting  $5824$  thousand from  $6706$  thousand, and bringing down the next term, we have  $882125$  units.

Taking  $3$  times the square of  $24$  tens (the new value of  $m$ ), we have  $1728$  hundred (the new trial divisor); dividing  $8821$  hundred by  $1728$  hundred, we have  $5$  ( $n$ ); finding as before the values of  $3mn$  and  $n^2$ , we have for the first correction  $360$  tens, for the second correction  $25$  units, and for the complete divisor  $176425$  units; multiplying by  $5$ , or  $n$ , we have  $882125$ , which, subtracted from  $882125$ , leaves no remainder. Therefore,  $\sqrt[3]{14706125} = 245$ .

#### Abbreviated Rule.

1. *Point off the number into terms of three figures each.*
2. *The cube root of the first term gives the first figure of the root.*
3. *Three times the square of the root already found always gives the trial divisor.*
4. *The remainder, exclusive of the two right-hand figures, divided by the trial divisor, gives the next figure of the root.*
5. *Three times the root previously found multiplied by the last figure found gives the first correction, and the square of the last figure found the second correction.*
6. *The right-hand figure of the first correction is placed one order to the right of the trial divisor, and that of the second correction one order to the right of the first correction.*
7. *The sum of the trial divisor and the two corrections gives the complete divisor.*

## EXERCISE 18.

Extract the cube root of :

- |           |             |                  |
|-----------|-------------|------------------|
| 1. 1728   | 5. 1953125  | 9. 64481201      |
| 2. 39304  | 6. 8242408  | 10. 144703125    |
| 3. 531441 | 7. 15625000 | 11. 1879080904   |
| 4. 857375 | 8. 48627125 | 12. 216648648216 |

## 6. To extract higher roots.

Illustration.—Find the value of  $\sqrt[6]{4096}$ .Solution:  $\sqrt[6]{4096} = \sqrt[3]{\sqrt{4096}} [124, P.] = \sqrt[3]{64} = 4$ .

## EXERCISE 19.

Find the value of :

- |   |                         |                                      |
|---|-------------------------|--------------------------------------|
| 1. $\sqrt[4]{65536}$  | 3. $\sqrt[5]{390625}$   | 5. $\sqrt[12]{68719476736}$          |
| 2. $\sqrt[5]{2985984}$  | 4. $\sqrt[3]{40353607}$ | 6. $\sqrt[18]{262144 a^{18} b^{36}}$ |
| 7. $\sqrt[4]{(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4)}$  |                         |                                      |
| 8. $\sqrt[3]{(x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64)}$  |                         |                                      |
| 9. $\sqrt[5]{(x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8)}$               |                         |                                      |
| 10. $\sqrt[3]{(x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9)}$ |                         |                                      |

## Exact Division.

## Definitions and Principles.

131. A *divisor* of a quantity is a quantity that will exactly divide it. Thus,  $a$  is a divisor of  $ab$ .

132. The *divisors* of a quantity are all the different quantities that will exactly divide it. Thus, the divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.

**133.** The *continued divisors* of a quantity are the quantities that will successively divide it. Thus, the continued divisors of 24 are 2, 2, 2, and 3, since  $24 \div 2 = 12$ ,  $12 \div 2 = 6$ ,  $6 \div 2 = 3$ , and  $3 \div 3 = 1$ .

**Note.**—The continued divisors of a quantity are the factors of the quantity.

**134.** Let us determine whether  $a^n - b^n$  is divisible by  $a - b$ :

$$\begin{array}{r}
 a - b \ ) \ a^n - b^n \ ( \ a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \text{etc.} \\
 \underline{a^n - a^{n-1}b} \\
 \text{1st Rem.} = a^{n-1}b - b^n \\
 \underline{a^{n-1}b - a^{n-2}b^2} \\
 \text{2d Rem.} = a^{n-2}b^2 - b^n \\
 \underline{a^{n-2}b^2 - a^{n-3}b^3} \\
 \text{3d Rem.} = a^{n-3}b^3 - b^n \\
 \therefore \text{nth Rem.} = a^{n-n}b^n - b^n = a^0b^n - b^n = b^n - b^n = 0
 \end{array}$$

Since the  $n$ th remainder reduces to zero, the division is complete. Now,  $a - b$  is the difference of two quantities, and  $a^n - b^n$  is the difference of the equal even or odd powers of those quantities. Therefore,

**Prin. 1.**—*The difference of the equal even or odd powers of two quantities is divisible by the difference of the quantities.*

**135.** Let us determine whether  $a^n + b^n$  is divisible by  $a + b$ :

$$\begin{array}{r}
 a + b \ ) \ a^n + b^n \ ( \ a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3 + \\
 \underline{a^n + a^{n-1}b} \\
 \text{1st Rem.} = -a^{n-1}b + b^n \\
 \underline{-a^{n-1}b - a^{n-2}b^2} \\
 \text{2d Rem.} = a^{n-2}b^2 + b^n \\
 \underline{a^{n-2}b^2 + a^{n-3}b^3} \\
 \text{3d Rem.} = -a^{n-3}b^3 + b^n \\
 \underline{-a^{n-3}b^3 - a^{n-4}b^4} \\
 \text{4th Rem.} = a^{n-4}b^4 + b^n
 \end{array}$$

Since the second term of each remainder is  $+b^n$ , the first term of some remainder must become  $-b^n$  for the division to terminate. This can only occur when  $a$  obtains an exponent of zero and  $b$  an exponent of  $n$ , which will evidently be in the  $n$ th remainder. We observe, moreover, that the first term of the odd remainders is negative, and the first term of the even remainders positive. Therefore, when  $n$  is an odd number, the

$$n\text{th Rem.} = -a^{n-n}b^n + b^n = -a^0b^n + b^n = -b^n + b^n = 0;$$

but, when  $n$  is an even number, the

$$n\text{th Rem.} = a^{n-n}b^n + b^n = a^0b^n + b^n = b^n + b^n = 2b^n.$$

Therefore,

*Prin. 2.*—1. The sum of the equal odd powers of two quantities is divisible by the sum of the quantities; but,  
2. The sum of the equal even powers of two quantities is not divisible by the sum of the quantities.

**136.** Let us determine whether  $a^n - b^n$  is divisible by  $a + b$ :

$$\begin{array}{r} a + b \overline{) a^n - b^n} \\ \underline{a^n + a^{n-1}b} \phantom{- b^n} \\ 1\text{st Rem.} = -a^{n-1}b - b^n \\ \underline{-a^{n-1}b - a^{n-2}b^2} \phantom{- b^n} \\ 2\text{d Rem.} = a^{n-2}b^2 - b^n \\ \underline{a^{n-2}b^2 + a^{n-3}b^3} \phantom{- b^n} \\ 3\text{d Rem.} = -a^{n-3}b^3 - b^n \\ \underline{-a^{n-3}b^3 - a^{n-4}b^4} \phantom{- b^n} \\ 4\text{th Rem.} = a^{n-4}b^4 - b^n \end{array}$$

Since the second term of each remainder is  $-b^n$ , the first term of some remainder must become  $+b^n$ , for the division to terminate. This can occur only when  $a$  obtains an exponent of zero and  $b$  an exponent of  $n$ , which will evidently be in the  $n$ th remainder. We observe, moreover, that the first term of the odd remainders is negative,

and the first term of the even remainders positive. Therefore, when  $n$  is an odd number, the

$$n\text{th Rem.} = -a^{n-1}b^n - b^n = -a^0b^n - b^n = -b^n - b^n = -2b^n;$$

but, when  $n$  is an even number, the

$$n\text{th Rem.} = +a^{n-1}b^n - b^n = a^0b^n - b^n = b^n - b^n = 0.$$

Therefore,

**Prin. 3.—1.** *The difference of the equal even powers of two quantities is divisible by the sum of the quantities ; but,*

*2. The difference of the equal odd powers of two quantities is not divisible by the sum of the quantities.*

**137.** Let us determine whether  $a^n + b^n$  is divisible by  $a - b$  :

$$\begin{array}{r} a - b \ ) \ a^n + b^n \ ( \ a^{n-1} + a^{n-2}b + a^{n-3}b^2 \\ \underline{a^n - a^{n-1}b} \\ 1\text{st Rem.} = a^{n-1}b + b^n \\ \underline{a^{n-1}b - a^{n-2}b^2} \\ 2\text{d Rem.} = a^{n-2}b^2 + b^n \\ \underline{a^{n-2}b^2 - a^{n-3}b^3} \\ 3\text{d Rem.} = a^{n-3}b^3 + b^n \end{array}$$

Since each remainder is the sum of two positive quantities, no remainder can reduce to zero ; therefore, the division can not terminate. Therefore,

**Prin. 4.—***The sum of the equal even or odd powers of two quantities is not divisible by the difference of the quantities.*

**138.** The Laws of the Quotient.

$$(x^5 - 32y^5) \div (x - 2y) = x^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 16y^4 \text{ [P. 1].}$$

$$(32x^5 + y^5) \div (2x + y) = 16x^4 - 8x^3y + 4x^2y^2 - 2xy^3 + y^4 \text{ [P. 2].}$$

$$(81x^4 - 16y^4) \div (3x - 2y) = 27x^3 + 18x^2y + 12xy^2 + 8y^3 \text{ [P. 1].}$$

$$(81x^4 - 16y^4) \div (3x + 2y) = 27x^3 - 18x^2y + 12xy^2 - 8y^3 \text{ [P. 3].}$$

By a careful inspection of the above quotients the following laws will appear :

1. *The number of terms equals the exponent of the power involved in the dividend.*

2. *The terms are all positive when the divisor is the difference of the quantities, and alternately positive and negative when it is the sum of the quantities.*

3. *The first term of the quotient is obtained by dividing the first term of the dividend by the first term of the divisor.*

4. *Each subsequent term of the quotient may be found by dividing the preceding term by the first term of the divisor and multiplying the quotient by the second term of the divisor, disregarding the sign.*

**Note.**—The last term of the quotient equals the last term of the dividend divided by the last term of the divisor, which fact will serve as a check on the preceding operations.

#### EXERCISE 20.

Four of the following examples will not produce entire quotients. Let the pupil point them out and give reasons.

Write the quotients of the following examples at sight, telling each time why the dividend is divisible by the divisor :

$$1. (x^3 + y^3) \div (x + y) = \quad 4. (x^3 - y^3) \div (x - y) =$$

$$2. (x^6 - y^6) \div (x + y) = \quad 5. (a^3 - 8b^3) \div (a - 2b) =$$

$$3. (x^6 - y^6) \div (x - y) = \quad 6. (x^4 - 16y^4) \div (x - 2y) =$$

$$7. (x^4 - 16y^4) \div (x + 2y) =$$

$$8. (8x^3 - 27y^3) \div (2x - 3y) =$$

$$9. (8x^3 + 27y^3) \div (2x + 3y) =$$

$$10. (x^4 + 16y^4) \div (x + 2y) =$$

11.  $(27m^3 - 64n^3) \div (3m - 4n) =$

12.  $(27m^3 - 64n^3) \div (3m + 4n) =$

13.  $(x^5 + 32y^5) \div (x + 2y) =$

14.  $(x^5 - 32y^5) \div (x - 2y) =$

15.  $(32a^5 - 243b^5) \div (2a - 3b) =$

16.  $(32a^5 + 243b^5) \div (2a - 3b) =$

17.  $(256m^4 - 625n^4) \div (4m - 5n) =$

18.  $(256m^4 - 625n^4) \div (4m + 5n) =$

19.  $(x^6 + y^3) \div (x + y) =$       20.  $(x^6 - y^6) \div (x + y) =$

21.  $(64a^6 - 729b^6) \div (2a - 3b) =$

22.  $(x^7 + 128y^7) \div (x + 2y) =$

23.  $(x^{3^n} + y^{3^n}) \div (x^n + y^n) =$

24.  $(x^{4^n} - y^{4^n}) \div (x^n - y^n) =$

## EXERCISE 21.

Divide  $x^6 + y^6$  by  $x^2 + y^2$ .

**Solution:**  $x^6 = (x^2)^3$  and  $y^6 = (y^2)^3$ ; therefore,  $x^6 + y^6$  is divisible by  $x^2 + y^2$  [P. 2]. The quotient is  $x^4 - x^2y^2 + y^4$  [138].

Among the following examples are four that will not give entire quotients. Point them out, and divide the rest :

1.  $(a^6 - b^3) \div (a^2 - b) =$       2.  $(a^3 + b^6) \div (a + b^2) =$

3.  $(8x^9 - 27y^6) \div (2x^3 - 3y^2) =$

4.  $(8x^6 + 27y^6) \div (2x^3 + 3y^2) =$

5.  $(x^6 + 8y^6) \div (x^3 + 2y^3) =$

6.  $(x^6 - 8y^6) \div (x^3 - 2y^3) =$

7.  $(a^{10} + 32b^5) \div (a^2 + 2b) =$

8.  $(16a^8 - 81b^4) \div (2a^2 + 3b) =$

9.  $(16a^8 - 81b^4) \div (2a^2 - 3b) =$

10.  $(16a^8 + 81b^4) \div (2a^2 + 3b) =$

$$11. (16a^8 + 81b^4) \div (2a^2 - 3b) =$$

$$12. (a^5x^{10} + b^{10}y^5) \div (ax^2 + b^2y) =$$

$$13. (m^9p^{12} + n^6q^9) \div (m^3p^4 + n^2q^3) =$$

$$14. (32x^{10}y^5 + 243x^5y^{10}) \div (2x^2y + 3xy^2) =$$

$$15. (8x^{6m} + 27y^{9m}) \div (2x^{2m} + 3y^{3m}) =$$

$$16. \{(a+b)^{10} + c^5\} \div \{(a+b)^2 + c\} =$$

$$\text{Since } (x^3 - y^3) \div (x - y) = x^2 + xy + y^2,$$

$$(x^3 - y^3) \div (x^2 + xy + y^2) = x - y$$

$$17. (a^3 - b^3) \div (a^2 + ab + b^2) =$$

$$18. (a^3 + b^3) \div (a^2 - ab + b^2) =$$

$$19. (x^3 - 8y^3) \div (x^2 + 2xy + 4y^2) =$$

$$20. (x^3 + 8y^3) \div (x^2 - 2xy + 4y^2) =$$

$$21. (x^6 - y^6) \div (x^4 + x^2y^2 + y^4) =$$

$$22. (a^4 - b^4) \div (a^3 + a^2b + ab^2 + b^3) =$$

$$23. (a^3x^3 + b^3y^3) \div (a^2x^2 - abxy + b^2y^2) =$$

$$24. (16x^4 - 81y^4) \div (8x^3 + 12x^2y + 18xy^2 + 27y^3) =$$

$$25. (x^5 + y^5) \div (x^4 - x^3y + x^2y^2 - xy^3 + y^4) =$$

$$26. (x^6 - y^6) \div (x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5) =$$

## Factoring.

### Definition and Principles.

**139. Factoring** is the process of finding the factors of a quantity.

Since the quotient times the divisor equals the dividend, the quotient and divisor are the two factors of the dividend. Therefore,

**Prin. 1.**—A divisor of a quantity is one factor of the quantity and the quotient is the other.

140. Since dividing every term of a quantity divides the quantity [99, P.], a divisor of every term of a quantity is a divisor of the quantity, and hence a factor of the quantity. Therefore,

*Prin. 2.*—*A divisor of every term of a quantity is a factor of the quantity.*

### Problems.

#### 1. To factor a polynomial containing a monomial factor.

**Illustration.**—Factor  $a^2b + ab^2 - a^2b^2$ .

**Solution:**  $ab$  is a divisor of every term of  $a^2b + ab^2 - a^2b^2$ ; it is, therefore, one factor of it, and  $(a^2b + ab^2 - a^2b^2) \div ab$  is the other factor [P. 1]. Therefore,  $a^2b + ab^2 - a^2b^2 = ab(a + b - ab)$ .

### EXERCISE 22.

Factor :

1.  $ax + b^2x$
2.  $6a^2 - 3ax$
3.  $2a^2b - 4ab^2 + 2abc$
4.  $3x^3y + 3x^2y^2 + 6xy^3$
5.  $2x^2z - 4xyz + 6xz^2$
6.  $ax^{2n} + bx^n + cx^{n+1}$
7.  $12am^3 - 18am^2n + 3am^2n^2$
8.  $6x^{m+n} - 12x^{m+1} + 18x^m$
9.  $10a^2bc + 15ab^2c - 10abc^2$
10.  $8x^2y - 12xy^2 + 20xyz$
11.  $20a^2b - 30ac + 20ab - 10a^3$
12.  $30acx - 15acy + 10acz + 5ac$
13.  $21c - 6c^2 - 15ac + 3cm^2$
14.  $10xy + 25x^2y - 35cxy + 5a^2xy$
15.  $10a^m b^n - 15a^{2m} b^{2n} - 5a^m b^{2n} + 20a^{2m} b^n$
16.  $6a^{m+3} + 12a^{m+2} - 18a^{m+1} - 24a^m$
17.  $a^m x^{3n} + a^{2m} x^{2n} - a^{3m} x^n + a^{4m} x^{2n}$
18.  $a^{n+4} y^{n-4} - a^{n+3} y^{n-3} + a^{n+2} y^{n-2}$

### 2. To factor binomials.

**Illustrations.**—1. Factor  $a^6 - b^6$ .

**Solution :**  $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$  [91, P. 1]

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \text{ [135, P. 2]}$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \text{ [134, P. 1]}$$

$$\therefore a^6 - b^6 = (a + b)(a^3 - ab^2 + b^3)(a - b)(a^3 + ab^2 + b^3).$$

2. Factor  $a^4 + 4$ .

**Solution :**  $a^4 + 4 = a^4 + 4a^2 + 4 - 4a^2$  [Ax. 9]

$$= (a^4 + 4a^2 + 4) - 4a^2 = (a^2 + 2)^2 - (2a)^2$$

$$= (a^2 + 2 + 2a)(a^2 + 2 - 2a) \text{ [91, P. 1]}$$

$$= (a^2 + 2a + 2)(a^2 - 2a + 2).$$

#### EXERCISE 28.

Factor :

1.  $4x^2 - y^2$

9.  $x^4 + 4y^4$

17.  $(a + b)^2 - c^2$

2.  $9x^2 - 16y^2$

10.  $4x^4 + 1$

18.  $(x - y)^2 - z^2$

3.  $16a^4 - 81b^4$

11.  $x^{4n} - y^{4n}$

19.  $32x^5 - 243$

4.  $a^3 - b^3$

12.  $x^{12} + y^{12}$

20.  $x^7 + 128$

5.  $x^5 + y^5$

13.  $x^{12} - y^{12}$

21.  $x^4 + 2500y^4$

6.  $x^3 - 8y^3$

14.  $a^4x^4 - 1$

22.  $a^3 + 4b^4$

7.  $64x^3 + 1$

15.  $x^4 + 64$

23.  $x^3 + 64y^4$

8.  $x^5 + 1$

16.  $64a^4 + b^4$

24.  $8x^{3n} + 27y^{3n}$

25.  $a^4x^4 + 64b^4$

26.  $a^5x^{5n} + b^{5n}$

### 3. To factor trinomials.

**Illustrations.**—1. Factor  $x^2 + 2xy + y^2$  and

$$x^2 - 2xy + y^2.$$

**Solution :** 1.  $x^2 + 2xy + y^2 = (x + y)(x + y)$ , since  $(x + y)^2 = (x^2 + 2xy + y^2)$  [112, P. 1].

2.  $x^2 - 2xy + y^2 = (x - y)(x - y)$ , since  $(x - y)^2 = x^2 - 2xy + y^2$  [113, P. 2].

**Note.**—A trinomial is a square when two of its terms are positive squares, and the other term is  $\pm$  twice the product of their square roots.

2. Factor  $x^2 + 5x + 6$ ,  $x^2 + 2x - 15$ , and  $x^2 - 2x - 15$ .

**Solution:** 1.  $x^2 + 5x + 6 = (x + 3)(x + 2)$  [92, P. 2].

2.  $x^2 + 2x - 15 = (x + 5)(x - 3)$  [92, P. 2].

3.  $x^2 - 2x - 15 = (x - 5)(x + 3)$  [92, P. 2].

**Note.**—A trinomial is the product of two binomials having a common term if the first term is a perfect square, and the last term is the algebraic product of two factors whose sum into the square root of the first term will give the middle term.

3. Factor  $6x^2 + 13x + 6$  and  $3x^2 + 16x - 35$ .

**Solution:** 1.  $6x^2 + 13x + 6 = (2x + 3)(3x + 2)$  [93, P. 3].

2.  $3x^2 + 16x - 35 = (3x - 5)(x + 7)$  [93, P. 3].

**Notice.**—The first terms of the factors are the factors of the first term of the trinomial; the last terms of the factors are the factors of the last term of the trinomial; and the last terms of the factors are so arranged with the first terms that the algebraic sum of the products obtained by a cross-multiplication of the terms will give the middle term of the trinomial.

4. Factor  $x^4 + x^2y^2 + y^4$ .

**Solution:**  $x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2$  [Ax. 9]

$= (x^2 + y^2)^2 - (xy)^2 = (x^2 + xy + y^2)(x^2 - xy + y^2)$  [91, P. 1]

#### EXERCISE 24.

Factor :

- |                                   |                                      |
|-----------------------------------|--------------------------------------|
| 1. $x^2 + 4x + 4$                 | 7. $2x^2 + 11x - 21$                 |
| 2. $9a^2 - 24ab + 16b^2$          | 8. $x^4y^4 + x^2y^2 + 1$             |
| 3. $16x^4 + 40x^2y^2 + 25y^4$     | 9. $16x^4 + 36x^2 + 81$              |
| 4. $x^2 + x - 132$                | 10. $10x^4 + 31x^2 - 14$             |
| 5. $x^2 - 3x - 18$                | 11. $16x^2 + 8xy - 3y^2$             |
| 6. $2x^2 + 13x + 21$              | 12. $a^2x^2 + 100ax + 99$            |
| 13. $121x^4 + 144y^4 + 264x^2y^2$ | 14. $x^{2n} + 2x^ny^n + y^{2n}$      |
| 15. $a^2y^2 + 3abyz + 2b^2z^2$    | 16. $6x^2 + 7ax + 2a^2$              |
| 17. $10x^2 - 19bx + 6b^2$         | 18. $x^8 + x^4y^4 + y^8$             |
| 19. $64x^3 + 128x^4y^2 + 81y^4$   | 20. $x^{4n} + x^{2n}y^{2n} + y^{4n}$ |



8. Factor  $64 + 48x + 12x^2 - 7x^3$ .

Form.

$$\begin{array}{r|l}
 64 + 48x + 12x^2 - 7x^3 & (4 + x) \\
 64 & \\
 \hline
 3 \times 4^2 = & 48 \\
 3 \times x \times 4 = & 12x \\
 (x)^2 = & x^2 \\
 \hline
 48 + 12x + x^2 & \\
 \hline
 & 48x + 12x^2 - 7x^3 \\
 & \hline
 & 48x + 12x^2 + x^3 \\
 & \hline
 & -8x^3
 \end{array}$$

**Solution:** By extracting the cube root of the polynomial we observe that it lacks  $8x^3$  of being  $(4 + x)^3$ . Therefore,  $64 + 48x + 12x^2 - 7x^3 = (4 + x)^3 - 8x^3$ , which is divisible by  $4 + x - 2x$ , or  $4 - x$  [134, P. 1]. The other factor may be obtained by dividing  $64 + 48x + 12x^2 - 7x^3$  by  $4 - x$ . It is found to be  $16 + 16x + 7x^2$ .

9. Factor  $4x^3 + 12xy + 9y^3 + 12x + 18y + 8$ .

$$\begin{aligned}
 \text{Solution: } 4x^3 + 12xy + 9y^3 + 12x + 18y + 8 &= \\
 (4x^3 + 12xy + 9y^3) + (12x + 18y) + 8 &= \\
 (2x + 3y)^3 + 6(2x + 3y) + 8 &= \\
 (2x + 3y + 4)(2x + 3y + 2) \text{ [92, P. 2]} &= \\
 (2x + 3y + 4)(2x + 3y + 2). &
 \end{aligned}$$

10. Factor  $6x^3 - xy - 2y^3 - 8x + 17y - 30$ .

$$\begin{aligned}
 \text{Solution: } 6x^3 - xy - 2y^3 - 8x + 17y - 30 &= \\
 (6x^3 - xy - 2y^3) + (-8x + 17y) - 30 &= \\
 (2x + y)(3x - 2y) + (-8x + 17y) - 30 &= \\
 (2x + y - 6)(3x - 2y + 5) \text{ [93, P. 3]}. &
 \end{aligned}$$

#### EXERCISE 25.

Factor :

1.  $ax - ay + bx - by$
2.  $cx^2 + c - dx^2 - d$
3.  $x^2 - 2xy + y^2 - z^2$
4.  $y^2 - x^2 - z^2 - 2xz$
5.  $25m^2 - n^2 + 4n - 4$
6.  $x^3 + 3x^2 + 3x + 1$
7.  $8x^3 - 36x^2 + 54x - 27$
8.  $1 - r^2 - 2rs - s^2$
9.  $ax^2 + axy - bx^2 - bxy$
10.  $x^3 + 3x^2y + 3xy^2 + y^3 - z^3$
11.  $x^4 + 4x^3 + 6x^2 + 4x + 1$

12.  $x^3 + 8x^2 + 24x + 32x^2 + 16$
13.  $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$
14.  $x^3 + y^3 + 3y^2 + 3y + 1$
15.  $9a^2b^2 - 9a^2 + 12ab - 4b^2$
16.  $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$
17.  $4xz + 6yz - 6x^2 - 9xy$
18.  $3a^2 - 15ay - 2ab + 10by$
19.  $3ac - 18bc - 2ax + 12bx$
20.  $x^4 + 6x^3 + x^2 - 12x + 4$
21.  $x^2 + 4y^2 + 4xy + 6xz + 12yz$
22.  $8x^3 + 8a^3 - 12a^2 + 6a - 1$
23.  $x^2 - 2xy + y^2 + 9x - 9y + 18$
24.  $6x^2 - 5xy - 6y^2 - 4x - 20y - 16$
25.  $2x^2 - 5xy - 3y^2 - x + 17y - 10$

### Miscellaneous Examples.

#### EXERCISE 26.

(Remove monomial factors first.)

Factor :

- |                               |                              |
|-------------------------------|------------------------------|
| 1. $6a^3b - 6a^2b^2 + 2ab^2c$ | 10. $6x^2 - 5xy - 21y^2$     |
| 2. $5ax^2 + 10ax + 5a$        | 11. $(a-x)^2 - (b+y)^2$      |
| 3. $3a^5x - 3ab^4x$           | 12. $(x+y)^4 - (x-y)^4$      |
| 4. $3x^4 + 3a^2x^2 + 3a^4$    | 13. $27 + (2x+y)^3$          |
| 5. $5x^2 + 20x - 105$         | 14. $(2a+3b)^3 + 27c^3$      |
| 6. $a^2 - x^2 - 2xz - z^2$    | 15. $4x^3 + 15x^4y^3 + 9y^6$ |
| 7. $2x^2 - 7xy - 15y^2$       | 16. $ac - bc + a^2 - ab$     |
| 8. $x^2 + 2xy + y^2 - x^4$    | 17. $a^2 - (b+c-d)^2$        |
| 9. $x^4 + 26x^2 + 25$         | 18. $x^2 - 4y^2 + 7x + 14y$  |

- |   |                            |
|---|----------------------------|
| 19. $x^3 + x^4 y^4 + y^3$                         | 29. $a^{10} x^2 y + x^2 y$ |
| 20. $27 x^3 + 27 x^2 + 9 x + 1$                   | 30. $a^5 x^3 - a$          |
| 21. $x^3 - 9 x^2 y + 27 x y^2 - 27 y^3$           | 31. $a^9 - a b^8$          |
| 22. $1 - 4 x^3 + 12 x y - 9 y^3$                  | 32. $x^{12} - z^{12}$      |
| 23. $(x + y)^2 - 4(x + y) + 4$                    | 33. $x^{12} + z^{12}$      |
| 24. $a^2 x^2 - b^2 x^2 - a^2 y^2 + b^2 y^2$       | 34. $8 x^6 + 27 y^9$       |
| 25. $x^4 + 7 x^2 + 1 - 6 x^3 + 6 x$               | 35. $x^8 + x y^7$          |
| 26. $a^2 x^3 + a^2 y^3 - b^2 x^3 - b^2 y^3$       | 36. $x^9 + y^9$            |
| 27. $x^4 + 6 x^3 + 5 x^2 - 12 x + 3$              | 37. $32 x^{10} - 1$        |
| 28. $3 x^6 + a y^6 + a x^6 + 3 y^6$               | 38. $1 - (x - y)^3$        |
| 39. $216 a^3 - 216 a^2 + 72 a - 7$                |                            |
| 40. $x^3 - 3 x^2 y + 3 x y^2 - y^3 + 64 x^6$      |                            |
| 41. $x^4 + 4 x^2 + y^3 + 4 x^3 + 2 x^2 y + 4 x y$ |                            |
| 42. $6 x^2 - x y - 2 y^2 + x + 11 y - 15$         |                            |
| 43. $3 a x + 3 b x + 3 c x - 2 y(a + b + c)$      |                            |
| 44. $(x + y)^3 + 3(x + y)^2 + 3(x + y) + 1$       |                            |
| 45. $5 x^2 - 34 x y - 7 y^2 - 9 x - 45 y - 18$    |                            |
| 46. $3 x^2 + 14 x y - 5 y^2 + 28 x + 28 y + 49$   |                            |
| 47. $4 x^2 - 12 x y + 9 y^2 + 28 x - 42 y + 24$   |                            |
| 48. $256 x^4 + 128 x^3 - 944 x^2 - 240 x + 900$   |                            |

---

### Cancellation.

#### Definitions and Principles.

141. Multiplying a quantity by a factor is called *inserting* a factor.

142. Dividing a quantity by a factor is called *eliminating* a factor.

**143.** Crossing out a quantity, and writing in its stead the result obtained by inserting or eliminating a factor, is *Cancellation*.

**144.** Let  $p = x \times y$ , and  $q = r \times s$ ;  
 then  $p q = r \times s \times x \times y$  [Ax. 4].  
 Now,  $p \div x = y$ , and  $q x = r s x$ ;  
 therefore,  $(p \div x) \times q x = r \times s \times x \times y$ .

Therefore,

**Prin. 1.**—*Dividing one quantity and multiplying another by the same factor does not change their product.*

**145.** Let  $D =$  a dividend,  $d =$  a divisor,  $q =$  the quotient of  $D$  and  $d$ , and  $n =$  any quantity.

Now,  $D \div d = q$ ; whence  $D = q \times d$ ,

and  $n D = n q \times d$  [Ax. 4];

or  $n D \div d = n q$  [Ax. 5]. (A)

Again,  $D = q \times d = n q \times (d \div n)$  [P. 1];

or  $D \div (d \div n) = n q$  [Ax. 5]. (B). Therefore,

**Prin. 2.**—*Multiplying the dividend or dividing the divisor multiplies the quotient.*

**146.** Since  $D = q \times d$ ,

$D \div n = (q \div n) \times d$  [Ax. 5];

and  $(D \div n) \div d = q \div n$  [Ax. 5]. (A)

Again,  $D = q \times d = (q \div n) \times n d$  [P. 1],

and  $D \div n d = q \div n$  [Ax. 5]. (B). Therefore,

**Prin. 3.**—*Dividing the dividend or multiplying the divisor divides the quotient.*

**147.** Since  $D = q \times d$ ,  $n D = q \times n d$  [Ax. 4];

and  $n D \div n d = q$  [Ax. 5]. (A)

Also,  $D \div n = q \times (d \div n)$  [Ax. 5];

and  $(D \div n) \div (d \div n) = q$  [Ax. 5]. (B). Therefore,

**Prin. 4.**—*Multiplying or dividing both dividend and divisor by the same quantity does not alter the quotient.*

**Problem.** To multiply or divide by cancellation.

**Illustrations.**—1. Multiply  $(a + b)^2$  by  $a - b$ .

$$\begin{array}{c} \text{Form.} \\ a + b \quad a^2 - b^2 \\ (a + b)^2 \times (a - b) = a^3 + a^2b - ab^2 - b^3 \end{array}$$

**Solution :** Since dividing one quantity and multiplying another by the same factor does not change their product [P. 1], we divide the first quantity by  $a + b$  and multiply the second by  $a + b$ , and obtain  $(a + b) \times (a^2 - b^2) = a^3 + a^2b - ab^2 - b^3$ .

2. Multiply  $(x^3 + y^3) \div (x^2 - y^2)$  by  $(x - y)$ .

$$\begin{array}{c} \text{Form.} \\ x^3 + y^3 \quad x^2 - y^2 \\ \frac{x^3 + y^3}{x^2 - y^2} \times (x - y) = \frac{x^3 + y^3}{x + y} = x^2 - xy + y^2 \end{array}$$

**Solution :** Since dividing the divisor multiplies the quotient [P. 2], we divide the divisor by  $x - y$  and obtain  $\frac{x^2 + y^2}{x + y}$ , which equals  $x^2 - xy + y^2$  [135, P. 2].

3. Find the value of  $\frac{x(x^3 + y^3)(x^2 + xy + y^2)}{(x^4 + x^2y^2 + y^4)(x + y)}$

**Solution :**

$$\frac{x(x^3 + y^3)(x^2 + xy + y^2)}{(x^4 + x^2y^2 + y^4)(x + y)} = \frac{x(\cancel{x + y})(\cancel{x^2 - xy + y^2})(\cancel{x^2 + xy + y^2})}{(\cancel{x^2 + xy + y^2})(\cancel{x^2 - xy + y^2})(x + y)} = x \text{ [P. 4].}$$

#### EXERCISE 27.

Find the value of :

1.  $(a^3 + ab + b^2)(a^2 - b^2)$

2.  $(x^3 - y^2)(x^3 + xy + y^2)(x^2 - xy + y^2)$

3.  $(ax + ay)\left(\frac{x}{a} - \frac{y}{a}\right)$       4.  $\frac{a+x}{a-x} \times (a^2 - x^2)$

5.  $(a^2 - b^2) \times \frac{a+b}{a-b}$       6.  $\frac{a^3 + x^3}{a^3 - x^3} \times (a^6 - x^6)$

7.  $\frac{(x^3 + y^3)(x^3 - y^3)}{x^4 + x^2y^2 + y^4}$       8.  $\frac{(x^6 + y^6)(x^6 - y^6)}{(x^3 + y^3)(x^4 - x^2y^2 + y^4)}$

9.  $\frac{x^4 + x^2 y^2 + y^4}{x^2 + x y + y^2} \div (x^2 - x y + y^2)$
10.  $\frac{\{(x+y)^2 - z^2\} \{(x-y)^2 - z^2\}}{\{(x-z)^2 - y^2\} (x+y+z)}$
11.  $\frac{(ax + bx + ay + by)(ax - ay - bx + by)}{(x^2 - y^2)(a^2 - b^2)}$
12.  $\frac{(x^2 + 2x^2 - 24x)(x^2 + x - 6)}{(x^2 + 9x + 18)(x^2 - 6x + 8)}$
13.  $\frac{(3x^2 - 3x - 36)(2x^2 - 98)}{(x^2 + 10x + 21)(x^2 - 11x + 28)}$
14.  $\frac{(6x^2 - x - 2)(15x^2 - 16x - 15)}{(10x^2 + 11x + 3)(9x^2 - 21x + 10)}$
15.  $\frac{(x^4 - x)(6x^2 - 10x - 24)}{(3x + 4)(x^2 + x + 1)(2x^2 - 8x + 6)}$
16.  $\frac{(ax + by)^2 (ax - by)^2}{(a^2 x^2 - b^2 y^2)(a^2 x^2 - b^2 y^2)}$
17.  $\frac{(x^2 + 2xy + y^2)(x^3 - 3x^2 y + 3xy^2 - y^3)}{x^4 - y^4}$
18.  $\frac{(a^2 - b^2)(a^3 - b^3)(a^4 - b^4)}{(b^2 - a^2)(b^3 - a^3)(b^4 - a^4)}$
19.  $\frac{(a+b)^3}{(a-b)^3} \times \frac{(a-b)^4}{(a+b)^2}$
20.  $\frac{\{(a+b)^2 - (a-b)^2\} \{(a-b)^2 - (a+b)^2\}}{4a^2 b^2}$

### Highest Common Divisor.

#### Definitions and Principles.

148. A *Common Divisor* of two or more quantities is an exact divisor of each of them.

149. The *Highest Common Divisor* of two or more quantities is that common divisor which is of the highest

degree—that is, which contains the greatest number of prime factors.

150. Quantities are *prime to each other* when they contain no common divisor, except unity.

151. The product of two or more common factors is a common divisor. Thus,  $a \times b$ , or  $ab$ , is a common divisor of  $abc$  and  $abd$ .

152. No common divisor can contain factors not common to the quantities, since it contains its own factors and is itself contained in each of the quantities.

153. Since the product of two or more common factors is a common divisor [151], and no common divisor can contain factors not common to the quantities [152], it follows that—

*Prin. 1.*—*The highest common divisor equals the product of all the common factors.*

154. The abbreviation H. C. D. stands for highest common divisor.

**Problem.** To find the highest common divisor of quantities whose common factors may be readily obtained.

**Illustrations.**—1. Find the H. C. D. of

$$3a^2b^3, 6a^3b^2c, \text{ and } 9a^4b^3c.$$

**Solution:** The common factors are 3,  $a^2$ , and  $b^2$ ; therefore, the H. C. D. is  $3 \times a^2 \times b^2 = 3a^2b^2$  [P. 1].

2. Find the H. C. D. of

$$a^2 - x^2, a^4 - x^4, \text{ and } a^3 + a^2x - ax^2 - x^3.$$

**Form.**

$$a^2 - x^2 = (a + x)(a - x)$$

$$a^4 - x^4 = (a^2 + x^2)(a + x)(a - x)$$

$$a^3 + a^2x - ax^2 - x^3 = a^2(a + x) - x^2(a + x)$$

$$= (a^2 - x^2)(a + x) = (a + x)(a - x)(a + x)$$

$$\text{H. C. D.} = (a + x)(a - x) = a^2 - x^2.$$

**Solution:** Resolving the quantities into their prime factors, we observe that  $a+x$  and  $a-x$  are the only common factors; hence their product,  $a^2-x^2$ , is the H. C. D. [P. 1].

## EXERCISE 28.

Find the H. C. D. of :

1.  $6x^3y^2z$ ,  $12x^4y^3z^2$ , and  $18x^3y^3z^3$
2.  $4a^2b^3c^3d$ ,  $8a^3b^3cx$ , and  $16a^4b^4c^3y$
3.  $15m^2(p+q)$ ,  $30m^3(p+q)^2$ , and  $45m^4(p+q)^3$
4.  $2(p+q)^2(p-q)^3$ ,  $4(p+q)^3(p-q)^3$ , and  $6(p+q)^4(p-q)^3$
5.  $a^2-b^2$  and  $a^2+2ab+b^2$
6.  $a^3-b^3$  and  $(a-b)^3$
7.  $a^3+b^3$  and  $a^2-b^2$
8.  $a^4-b^4$  and  $a^6-b^6$
9.  $(x-y)^4$  and  $x^4-y^4$
10.  $x^3+y^3$  and  $x^2-xy+y^2$
11.  $x^4+x^2y^2+y^4$  and  $x^3+y^3$
12.  $abx-aby$  and  $m^3x-m^2y$
13.  $x^2+5x+6$ ,  $x^2+7x+10$ , and  $x^2-x-6$
14.  $x^3+y^3$ ,  $x^4+x^2y^2+y^4$ , and  $x^2-xy+y^2$
15.  $9x^2-3x-2$ ,  $3x^2-17x+10$ , and  $6x^2+17x-14$
16.  $2x^2-3x-35$ ,  $6x^2+19x-7$ , and  $4x^2+22x+28$
17.  $ax+ay+bx+by$  and  $cx+cy+dx+dy$
18.  $9a^2-6ab-6ac+4bc$  and  $9a^3-12ab+4b^3$
19.  $x^3+3x^2y+3xy^2+y^3$  and  $abx+aby-acx-acy$
20.  $(a+b)^2-(c+d)^2$  and  $ax+bx+cx+dx$
21.  $x^{12}+y^{12}$ ,  $x^8-x^4y^4+y^8$ , and  $x^{16}+x^8y^8+y^{16}$
22.  $4x^2+12xy+9y^2-16$  and  $4x^2+12xy+9y^2+14x+21y+12$
23.  $x^4+4y^4$ ,  $x^2+2xy+2y^2$ , and  $x^3+2x^2y+2xy^2+x^2z+2xyz+2y^2z$

24.  $x^3 + 6xy + 9y^3 + 2x + 6y - 24$  and  
 $x^3 + 6xy + 9y^3 + x + 3y - 30$
25.  $x^3 + 2xy + y^3 - x^2$  and  
 $x^3 + y^3 + x^2 + 2xy + 2xz + 2yz$
26.  $a^3x^3 + a^2y^3 - a^2x^3 + b^3x^3 + b^2y^3 - b^2x^3$  and  
 $x^3 + 2x^2y + y^3 - x^3$
27.  $(p+q)^4 - 2(p+q)^3r^2 + r^4$  and  
 $(p+r)^4 - 2(p+r)^3q^2 + q^4$
28.  $x^{16} - 4096y^{16}$  and  $x^5y - 4x^3y^3 + 8xy^5$

### Highest Common Divisor by Successive Division.

#### Definitions and Principles.

155. If one quantity be divided by another, then the divisor by the remainder, then the next divisor by the next remainder, and so on, until the division terminates, the process is *Successive Division*.

156. Let  $d$  be a divisor of  $a$ , and  $\frac{a}{d} = q$ ,

then  $a = q \times d$ , and  $na = nqd$  [Ax. 4];

whence  $\frac{na}{d} = qn$ , a whole quantity. Therefore,

**Prin. 1.**—*A divisor of a quantity is also a divisor of any number of times the quantity.*

157. Let  $d$  be a common divisor of  $a$  and  $b$ ,

and  $\frac{a}{d} = q$ , and  $\frac{b}{d} = r$ ;

then  $a = qd$ , and  $b = rd$  [Ax. 4].

$\therefore a \pm b = qd \pm rd$  [Ax. 2 and 3]  $= (q \pm r)d$ ;

and  $\frac{a \pm b}{d} = q \pm r$ , a whole quantity. Therefore,

**Prtn. 2.**—*A common divisor of two quantities is also a divisor of their sum and of their difference.*

**158. Theorem.**—*The last divisor obtained by the successive division of two quantities is their highest common divisor.*

**Demonstration.**—Let  $A$  and  $B$  represent any two quantities. Divide  $B$  by  $A$ , and let the quotient be  $q$  and the remainder  $R$ ; divide  $A$  by  $R$ , and let the quotient be  $q'$  and the remainder  $R'$ ; divide  $R$  by  $R'$ , and let the quotient be  $q''$  and the remainder zero. Prove  $R'$  the H. C. D. of  $A$  and  $B$ .

$$\begin{array}{r}
 A) B (q \\
 \underline{Aq} \\
 R) A (q' \\
 \underline{Rq'} \\
 R') R (q'' \\
 \underline{R'q''} \\
 0
 \end{array}$$

1.  $R'$  is a divisor of  $R$ , since the division has terminated; hence it is also a divisor of  $Rq'$  [P. 1] and of  $R' + Rq'$ , or  $A$  [P. 2], and therefore of  $Aq$  [P. 1] and of  $R + Aq$ , or  $B$  [P. 2]; therefore,  $R'$  is a common divisor of  $A$  and  $B$ .

2. There can be no higher common divisor of  $A$  and  $B$  than  $R'$ ; for, if there could be, it would be a divisor of  $A$  and  $B$ , and hence, too, of  $Aq$  [P. 1] and of  $B - Aq$ , or  $R$  [P. 2], and therefore of  $Rq'$  [P. 1] and of  $A - Rq'$ , or  $R'$  [P. 2]; that is, a quantity of a higher degree than  $R'$  would be a divisor of  $R'$ , which is impossible. Therefore,  $R'$  is the H. C. D. of  $A$  and  $B$ .

**159.** *Either of two quantities may be multiplied by a factor not found in the other without changing their highest common divisor [153, P. 1].*

**160.** *Either of two quantities may be divided by a factor not common to both without changing their highest common divisor [153, P. 1].*

**161.** *Quantities have two highest common divisors, a positive one and a negative one.*

Thus,  $a - b$  and  $b - a$  are both highest common divisors of  $a^2 - b^2$  and  $a^3 - 2ab + b^3$ . If  $a > b$ ,  $a - b$  is positive and  $b - a$  negative; if  $b > a$ ,  $a - b$  is negative and  $b - a$  positive.

**Problem.** To find the highest common divisor by successive division.

**Illustration.**—Find the H. C. D. of

$$2x^4 - 9x^3 - 14x + 3 \text{ and } 3x^4 - 14x^3 - 9x + 2.$$

**Form.**

$$2x^4 - 9x^3 - 14x + 3 \quad 3x^4 - 14x^3 - 9x + 2$$

2

$$\begin{array}{r} 6x^4 - 28x^3 - 18x + 4 \quad 3 \\ 6x^4 - 27x^3 - 42x + 9 \quad - \\ \hline -1 \quad -x^3 + 24x - 5 \\ \hline x^3 - 24x + 5 \end{array}$$

$$x^3 - 24x + 5 \quad 2x^4 - 9x^3 - 14x + 3 \quad (2x - 9$$

$$2x^4 - 48x^3 + 10x$$

$$\begin{array}{r} -9x^3 + 48x^3 - 24x + 3 \\ -9x^3 \quad \quad + 216x - 45 \\ \hline \end{array}$$

$$\begin{array}{r} 48 \quad 48x^3 - 240x + 48 \\ \hline x^3 - 5x + 1 \end{array}$$

$$x^3 - 5x + 1 \quad x^3 - 24x + 5 \quad (x + 5$$

$$x^3 - 5x^3 + x$$

$$5x^3 - 25x + 5$$

$$5x^3 - 25x + 5$$

$$\therefore \text{H. C. D.} = x^3 - 5x + 1 \quad [158, \text{T.}]$$

**Explanation.**—Assume  $2x^4 - 9x^3 - 14x + 3$  as the divisor. Multiply the dividend by 2 to make the first term of it divisible by the first term of the divisor [159]. Divide until the remainder is of lower degree than the divisor. Remove the factor  $-1$  from the remainder to make the first term positive. This will not change the numerical value of the H. C. D. [160]. Use the remainder as a second divisor, and the first divisor as a second dividend. Divide again until the remainder is of lower degree than the divisor. Remove the factor 48 from the remainder to make it as simple as possible. This will not affect the H. C. D. [160]. Use the second remainder as a third divisor, and the second divisor as a third dividend. The division terminates without a remainder. Therefore,  $x^3 - 5x + 1$ , the last divisor, is the H. C. D. [158].



**Illustration.**—Find the H. C. D. of  $x^3 + 2x^2 - 5x - 10$   
and  $x^3 - 7x^2 + 3x^2 + 10x - 15$ .

**Solution:**  $x^3 + 2x^2 - 5x - 10 = x^2(x+2) - 5(x+2) = (x^2-5)(x+2)$ .  
Now,  $x+2$  is not a factor of  $x^3 - 7x^2 + 3x^2 + 10x - 15$ , since 2 is  
not a factor of 15. So we proceed to find the H. C. D. of  $x^2 - 5$  and  
 $x^3 - 7x^2 + 3x^2 + 10x - 15$ . This we find by division to be  $x^2 - 5$ .

**164.** Sometimes a remainder can be factored and factors not common rejected.

**Illustration.**—Find the H. C. D. of

$$x^3 + 10x^2 + 31x + 30 \text{ and } x^3 + 12x^2 + 44x + 48.$$

**Form.**

$$\begin{array}{r} x^3 + 10x^2 + 31x + 30 \ ) \ x^3 + 12x^2 + 44x + 48 \ (1 \\ \underline{x^3 + 10x^2 + 31x + 30} \phantom{00} \\ 2x^2 + 13x + 18 \\ = (x+2)(2x+9) \end{array}$$

Now,  $2x+9$  is not a factor of either polynomial, since  $2x$  is not  
a factor of  $x^2$ , nor 9 a factor of 30 or 48.

Dividing  $x^3 + 10x^2 + 31x + 30$  by  $x+2$  we find that the division is complete. Therefore,  $x+2$  is the H. C. D.

#### EXERCISE 29.

Find the H. C. D. of :

1.  $x^3 - 19x - 30$  and  $x^3 + 10x^2 + 31x + 30$

2.  $x^3 - 2x^2 - 15x + 36$  and  $x^3 - 11x^2 + 39x - 45$

3.  $x^3 + 2x^2 + x + 2$  and  $2x^4 - 8x^2 - 2x - 4$

4.  $x^5 + 2x^4 - x^3 - 2x$  and  $ax^4 - ax^3 + 2ax^2 + ax + 3a$

5.  $x^4 - 4x^3 - 16x^2 + 7x + 24$  and  
 $2x^3 - 15x^2 + 9x + 40$

6.  $2x^4 - 5x^3y - 3x^2y^2 + 7xy^3 + 3y^4$  and  
 $8x^3 - 4x^2y - 8xy^2 - 6y^3$

7.  $x^4 + 2x^2 + 9$  and  $7x^3 - 11x^2 + 15x + 9$

8.  $7a^2 - 23ab + 6b^2$  and  $5a^3 - 18a^2b + 11ab^2 - 6b^3$

9.  $6x^3 - 21x^2 - 138x - 63$  and  
 $6x^4 + 33x^3 - 39x^2 - 297x - 135$

$$10. 6x^5 + 15x^4y - 4x^3z^2 - 10x^2yz^2 \text{ and} \\ 9x^3 - 27x^2z - 6xz^2 + 18z^3$$

$$11. 6x^2 + x - 2 \text{ and } 9x^3 + 48x^2 + 52x + 16$$

$$12. x^4 - 9x^2 - 30x - 25 \text{ and } x^5 + x^4 - 7x^2 + 5x$$

$$13. 35x^3 + 47x^2 + 13x + 1 \text{ and} \\ 42x^4 + 41x^3 - 9x^2 - 9x - 1$$

$$14. 2x^4 - 6x^3 + 3x^2 - 3x + 1 \text{ and} \\ x^7 - 3x^6 + x^5 - 4x^3 + 12x - 4$$

165. If  $x$  is the H. C. D. of  $A$  and  $B$ , and  $y$  is the H. C. D. of  $x$  and  $C$ , then will  $y$  be the H. C. D. of  $A$ ,  $B$ , and  $C$ . For,

Let  $A = mx$  and  $B = nx$ , then  $m$  and  $n$  are prime to each other.

Let  $x = py$  and  $C = qy$ , then  $p$  and  $q$  are prime to each other.

$$\text{Now, } A = mx = mpy$$

$$B = nx = npy$$

$$C = qy$$

$\therefore y$  = H. C. D. of  $A$ ,  $B$ , and  $C$ . Therefore,

**Prin. 3.**—*The highest common divisor of three quantities may be obtained by finding the highest common divisor of two of them, then the highest common divisor of that and the third quantity.*

**Note.**—This principle may be extended so as to apply to any number of quantities.

Find the H. C. D. of :

$$15. x^3 + 6x^2 + 11x + 6, x^3 + 7x^2 + 14x + 8, \\ x^3 + 8x^2 + 17x + 10$$

$$16. x^3 + 9x^2 + 26x + 24, x^3 + x^2 - 14x - 24, \\ x^3 - 5x^2 - 2x + 24$$

$$17. x^2 - 11x + 30, x^3 - 12x^2 + 41x - 30, \\ x^4 - 12x^3 + 47x^2 - 72x + 36$$

18.  $x^2 + 11x + 30$ ,  $2x^2 + 21x + 54$ ,  
 $9x^2 + 53x - 9x - 18$
19.  $x^4 + x^2y^2 + y^4$ ,  $x^3 + x^4y^4 + y^3$ ,  $x^{16} + x^3y^3 + y^{16}$
20.  $3a^3 - 7a^2b + 5ab^2 - b^3$ ,  $3a^3 + 5a^2b + ab^3 - b^3$ ,  
 and  $a^2b + 3ab^2 - 3a^3 - b^3$
21.  $x^3 + 11x^2 + 40x + 48$ ,  $x^3 + 10x^2 + 33x + 36$ ,  
 $x^3 + x^2 - 21x - 45$ ,  $x^3 + 2x^2 - 23x - 60$ ,  
 and  $x^3 - 6x^2 - 7x + 60$

### Lowest Common Multiple.

#### Definitions and Principles.

**166.** A *Multiple* of a quantity is an entire number of times the quantity.

**Note.**—A multiple of a quantity contains the quantity as a divisor.

**167.** A *Common Multiple* of two or more quantities is an entire number of times each of them.

**Note.**—A common multiple of two or more quantities contains each of them as a divisor.

**168.** The *Lowest Common Multiple* is the common multiple which is of the lowest degree.

**169.** It is obvious that, to contain each of two or more quantities, a common multiple must contain all the different prime factors of those quantities. It must, moreover, contain each prime factor the greatest number of times it occurs in any of the quantities, for the same reason. It need, however, contain no other prime factors than these to contain each of the quantities. Therefore,

**Prin. 1.**—*The lowest common multiple equals the product of all the different prime factors, each taken the greatest number of times it occurs in any quantity.*

Problems.

1. To find the lowest common multiple of quantities readily factored.

**Illustrations.**—1. Find the L. C. M. of

$$a + b, a - b, \text{ and } a^2 - b^2.$$

**Solution:**  $a^2 - b^2$  is the lowest quantity that contains  $a^2 - b^2$ ; it, moreover, contains  $a + b$  and  $a - b$ ; it is, therefore, the L. C. M. of the three quantities.

2. Find the L. C. M. of

$$a^2 - b^2, a^2 + 2ab + b^2, \text{ and } a^3 + b^3.$$

**Solution:**  $a^2 - b^2 = (a + b)(a - b)$

$$a^2 + 2ab + b^2 = (a + b)(a + b)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$\therefore \text{L. C. M.} = (a + b)^2(a - b)(a^2 - ab + b^2) \text{ [P. 1].}$$

EXERCISE 80.

Find the L. C. M. of :

1.  $4x^2y^3$ ,  $6x^3y^2z$ , and  $9xy^3z^2$
2.  $12a^3b^2c$ ,  $16a^2b^3c^2$ , and  $24a^3b^2c^3$
3.  $60x^2yz^2$ ,  $80xy^2z^3$ , and  $90x^3y^2z$
4.  $x^2 + 1$  and  $x^4 - 1$
5.  $ab + ac$  and  $ab - ac$
6.  $a^2b + a^2$  and  $b^2 + 2b + 1$
7.  $2(a - 1)$ ,  $4(a^2 - 1)$ , and  $6(a - 1)^2$
8.  $a(x + y)$ ,  $ab(x - y)$ , and  $abc(x^2 - y^2)$
9.  $a^2 + b^2$ ,  $(a + b)^2$ , and  $(a - b)^2$
10.  $(x + y)^2$ ,  $x^2 + y^2$ , and  $x^4 - y^4$
11.  $x^2 + xy + y^2$ ,  $x^2 - xy + y^2$ , and  $x^4 + x^2y^2 + y^4$
12.  $x^3 + y^3$ ,  $x^3 - y^3$ , and  $x^4 - x^2y^2 + y^4$
13.  $16(x^3 - y^3)$ ,  $24(x^4 - y^4)$ , and  $36(x^3 + y^3)$
14.  $(x + y)^2$ ,  $(x - y)^2$ , and  $y^2 - x^2$
15.  $ax - bx$ ,  $ay - by$ , and  $a^2xy - b^2xy$
16.  $x^2 + 7x + 12$ ,  $x^2 + x - 12$ , and  $x^2 + 3x - 4$

17.  $x^2 - 10x + 25$ ,  $x^2 - x - 20$ , and  $x^2 + x - 12$

18.  $2x^2 + 11x + 12$ ,  $2x^2 - 7x - 15$ , and  $4x^2 + 12x + 9$

19.  $6x^2 + 10x - 24$ ,  $8x^2 + 30x + 18$ , and

$$8x^2 - 14x - 15$$

20.  $x^2 - y^2$ ,  $y^2 - x^2$ ,  $x^2 - y^2$ , and  $y^2 - x^2$

21.  $x^2 + 2x^2y + xy^2$ ,  $x^4y + xy^4$ , and

$$x^2 + 3x^2y + 3xy^2 + y^2$$

22.  $ac + ad + bc + bd$ ,  $ac - ad + bc - bd$ , and

$$ac - ad - bc + bd$$

23.  $2a^2 + 5ab + 3b^2$ ,  $2a^2 + ab - 3b^2$ , and

$$2a^2 - 5ab + 3b^2$$

24.  $a^2 + a^2b + a^2c$ ,  $ac^2 + bc^2 - c^2$ , and

$$a^2 + 2ab + b^2 - c^2$$

25.  $(x + y)^2 - x^2$ ,  $(x - y)^2 - x^2$ ,  $(x + z)^2 - y^2$ , and

$$(x - z)^2 - y^2$$

2. To find the lowest common multiple of quantities not readily factored.

170. If  $A = mx$  and  $B = nx$ , and  $m$  and  $n$  are prime to each other,

the H. C. D. of  $A$  and  $B = x$ , and

the L. C. M. of  $A$  and  $B = mnx = m \times B =$

$$\frac{A}{x} \times B = \frac{A \times B}{x}. \text{ Therefore,}$$

*Prin. 2.*—The lowest common multiple of two quantities equals their product divided by their highest common divisor.

**Illustrative Example.**—Find the L. C. M. of

$$x^3 - 19x - 30 \text{ and } x^3 + 10x^2 + 31x + 30.$$

**Solution:** The H. C. D. of the two polynomials, as found by successive division, is  $x^2 + 5x + 6$ ,

$$\therefore \text{ L. C. M. } = \frac{(x^3 - 19x - 30)(x^3 + 10x^2 + 31x + 30)}{x^2 + 5x + 6} =$$

$$(x - 5)(x^3 + 10x^2 + 31x + 30) \text{ by cancellation.}$$

EXERCISE 81.

Find the L. C. M. of :

1.  $3x^2 - 14ax + 15a^2$  and  $x^3 - 6ax^2 + 10a^2x - 3a^3$
2.  $4a^2 - 7ab + 3b^2$  and  $3a^3 - 4a^2b + 3ab^2 - 2b^3$
3.  $2y^3 - 15y^2 + 9y + 40$  and  $y^4 - 4y^3 - 16y^2 + 7y + 24$
4.  $a^4 - a^3 + 2a^2 + a + 3$  and  $a^4 + 2a^3 - a - 2$
5.  $3a^2b + 2ab^2 - b^3$  and  $2a^3 + a^2b - ab^2$
6.  $x^2 + 12x - 28$  and  $x^3 + 9x^2 + 27x - 98$
7.  $2x^3 + 5 - 8x + x^2$  and  $42x^3 + 30 - 72x$
8.  $x^2 - 3x - 70$  and  $x^3 - 39x + 70$
9.  $x^3 - 2x - 1$  and  $x^3 + 2x^2 + 2x + 1$
10.  $4x^3 - 5xy + y^2$  and  $3x^3 - 3x^2y + xy^2 - y^3$
11.  $2x^5 - 16x + 6$  and  $5x^6 + 15x^5 + 5x + 15$
12.  $x^4 - 2x^3y + 2xy^3 - y^4$  and  
 $x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4$

171. If  $x$  is the L. C. M. of  $A$  and  $B$ ,

and  $y$  is the L. C. M. of  $x$  and  $C$ ,

then  $x$  = the lowest quantity that contains  $A$  and  $B$ ,

and  $y$  = " " "  $x$  and  $C$ .

$\therefore y$  = " " "  $A, B$ , and  $C$ ,

or,  $y$  = L. C. M. of  $A, B$ , and  $C$ . Therefore,

**Prin. 3.**—*The lowest common multiple of three quantities may be obtained by finding the lowest common multiple of two of them, then the lowest common multiple of that and the third quantity, etc.*

Find the L. C. M. of :

13.  $x^3 + 1$ ,  $x^3 + 2x^2 + 2x + 1$ , and  
 $x^4 - 2x^3 + 3x^2 - 2x + 1$
14.  $2x^3 + 7x^2 + 2x - 3$ ,  $6x^3 + 7x^2 - x - 2$ , and  
 $8x^3 + 6x^2 - 3x - 1$

15.  $x^2 + 5x + 6$ ,  $x^2 + 6x + 8$ , and  $x^2 - 3x - 10$
16.  $3x^3 + 2x^2 - 5x - 4$ ,  $3x^3 - 10x^2 + 11x - 4$ , and  
 $3x^5 - 4x^4 - 6x^3 + 8x^2 + 3x - 4$
17.  $x^3 + 4x^2y + 4xy^2 + 3y^3$ ,  $x^3 - 4x^2y + 4xy^2 - 3y^3$ ,  
and  $x^5 - 8x^4y^2 - 8x^3y^4 - 9y^5$
18.  $2x^3 - x^2 - 2x + 1$ ,  $2x^4 - 3x^3 + 3x^2 - 3x + 1$ ,  
and  $2x^5 - 3x^4 + x^3 + 2x^2 - 3x + 1$
19.  $a^3 - 7a^2 + 16a - 12$ ,  $3a^3 - 14a^2 + 16a$ , and  
 $5a^3 - 10a^2 + 7a - 14$
20.  $x^3 - 5x^2 + 11x - 15$ ,  $x^3 - x^2 + 3x + 5$ , and  
 $2x^3 - 7x^2 + 16x - 15$

21. The H. C. D. of two quantities is  $x^2 - xy + y^2$  and their L. C. M. is  $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ . One of the quantities is  $x^3 + y^3$ , what is the other?

22. Show that the H. C. D. of two quantities is the L. C. M. of all their common divisors.

23. The sum of two polynomials is  $2x^3 + 12x^2 + 19x + 3$ , their H. C. D. is  $x + 3$ , and the difference of the other two factors is  $10x + 31$ . Find the polynomials.

24. Find the lowest polynomial which, when divided by  $x^2 + 5x + 6$ ,  $x^2 + 7x + 12$ , or  $x^2 + 9x + 20$ , gives in each case a remainder of  $4x^2 - 150x - 100$ .

25. Show that the product of the H. C. D. and L. C. M. of two quantities equals the product of the quantities.

26. The product of the H. C. D. and L. C. M. of two quantities is  $x^5 + 2x^5y + 2x^4y^2 - 2x^2y^4 - 2xy^5 - y^5$ . One of the quantities is  $x^3 - y^3$ , what is the other?

27. The quotient of two quantities is  $(a + x) \div (a - x)$ , and their L. C. M. is  $a^4 - x^4$ . Required the quantities.

28. The product of two polynomials is  $x^4 + 12x^3 + 51x^2 + 92x + 60$ , their L. C. M. is  $x^3 + 10x^2 + 31x + 30$ , and their unlike factors are  $x + 3$  and  $x + 5$ . What are the polynomials?

## CHAPTER II.

### ALGEBRAIC FRACTIONS.

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#### Preliminary Definitions.

**172.** An algebraic fraction, in the most general sense, is an expression denoting that one algebraic quantity is to be divided by another. The dividend, called the *Numerator*, is written above, and the divisor, called the *Denominator*, below, a horizontal line.

Thus,  $\frac{a-b}{c+d}$ , read  $a-b$  divided by  $c+d$ , in which  $a$ ,  $b$ ,  $c$ , and  $d$  may have any values, positive or negative, integral or fractional, rational or irrational, is an algebraic fraction.

**173.** The numerator and denominator are called the *terms* of the fraction.

**174.** An algebraic fraction is usually preceded by the positive or negative sign, to denote whether it is to be taken *additively* or *subtractively*.

**175.** The *value* of a fraction is the result of performing all the operations indicated.

Thus, the value of  $-\frac{x-y}{x+y}$  when  $x = -4$  and  $y = +6$  is  $-\frac{-4-(+6)}{-4+(+6)} = -\frac{-4-6}{-4+6} = -\frac{-10}{+2} = -(-5) = +5$ .

**176.** The *apparent* sign of a fraction is the sign preceding it; the *real* sign the sign of its value.

**177.** An *Integral Quantity*, in the literal notation, is a quantity that has the form of an entire quantity. In value it may be integral or fractional; as,  $a = 8$ , or  $a = \frac{7}{8}$ .

**178.** A *Mixed Quantity* is one that is partly integral and partly fractional in form; as,  $x \pm \frac{y}{z}$ .

**179.** A *Proper Fraction* is one that can not be reduced to the integral or mixed form; as,  $\frac{x+y}{z}$ .

**180.** An *Improper Fraction* is one that can be reduced to the integral or mixed form; as,  $\frac{x^2+y^2}{x} = x + \frac{y^2}{x}$ .

**181.** In a very limited sense, in which the terms are restricted to arithmetical integers, an algebraic fraction may denote "a number of equal parts of a unit."

Thus, if  $a = 3$  and  $b = 5$ ,  $\frac{a}{b}$  may denote  $a$  of the  $b$  equal parts of a unit, and may be read  $a$  bth. But as  $a$  of the  $b$  equal parts of a unit is equivalent to one of the  $b$  equal parts of  $a$  units,  $\frac{a}{b}$  in the more restricted sense may still be read  $a$  divided by  $b$ .

**182.** A *Compound Fraction* is a fractional part of an integral or fractional quantity; as,  $\frac{a}{b}$  of  $c$  (read  $a$  bth of  $c$ ), or  $\frac{a}{b}$  of  $\frac{c}{d}$  (read  $a$  bth of  $c$  divided by  $d$ ).

**183.** A *Complex Fraction* is a fraction one or both of whose terms are fractional in form.

**184.** The *inverse of a fraction* is the fraction resulting from an interchange of its terms.  $\frac{b}{a}$  is the inverse of  $\frac{a}{b}$ .

**185.** The *reciprocal* of a quantity is unity divided by the quantity.

## Reduction of Fractions.

## Definition.

**186.** Reduction is the process of changing the form of an expression without altering its value.

## Principles.

**187.** Let  $\frac{x}{y}$  be any fraction,  $v$  its value, and  $n$  any quantity, then

$$\frac{x}{y} = v \quad (1)$$

Now, since the dividend equals the product of the divisor and quotient,

$$x = v y \quad (2)$$

and  $nx = nv y$  [Ax. 4];

whence  $\frac{nx}{y} = nv$  [Ax. 5]. Therefore,

**Prtn. 1.**—*Multiplying the numerator multiplies the value of a fraction.*

---

**188.** Let  $\frac{x}{y} = v$

then  $x = v y$

and  $x = nv \times y \div n$  [144, P. 1];

whence  $\frac{x}{y \div n} = nv$  [Ax. 5]. Therefore,

**Prtn. 2.**—*Dividing the denominator multiplies the value of a fraction.*

---

**189.** Let  $\frac{x}{y} = v$

then  $x = v y$

and  $x \div n = (v \div n) \times y$  [Ax. 5], [98, P. 1];

whence  $\frac{x \div n}{y} = v \div n$  [Ax. 5]. Therefore,

**Prtn. 3.** *Dividing the numerator divides the value of a fraction.*

190. Let  $\frac{x}{y} = v$

then  $x = vy$

and  $x = (v \div n) \times ny$  [144, P. 1];

whence  $\frac{x}{ny} = v \div n$  [Ax. 5]. Therefore,

**Prin. 4.**—*Multiplying the denominator divides the value of a fraction.*

---

191. Let  $\frac{x}{y} = v$

then  $\frac{nx}{y} = nv$  [P. 1],

and  $\frac{nx}{ny} = nv \div n = v$  [P. 4].

$\therefore \frac{x}{y} = \frac{nx}{ny}$  [Ax. 1]. Therefore,

**Prin. 5.**—*Multiplying both terms of a fraction by the same quantity does not alter its value.*

---

192. Let  $\frac{x}{y} = v$

then  $\frac{x \div n}{y} = v \div n$  [P. 3],

and  $\frac{x \div n}{y \div n} = (v \div n) \times n = v$  [P. 2]. Therefore,

**Prin. 6.**—*Dividing both terms of a fraction by the same quantity does not alter its value.*

---

193. Multiplying both terms of a fraction by  $-1$  changes the signs of both terms, but does not alter the value of a fraction [P. 5]; therefore,

**Prin. 7.**—*Changing the signs of both terms of a fraction does not alter the value of a fraction.*

**Illustration.**—Thus,  $\frac{a-x}{a-y} = \frac{(a-x) \times (-1)}{(a-y) \times (-1)} = \frac{x-a}{y-a}$ .

194. Let  $\frac{a-x}{b-y} = v$

then  $a-x = v(b-y)$

and  $-1(a-x) = -1 \times v(b-y)$  [Ax. 4],

or  $-(a-x) = -v(b-y)$

and  $\frac{-(a-x)}{b-y} = -v$  [Ax. 5],

whence  $-\frac{(a-x)}{b-y} = v$ .

$\therefore \frac{a-x}{b-y} = -\frac{(a-x)}{b-y}$ .

In a similar manner it may be shown that :

$$\frac{a-x}{b-y} = -\frac{a-x}{-(b-y)}; \quad -\frac{a-x}{b-y} = +\frac{-(a-x)}{b-y}; \quad \text{and}$$

$$-\frac{a-x}{b-y} = \frac{a-x}{-(b-y)}. \quad \text{Therefore,}$$

**Prin. 8.**—*Changing the apparent sign and the sign of either term of a fraction does not change the value of the fraction.*

### Problems.

#### 1. To reduce a fraction to its lowest terms.

**195. Definition.**—A fraction is in its lowest terms when the numerator and denominator are prime to each other.

**196.** Since a fraction is in its lowest terms when the numerator and denominator are prime to each other [195], and two quantities are prime to each other when they have no common factor [150], and the highest common divisor of two quantities is the product of all their common factors [153, P. 1], it follows that, if both terms of a fraction be divided by their highest common divisor, the fraction will be reduced to its lowest terms, and its value will not be changed [192, P. 6]. Therefore,

**Rule.**—Divide both terms of the fraction by their highest common divisor.

**Illustration.**—Reduce  $\frac{a^2 + 2ab + b^2}{a^2 - b^2}$  to its lowest terms.

$$\text{Solution: } \frac{a^2 + 2ab + b^2}{a^2 - b^2} = \frac{(a+b)(a+b)}{(a+b)(a-b)} = \frac{a+b}{a-b}.$$

## 2. To reduce a mixed quantity to an improper fraction.

**Illustration.**—

Reduce  $a - x - \frac{a^2 + x^2}{a + x}$  to an improper fraction.

$$\begin{aligned} \text{Solution: } a - x - \frac{a^2 + x^2}{a + x} &= \frac{a - x}{1} - \frac{a^2 + x^2}{a + x} \\ &= \frac{a^2 - x^2}{a + x} - \frac{a^2 + x^2}{a + x} = \frac{1}{a + x} (a^2 - x^2 - a^2 - x^2) \text{ [140, P. 2]} \\ &= \frac{1}{a + x} (a^2 - x^2 - a^2 - x^2) = \frac{1}{a + x} (-2x^2) = \frac{-2x^2}{a + x} \\ &= -\frac{2x^2}{a + x} \text{ [194, P. 8].} \end{aligned}$$

## 3. To reduce improper fractions to integral or mixed quantities.

**Illustration.**—

Reduce  $\frac{a^2 + ab - 4b^2}{a - b}$  to a mixed quantity.

$$\begin{aligned} \text{Form.} \\ \frac{a^2 + ab - 4b^2}{a - b} &= (a^2 + ab - 4b^2) \div (a - b) = \\ a + 2b + \frac{-2b^2}{a - b} &= a + 2b - \frac{2b^2}{a - b} \text{ [194, P. 8].} \end{aligned}$$

**Solution:** Since a fraction indicates division, and the numerator is partly divisible by the denominator, we perform the division and obtain a quotient of  $a + 2b$  and a remainder of  $-2b^2$ . As  $-2b^2$  is not divisible by  $a - b$ , we simply indicate the division and add the result to  $a + 2b$ , then change the apparent sign and the sign of the numerator of the fractional part.

## 4. To reduce fractions to similar forms.

**197. Definition.**—Fractions are *similar* when they have a common denominator.

Let  $\frac{a}{b}$ ,  $\frac{c}{d}$ , and  $\frac{e}{f}$  be any three fractions.

Suppose  $m$  any common multiple of  $b$ ,  $d$ , and  $f$ , and let  $m = b \times x = d \times y = f \times z$ .

$$\text{Now, } \frac{a}{b} = \frac{a \times x}{b \times x} [191, \text{P. 5}] = \frac{ax}{m},$$

$$\frac{c}{d} = \frac{c \times y}{d \times y} [191, \text{P. 5}] = \frac{cy}{m},$$

$$\text{and } \frac{e}{f} = \frac{e \times z}{f \times z} [191, \text{P. 5}] = \frac{ez}{m}.$$

Here we observe that  $m$ , a common multiple of the denominators of the fractions, forms a common denominator of the fractions. Therefore,

**Prin. 1.**—*Any common multiple of the denominators of two or more fractions may be taken as a common denominator of the fractions.*

**198.** If dissimilar fractions are in their lowest terms, it is evident that they must be reduced to higher terms to have a common denominator, and this can only be done by inserting a common factor in both terms of each fraction; hence, every common denominator *must be* a common multiple of the denominators, and the lowest common multiple of the denominators will be the lowest common denominator. Therefore,

**Prin. 2.**—*The lowest common multiple of the denominators of two or more fractions in their lowest terms is their lowest common denominator.*

**Schollum.**—*Fractions not in their lowest terms may sometimes be reduced to similar forms by reducing them to lower terms, in which case their common denominator is not a common multiple of the denominators.* Thus,

$$\frac{a^2-x^2}{(a-x)^2} \text{ and } \frac{a^3+x^3}{a^2-x^2} \text{ are equivalent to } \frac{a+x}{a-x} \text{ and } \frac{a^2-ax+x^2}{a-x}.$$

**Illustrative Example.—**

Reduce  $\frac{a-x}{a+x}$ ,  $\frac{a+x}{a-x}$ , and  $\frac{a^2+x^2}{a^2-x^2}$  to their lowest similar forms.

**Solution:** The L. C. M. of  $a+x$ ,  $a-x$ , and  $a^2-x^2$ , is  $a^2-x^2$ , which is the L. C. D. of the fractions [P. 2].

$$\begin{aligned}\frac{a-x}{a+x} &= \frac{(a-x)(a-x)}{(a+x)(a-x)} = \frac{a^2-2ax+x^2}{a^2-x^2} \\ \frac{a+x}{a-x} &= \frac{(a+x)(a+x)}{(a-x)(a+x)} = \frac{a^2+2ax+x^2}{a^2-x^2}, \\ \text{and } \frac{a^2-x^2}{a^2-x^2} &= \frac{a^2-x^2}{a^2-x^2}.\end{aligned}$$

**Note.**—To determine the factor to be inserted in both terms of any fraction, divide the L. C. D. by the denominator of that fraction.

**EXERCISE 82.**

1. Reduce  $\frac{9x^4y^{2n}z^3}{15x^5y^nz^4}$  and  $\frac{6a^{m+n}b^{r+1}}{9a^{n+1}b^{r-1}}$  to lowest terms.
2. Reduce  $\frac{x^2+5x+6}{x^2+6x+9}$  and  $\frac{x^4+x^2y^2+y^4}{x^3-x^2y+xy^2}$  to lowest terms.
3. Reduce  $\frac{6x^2+13x+6}{10x^2+13x-3}$  and  $\frac{16x^4+4x^2+1}{4x^2+2x+1}$  to lowest terms.
4. Reduce  $2x+7-\frac{3x-5}{4x+3}$  to an improper fraction.
5. Reduce  $x-2+\frac{11x+22}{x^2+7x+10}$  to an improper fraction.
6. Reduce  $\frac{12x^2-5x-6}{3x^2+2}$  to a mixed quantity.
7. Reduce  $\frac{a^3+a^2b+ab^2+b^3}{a^3+ab+b^2}$  to a mixed quantity.
8. Reduce  $\frac{a^3-a^2b+ab^2-b^3}{a^3-b^3}$  to a mixed quantity.
9. Reduce  $\frac{x}{1-x}$ ,  $\frac{y}{1-x^2}$ , and  $\frac{z}{1-x^3}$  to lowest similar forms.

10. Reduce  $\frac{x+y}{x^3-y^3}$ ,  $\frac{x-y}{x^3+y^3}$ , and  $\frac{x^3+y^3}{x^6-y^6}$   
to lowest similar forms.

11. Show that changing the signs of an even number of factors in either term of a fraction does not change the real sign of the fraction.

12. Show that changing the signs of an odd number of factors in either term of a fraction changes the real sign of the fraction.

13. Reduce  $\frac{5}{2-2x^2}$ ,  $\frac{3}{3-3x^2}$ , and  $\frac{2}{3x-3}$   
to lowest similar forms.

14. Reduce  $\frac{3x^3+x^2+6x+2}{3x^3+x^2-9x-3}$  to its lowest terms.

15. Reduce  $\frac{2a^3-4a^2-13a-7}{6a^3-11a^2-37a-20}$  to its lowest terms.

16. Reduce  $\frac{x+y}{(x-y)(z-x)}$ ,  $\frac{x+z}{(y-x)(x-z)}$ , and  $\frac{y+z}{(x-y)(z-y)}$   
to lowest similar forms.

17. Reduce  $x^{2n}+x^ny^n+y^{2n}-\frac{x^{4n}+y^{4n}}{x^{2n}-x^ny^n+y^{2n}}$   
to an improper fraction.

18. Reduce  $\frac{a^nc^n+a^nd^n+b^nc^n+b^nd^n}{a^nc^n-a^nd^n+b^nc^n-b^nd^n}$  to its lowest terms.

## Addition and Subtraction of Fractions.

### Principles.

$$199. \frac{a}{x} + \frac{b}{x} + \frac{c}{x} = \frac{1}{x} (a+b+c) \text{ [140, P. 2]} = \frac{a+b+c}{x}.$$

Therefore,

*Prin. 1.*—The sum of two or more similar fractions equals the sum of their numerators divided by their common denominator.

$$200. \frac{a}{x} + \frac{a}{y} = \frac{ay}{xy} + \frac{ax}{xy} = \frac{1}{xy} (ay + ax) = \frac{a(x+y)}{xy}.$$

Therefore,

**Prin. 2.**—*The sum of two fractions having a common numerator equals the common numerator into the sum of the denominators, divided by the product of the denominators.*

$$201. \frac{a}{x} - \frac{b}{x} = \frac{1}{x} (a - b) = \frac{a - b}{x}. \quad \text{Therefore,}$$

**Prin. 3.**—*The difference of two similar fractions equals the difference of their numerators divided by their common denominator.*

$$202. \frac{a}{x} - \frac{a}{y} = \frac{ay}{xy} - \frac{ax}{xy} = \frac{1}{xy} (ay - ax) = \frac{a(y-x)}{xy}.$$

Therefore,

**Prin. 4.**—*The difference of two fractions having a common numerator equals the common numerator into the difference between the second and first denominators, divided by the product of the denominators.*

**Problem.** To add and subtract fractions.

**Illustrations.**—

$$1. \text{ Find the value of } \frac{a+b}{a-b} + \frac{a-b}{a+b} - \frac{4ab}{a^2-b^2}.$$

**Solution :** L. C. D. =  $a^2 - b^2$ .

$$\begin{aligned} \frac{a+b}{a-b} &= \frac{(a+b)(a+b)}{(a-b)(a+b)} = \frac{a^2 + 2ab + b^2}{a^2 - b^2} \\ \frac{a-b}{a+b} &= \frac{(a-b)(a-b)}{(a+b)(a-b)} = \frac{a^2 - 2ab + b^2}{a^2 - b^2} \\ &\quad - \frac{4ab}{a^2 - b^2} = + \frac{-4ab}{a^2 - b^2} \\ \frac{a^2 + 2ab + b^2}{a^2 - b^2} + \frac{a^2 - 2ab + b^2}{a^2 - b^2} + \frac{-4ab}{a^2 - b^2} &= \frac{2(a^2 - 2ab + b^2)}{a^2 - b^2} = \frac{2(a-b)}{a+b} \\ \text{Therefore, } \frac{a+b}{a-b} + \frac{a-b}{a+b} - \frac{4ab}{a^2 - b^2} &= \frac{2(a-b)}{a+b}. \end{aligned}$$

2. Find the value of

$$\frac{1}{(x-z)(z-y)} + \frac{1}{(x-y)(y-z)} + \frac{1}{(y-x)(z-x)}.$$

Solution :

$$\frac{1}{(x-z)(z-y)} = \frac{-1}{(x-z)(y-z)}$$

$$\frac{1}{(y-x)(z-x)} = \frac{1}{(x-y)(x-z)}$$

$$\therefore \frac{1}{(x-z)(z-y)} + \frac{1}{(x-y)(y-z)} + \frac{1}{(y-x)(z-x)} =$$

$$\frac{-1}{(x-z)(y-z)} + \frac{1}{(x-y)(y-z)} + \frac{1}{(x-y)(x-z)} =$$

$$\frac{-1(x-y)}{(x-y)(x-z)(y-z)} + \frac{x-z}{c.d} + \frac{y-z}{c.d} = \frac{2(y-z)}{(x-y)(x-z)(y-z)} = \frac{2}{(x-y)(x-z)}$$

**Notes.**—1. Sometimes it is better to combine two fractions and the sum with the third, etc.

2. Mixed quantities are most readily added by adding first the integral parts, then the fractions.

#### EXERCISE 88.

Find the value of :

$$1. \frac{7x-5}{3x} + \frac{6x-4}{4x} + \frac{10-3x}{6x}$$

$$2. \frac{a^3 - b^3 + 2ab + ab^2}{ab^2} - \frac{a-b^2}{ab} - \frac{a^2+b^2}{b^2}$$

$$3. \frac{7}{x+2} + \frac{5}{x-3} + \frac{2}{x^2-x-6} \quad 4. \frac{5}{x-1} - \frac{2}{1-x^2}$$

$$5. \frac{1}{x-2} + \frac{2}{x-3} + \frac{3x+7}{x^2-5x+6}$$

$$6. \frac{1}{x+1} - \frac{x-2}{x^2-x+1} - \frac{3}{x^2+1}$$

$$7. \frac{3}{x-1} + \frac{7}{x-3} - \frac{9x-13}{x^2-4x+3}$$

$$8. \frac{5}{x-2} + \frac{6}{x-5} - \frac{4x-2}{x^2-7x+10}$$

$$9. \frac{1}{4(a-1)} - \frac{1}{4(a+1)} - \frac{1}{2(a^2+1)}$$

$$10. \frac{x+2}{x^3-x} - \frac{1}{2(x+1)} - \frac{3}{2(x-1)} + \frac{2}{x}$$

$$11. \frac{1}{1+n} - \frac{1}{1-n} + \frac{1}{1+2n} + \frac{1}{1-2n}$$

$$12. \frac{1}{x} - \frac{2}{x^2} + \frac{3}{x+1} - \frac{4}{(x+1)^2}$$

$$13. \frac{x+y}{x-y} - \frac{x-y}{y+x} + \frac{x^2-y^2}{y^2+x^2} + \frac{x^2+y^2}{y^2-x^2}$$

$$14. \frac{2}{5-5x^2} + \frac{3}{2x-2} - \frac{5}{3x^2-3} + \frac{4}{5x+5}$$

$$15. \frac{a+b}{(c-b)(a-c)} - \frac{b+c}{(b-a)(c-a)} - \frac{c+a}{(a-b)(c-b)}$$

$$16. \frac{x^2-1}{x^4+x^2+1} - \frac{x+1}{x^2-x+1} + \frac{x-1}{x^2+x+1}$$

$$17. \frac{x^2}{(a-b)(c-a)} + \frac{x^2}{(c-b)(b-a)} - \frac{x^2}{(a-c)(c-b)}$$

$$18. \frac{1}{(a-x)(x-b)} - \frac{1}{(c-x)(b-x)} - \frac{1}{(x-a)(c-x)} \\ - \frac{1}{(b-x)(x-c)}$$

$$19. \frac{4x^3}{x^3+x^4+1} + \frac{1}{x^2-x+1} - \frac{2x}{x^4-x^2+1} - \frac{1}{x^2+x+1}$$

$$20. \frac{1}{x^2-1} + \frac{1}{(x+1)^2} + \frac{x(x^2+3)}{(x^2-1)^2} - \frac{x}{(x-1)^2}$$

$$21. \frac{x}{a-b-(a-b)x} + \frac{x}{a+b+(a+b)x}$$

$$22. \frac{a^3-b^3}{a^4-b^4} - \frac{a-b}{a^2-b^2} - \frac{1}{2} \left\{ \frac{a+b}{a^2+b^2} - \frac{1}{a+b} \right\}$$

$$23. \frac{p+q}{p^3-q^3} + \frac{p-q}{p^3+q^3} + \frac{2(q^2-p^2)}{p^4+p^2q^2+q^4}$$

$$24. \frac{1}{2a(c-a)(a-x)} - \frac{1}{(a^2-c^2)(c+x)} + \frac{1}{2a(a+x)(c+a)}$$

## Multiplication and Division of Fractions.

### Definition and Principles.

**203.** To multiply or divide by a fraction is to multiply or divide by the quotient of two quantities.

$$\begin{aligned} 204. \quad \frac{a}{b} \times \frac{c}{d} &= \left( \frac{a}{b} \div d \right) \times \left( \frac{c}{d} \times d \right) [144, \text{P. 1}] = \\ &\frac{a}{b d} \times c [190, \text{P. 4}], [188, \text{P. 2}] = \frac{a c}{b d} [187, \text{P. 1}]. \end{aligned}$$

Similarly,  $\frac{a c}{b d} \times \frac{e}{f} = \frac{a c e}{b d f}$ , etc. Therefore,

**Prin. 1.**—*The product of two or more fractions equals the product of their numerators divided by the product of their denominators.*

**205.** Since dividing the numerator of one fraction divides that fraction, and dividing the denominator of another fraction multiplies that fraction, dividing the numerator of one fraction and the denominator of another fraction by the same quantity does not alter their product [144, P. 1]. Therefore,

**Prin. 2.**—*Canceling a factor common to the numerator of one fraction and the denominator of another does not alter their product.*

$$\begin{aligned} 206. \quad \frac{a}{b} \div \frac{c}{d} &= \left( \frac{a}{b} \times d \right) \div \left( \frac{c}{d} \times d \right) [147, \text{P. 4}] = \\ &\left( \frac{a}{b} \times d \right) \div c [188, \text{P. 2}] = \frac{a}{b} \times \frac{d}{c} [98, \text{P. 1}]. \end{aligned}$$

Therefore,

**Prin. 3.**—*The quotient of two fractions equals the dividend multiplied by the inverse of the divisor.*

**207.** Since dividing the numerators of two fractions by any quantity divides the values of the fractions by that quantity [189, P. 3], and dividing the denominators of two

fractions by any quantity multiplies the values of the fractions by that quantity [188, P. 2], and neither of these operations changes the value of the quotient of the fractions [147, P. 4], it follows that,

**Prin. 4.**—*Canceling a factor common to the numerators or the denominators of two fractions does not alter their quotient.*

### Problems.

1. To find the product of two quantities, one or both of which are fractions.

**Illustrations.**—1. Multiply  $\frac{a-x}{a+b}$  by  $\frac{a+x}{a-b}$ .

$$\text{Solution: } \frac{a-x}{a+b} \times \frac{a+x}{a-b} = \frac{(a-x)(a+x)}{(a+b)(a-b)} [\text{P. 1}] = \frac{a^2-x^2}{a^2-b^2}.$$

2. Multiply  $\frac{a^3-x^3}{a+b}$  by  $\frac{a^2+2ab+b^2}{(a-x)^2}$ .

$$\begin{aligned} \text{Solution: } \frac{a^3-x^3}{a+b} \times \frac{a^2+2ab+b^2}{(a-x)^2} &= \frac{(a+x)\cancel{(a-x)}}{a+b} \times \frac{\cancel{(a+b)}(a+b)}{(a-x)\cancel{(a-x)}} \\ &= \frac{(a+x)}{1} \times \frac{a+b}{a-x} [\text{P. 2}] = \frac{a^2+ab+ax+bx}{a-x} = [\text{P. 1}]. \end{aligned}$$

3. Multiply  $\frac{3x}{2a} + \frac{y}{3b}$  by  $\frac{2x}{3a} - \frac{3y}{2b}$ .

**Form.**

$$\begin{array}{r} \frac{3x}{2a} + \frac{y}{3b} \\ \frac{2x}{3a} - \frac{3y}{2b} \\ \hline \frac{x^2}{a^2} + \frac{2xy}{9ab} \\ \quad - \frac{9xy}{4ab} - \frac{y^2}{2b^2} \\ \hline \frac{x^2}{a^2} - \frac{73xy}{36ab} - \frac{y^2}{2b^2} \end{array}$$

2. To find the quotient of two quantities, one or both of which are fractions.

Illustrations.—1. Divide  $\frac{a+x}{a-x}$  by  $\frac{a-x}{a+x}$ .

$$\begin{aligned}\text{Solution: } \frac{a+x}{a-x} \div \frac{a-x}{a+x} &= \frac{a+x}{a-x} \times \frac{a+x}{a-x} \text{ [P. 8]} = \\ &= \frac{(a+x)^2}{(a-x)^2} \text{ [P. 1]} = \frac{a^2 + 2ax + x^2}{a^2 - 2ax + x^2}.\end{aligned}$$

2. Divide  $\frac{a+x}{a-x}$  by  $a^2 - x^2$ .

$$\begin{aligned}\text{Solution: } \frac{a+x}{a-x} \div \frac{a^2 - x^2}{1} &= \frac{a+x}{a-x} + \frac{a-x}{1} \text{ [P. 4]} = \\ &= \frac{1}{a-x} \times \frac{1}{a-x} \text{ [P. 8]} = \frac{1}{a^2 - 2ax + x^2} \text{ [P. 1]}.\end{aligned}$$

3. Divide  $\frac{x^3}{y^3} - z^3$  by  $\frac{x}{y} - z$ .

$$\text{Solution: } \left(\frac{x^3}{y^3} - z^3\right) \div \left(\frac{x}{y} - z\right) = \frac{x^3}{y^3} + \frac{xz}{y} + z^3 \text{ [194, P. 1]}.$$

4. Find the value of  $\frac{x^3 - y^3}{(a-b)^2} \times \frac{a^2 - b^2}{(x-y)^2} \div \frac{x+y}{x-y}$ .

$$\begin{aligned}\text{Solution: } \frac{x^3 - y^3}{(a-b)^2} \times \frac{a^2 - b^2}{(x-y)^2} \div \frac{x+y}{x-y} &= \\ &= \frac{\cancel{(x-y)}(\cancel{x+y})}{(a-b)(\cancel{a-b})} \times \frac{(a+b)\cancel{(a-b)}}{\cancel{(x-y)}(\cancel{x-y})} \times \frac{\cancel{x-y}}{x+y} = \frac{a+b}{a-b}.\end{aligned}$$

#### EXERCISE 24.

Find the value of :

1.  $\frac{xy}{x^2 - y^2} \times (x+y)$
2.  $\frac{x+y}{x-y} \times \{-(x^2 - y^2)\}$
3.  $(x^3 - y^3) \div \frac{x^2 - y^2}{4xy}$
4.  $\frac{a^3 - b^3}{a^3 + b^3} \div \frac{a^2 - b^2}{2ab}$
5.  $\frac{a^4 - b^4}{x^3 - y^3} \times \frac{x^2 + xy + y^2}{a^2 + b^2}$
6.  $\left(\frac{x}{a} + \frac{y}{b}\right) \left(\frac{a}{x} + \frac{b}{y}\right)$
7.  $\left(\frac{a}{x} + \frac{b}{y}\right)^2$
8.  $\frac{x^3 + y^3}{x^3 - y^3} \div \frac{x^3 - xy + y^3}{x^3 + xy + y^3}$

$$9. \frac{x^2 + 7x + 12}{x^2 - x - 12} \div \frac{x^2 + 6x + 8}{x^2 - 2x - 8}$$

$$10. \left( \frac{x^5}{y^5} - \frac{y^5}{x^5} \right) \div \left( \frac{x}{y} - \frac{y}{x} \right) \quad 11. \frac{x^{12} + y^{12}}{x^{12} - y^{12}} \div \frac{x^4 + y^4}{x^8 - y^8}$$

$$12. \left( \frac{x^6}{y^3} + 1 + \frac{y^3}{x^3} \right) \div \left( \frac{x^3}{y} + 1 + \frac{y}{x^3} \right)$$

$$13. \left( \frac{a+x}{a-x} - \frac{a-x}{a+x} \right) \times (a^2 - x^2)$$

$$14. \frac{x^3 - 3xy + 2y^3}{x^3 + 3xy + 2y^3} \times \frac{x^3 + xy - 2y^3}{x^3 - xy - 2y^3}$$

$$15. \frac{a^4 + 9a^2 + 81}{a^3 - 2a + 4} \times \frac{a^4 + 4a^2 + 16}{a^3 + 3a + 9}$$

$$16. \frac{x^3 - (y-z)^3}{y^3 - (x+z)^3} \div \frac{ay - ax - az}{bx + by + bz}$$

$$17. \left( \frac{x}{x+y} + \frac{y}{x-y} \right) \div \left( \frac{x}{x-y} - \frac{y}{x+y} \right)$$

$$18. \frac{x^3 - y^3}{x^3 + y^3} \times \frac{(x+y)^3}{(x-y)^3} \div \frac{x^3 + xy + y^3}{x^3 - xy + y^3}$$

$$19. \frac{ac + ad + bc + bd}{ax + ay - bx - by} \times \frac{ax + ay + bx + by}{ac + ad - bc - bd}$$

$$20. \left( \frac{x^2}{yz} - \frac{2x}{a} + \frac{xz}{by} + \frac{yz}{a^2} - \frac{z^2}{ab} \right) \div \left( \frac{x}{y} - \frac{z}{a} \right)$$

### Reduction of Complex Fractions.

**208.** A complex fraction may be reduced to a simple fraction by dividing the numerator by the denominator, or by multiplying both terms by the L. C. D. of the partial fractions it contains.

**Illustration.**—Simplify  $\frac{a + \frac{b}{c}}{b + \frac{c}{d}}$ .

**Solution 2:**  $\frac{a + \frac{b}{c}}{b + \frac{c}{d}} = \frac{(a + \frac{b}{c})cd}{(b + \frac{c}{d})cd} = \frac{acd + bd}{bcd + c^2}.$

**EXERCISE 85.**

Reduce to simplest forms :

$$1. \frac{\frac{a}{b} - \frac{c}{d}}{\frac{e}{f} - \frac{g}{h}} \quad 2. \frac{\frac{x+1}{x} + \frac{y+1}{y}}{\frac{1}{x} - \frac{1}{y}} \quad 3. 1 + \frac{1}{x + \frac{1}{x}}$$

$$4. \frac{\frac{x}{x+y} + \frac{y}{x-y}}{\frac{x}{x-y} - \frac{y}{x+y}} \quad 5. \frac{\frac{a+b}{a-b} + \frac{a-b}{a+b}}{\frac{a+b}{a-b} - \frac{a-b}{a+b}}$$

$$6. \frac{\frac{x^2 - y^2}{(x-y)^2}}{x^2 + xy + y^2} \quad 7. \frac{\frac{x-y}{x+y} - \frac{x^3 - y^3}{x^2 + y^2}}{\frac{x+y}{x-y} + \frac{x^2 + y^2}{x^2 - y^2}}$$

$$8. \frac{\frac{a+1}{ab+1} + \frac{ab+a}{ab+1} - 1}{\frac{a+1}{ab+1} - \frac{ab+a}{ab+1} + 1}$$

$$\frac{\frac{1}{a} - \frac{1}{b}}{\frac{(a+b}{a-b} + \frac{a-b}{a+b}) \left( \frac{a^2}{b^2} + \frac{b^2}{a^2} - 2 \right)}$$

$$9. \frac{\frac{\frac{1}{a^2} + \frac{1}{b^2}}{\frac{1}{a^2} - \frac{1}{b^2}} - \frac{\frac{1}{a^2} - \frac{1}{b^2}}{\frac{1}{a^2} + \frac{1}{b^2}}}{\frac{1}{a^2} - \frac{1}{b^2}}$$

### Highest Common Divisor and Lowest Common Multiple of Fractions.

#### Definitions and Principles

209. A *Divisor* of a fraction is a fraction that is contained in the fraction an integral number of times.

Thus,  $\frac{a}{b^2}$  is a divisor of  $\frac{a^2}{b}$ , since  $\frac{a^2}{b} \div \frac{a}{b^2} = \frac{a^2}{b} \times \frac{b^2}{a} = ab$ .

210. A *Multiple* of a fraction is any integral number of times the fraction, and, therefore, contains the fraction an integral number of times.

Thus,  $\frac{a^2}{b}$  is a multiple of  $\frac{a}{b^2}$ , since  $\frac{a^2}{b} \div \frac{a}{b^2} = ab$ .

211. The terms common divisor, common multiple, highest common divisor, and highest common multiple, have the same general meaning when applied to fractions as when applied to integral quantities.

212. Let  $\frac{a}{b}$ ,  $\frac{c}{d}$ , and  $\frac{e}{f}$  be any three fractions in their lowest terms, and  $\frac{x}{y}$  any common divisor of them in its lowest terms; then

$$\frac{a}{b} \div \frac{x}{y}, \text{ or } \frac{a}{b} \times \frac{y}{x} = \text{an integer,}$$

$$\frac{c}{d} \div \frac{x}{y}, \text{ or } \frac{c}{d} \times \frac{y}{x} = \text{an integer,}$$

$$\text{and } \frac{e}{f} \div \frac{x}{y}, \text{ or } \frac{e}{f} \times \frac{y}{x} = \text{an integer.}$$

Since all the fractions are in their lowest terms, it is evident that the results can be integral only, if  $x$  is a divisor of  $a$ ,  $c$ , and  $e$ , and  $y$  is a multiple of  $b$ ,  $d$ , and  $f$ ; therefore,

**Prin. 1.**—Any common divisor of two or more fractions in their lowest terms is a fraction whose numerator is a common divisor of their numerators, and whose denominator is a common multiple of their denominators.

213. If  $\frac{x}{y}$  is to be the H. C. D. of the fractions, it is evident that  $x$  must be taken of as high a degree as possible, and  $y$  of as low a degree as possible; therefore,

**Prin. 2.**—The highest common divisor of two or more fractions in their lowest terms equals the highest common divisor of their numerators divided by the lowest common multiple of their denominators.

214. Let  $\frac{a}{b}$ ,  $\frac{c}{d}$ , and  $\frac{e}{f}$  be three fractions in their lowest terms, and  $\frac{x}{y}$  be any common multiple of them in its lowest terms; then

$$\frac{x}{y} \div \frac{a}{b}, \text{ or } \frac{x}{y} \times \frac{b}{a} = \text{an integer,}$$

$$\frac{x}{y} \div \frac{c}{d}, \text{ or } \frac{x}{y} \times \frac{d}{c} = \text{an integer,}$$

$$\text{and } \frac{x}{y} \div \frac{e}{f}, \text{ or } \frac{x}{y} \times \frac{f}{e} = \text{an integer.}$$

These integral results are possible only, if  $x$  is a common multiple of  $a$ ,  $c$ , and  $e$ , and  $y$  is a common divisor of  $b$ ,  $d$ , and  $f$ . Therefore,

**Prin. 3.**—Any common multiple of two or more fractions in their lowest terms is a common multiple of their numerators divided by a common divisor of their denominators.

215. If  $\frac{x}{y}$  is to be the L. C. M.,  $x$  must be made of as low a degree as possible, and  $y$  of as high a degree as possible. Therefore,

**Prin. 4.**—The lowest common multiple of two or more fractions in their lowest terms is the lowest common multiple of their numerators divided by the highest common divisor of their denominators.

## EXERCISE 86.

Find the H. C. D. and L. C. M. of :

1.  $\frac{a^2}{b^2}$ ,  $\frac{a^3}{b^3}$ , and  $\frac{a^4}{b^4}$     2.  $\frac{(a-x)^2}{(a+x)^2}$ ,  $\frac{(a-x)^3}{(a+x)^3}$ , and  $\frac{(a-x)^4}{(a+x)^4}$

3.  $\frac{x^2 - a^2}{x + y}$ ,  $\frac{x^2 - 2ax + a^2}{x^2 + y^2}$ , and  $\frac{x^3 - a^3}{x^2 + 2xy + y^2}$

4.  $\frac{ac + ad + bc + bd}{ac - ad - bc + bd}$ ,  $\frac{a^3 + b^3}{a^3 - b^3}$ , and  $\frac{a^2x^3 + 2abx^2 + b^2x^2}{am - bm}$

5. The product of the H. C. D. and L. C. M. of two fractions is  $\frac{(a-b)^3}{(x-y)^3}$ , and one of them is  $\frac{a-b}{x-y}$ . What is the other?

6. The L. C. M. of two fractions is  $\frac{a}{b}$ , and their H. C. D. is  $\frac{c}{d}$ ; one of the fractions is  $\frac{m}{n}$ . What is the other fraction?

7. The sum of two numbers is  $a$ , their H. C. D. is  $c$ , the difference between the other two factors is  $d$ . What are the numbers?

## Involution and Evolution of Fractions.

## Principles.

$$216. \left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \dots \text{ to } n \text{ factors} = \frac{a^n}{b^n}.$$

Therefore,

**Prin. 1.**—Any power of a fraction equals that power of the numerator divided by that power of the denominator. Or,

**Prin. 2.**—Any power of the quotient of two quantities equals the quotient of the like powers of the quantities.

$$217. \text{ Since } \left(\frac{a}{b}\right)^n = \frac{(a)^n}{(b)^n} = \frac{a^n}{b^n}, \quad \sqrt[n]{\frac{a^n}{b^n}} = \frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}} = \frac{a}{b}.$$

Therefore,

**Prin. 3.**—Any root of a fraction equals that root of the numerator divided by that root of the denominator. Or,

**Prin. 4.**—Any root of the quotient of two quantities equals the quotient of the like roots of the quantities.

218. Any quantity that may be obtained by raising some other quantity to a power of a given degree is a *perfect power* of that degree.

219. A quantity that can not be obtained by raising some other quantity to a power of a given degree is an *imperfect power* of that degree.

#### EXERCISE 87.

Find the value of :

$$1. \left(-\frac{a^2 b^3}{c^2 d^4}\right)^4 \quad 2. \left(\frac{3 a^2 b^2 c^3}{4 d^2 e^3 f}\right)^4 \quad 3. \left(\frac{a x - b x}{a y + b y}\right)^2$$

$$4. \left(\frac{a}{x} + \frac{x}{a}\right)^2 \quad 5. \left(\frac{3 x^2}{5 y} - \frac{5 y}{3 x^2}\right)^2$$

$$6. \left(\frac{x}{y} + \frac{y}{x} + 2\right)^2 \quad 7. \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^2$$

$$8. \left(x^2 - 1 + \frac{1}{x^2}\right)^2 \quad 9. \left(\frac{a x + b y}{a x - b y}\right)^2 \times \left(\frac{a x - b y}{a x + b y}\right)^2$$

$$10. \sqrt[3]{\frac{1728}{4096}}$$

$$11. \sqrt{\frac{553536}{5764801}}$$

$$12. \sqrt[3]{\frac{127263527}{403583419}}$$

$$13. \sqrt[6]{\frac{2985984}{11390625}}$$

$$14. \sqrt[3]{-\frac{8a^6b^9}{27x^{12}y^{16}}}$$

$$15. \sqrt[3]{\frac{(a+x)^6(a-x)^3}{(x+y)^9}}$$

$$16. \sqrt[4]{\frac{a^{2n}b^{3n}c^{4n}}{x^{5n}y^nz^{4n}}}$$

$$17. \sqrt[3]{a^3 + 3a^2 - 5 + \frac{3}{a^2} - \frac{1}{a^3}}$$

$$18. \sqrt{\frac{(x+2)(x+5)(x^2+7x+10)}{(x-2)(x-5)(x^2-7x+10)}}$$

$$19. \sqrt{\frac{m^3}{n^3} + \frac{n^3}{m^3} + \frac{2m^3}{n} - 2n + m^2 - 2}$$

$$20. \sqrt[4]{\frac{a^4}{b^4} + \frac{4a^2}{b^2} + \frac{b^4}{a^4} + \frac{4b^2}{a^2} + 6}$$

$$21. \sqrt[3]{1 - \frac{1}{x} - \frac{1}{x^2}} \text{ to four terms.}$$

### Evolution of Decimals.

**220.** *The square of a tenth is a hundredth, the square of a hundredth is a ten-thousandth, the square of a thousandth is a millionth, etc. Therefore,*

**221.** *The square root of a hundredth is a tenth, the square root of a ten-thousandth is a hundredth, the square root of a millionth is a thousandth, etc.*

**Note.**—The square decimal units below *one* are the *hundredth*, the *ten-thousandth*, the *millionth*, the *hundredth-millionth*, etc.

**222.** *The cube of a tenth is a thousandth, the cube of a hundredth is a millionth, the cube of a thousandth is a billionth, etc. Therefore,*

**223.** *The cube root of a thousandth is a tenth, the cube root of a millionth is a hundredth, the cube root of a billionth is a thousandth, etc.*

**Note.**—The cubic decimal units below *one* are the *thousandth*, the *millionth*, the *billionth*, the *trillionth*, etc.

**Illustrations.**—1. Extract the square root of  $\cdot 015376$ .

**Solution:**  $\sqrt{\cdot 015376} = \sqrt{15376 \times 1 \text{ millionth}} = 124 \times 1 \text{ thousandth} = \cdot 124$ .

2. Extract the cube root of  $\cdot 015625$ .

**Solution:**  $\sqrt[3]{\cdot 015625} = \sqrt[3]{15625 \times 1 \text{ millionth}} = 25 \times 1 \text{ hundredth} = \cdot 25$ .

3. Extract the square root of  $\cdot 3$  to within a thousandth.

**Solution:** Since the square of a thousandth is a millionth, we reduce  $\cdot 3$  to millionths.  $\sqrt{\cdot 3} = \sqrt{300000} = \sqrt{300000 \times 1 \text{ millionth}} = (547+) \times 1 \text{ thousandth, or } (548-) \times 1 \text{ thousandth} = \cdot 547+ \text{ or } \cdot 548-.$

4. Extract the cube root of  $\frac{7}{8}$  to within a hundredth.

**Solution:** Since the cube of a hundredth is a millionth, we reduce  $\frac{7}{8}$  to millionths.  $\sqrt[3]{\frac{7}{8}} = \sqrt[3]{875000} = \sqrt[3]{875000 \times 1 \text{ millionth}} = 95+ \times 1 \text{ hundredth or } 96- \times 1 \text{ hundredth} = \cdot 95+ \text{ or } \cdot 96-.$

**Note.**—If a decimal contains 2, 4, 6, 8, etc., figures, the unit of its denomination is a perfect square. If it contains 3, 6, 9, 12, etc., figures, the unit of its denomination is a perfect cube.

#### EXERCISE 38.

1. Extract the square root of  $\cdot 0576$ ,  $5\cdot 6644$ ,  $1190\cdot 25$ .
2. Extract the cube root of  $\cdot 000008$ ,  $300\cdot 763$ ,  $66\cdot 923416$ .
3. Extract the square root of  $\cdot 2$ ,  $\cdot 35$ ,  $5$ ,  $\cdot 4$ ,  $1\cdot 06$  to thousandths.
4. Extract the cube root of  $\cdot 8$ ,  $\cdot 08$ ,  $\cdot 3$ ,  $5$ ,  $7$  to thousandths.
5. Find the value of  $\sqrt[4]{2}$  and  $\sqrt[4]{3}$  to within a hundredth.
6. Find the value of  $\sqrt[4]{\frac{2}{9}}$  to within a thousandth.

**224.** *If  $r$  figures of a square root have been obtained by the usual method,  $r - 1$  more figures may be obtained by simply dividing the remainder by the next trial divisor.*

This may be proved as follows: Let it be required to extract the square root of  $4023456789$ .

**Solution :** We find the first three figures of the root by the usual method. They represent 63400. The remainder is 8896789. Let  $n$  represent the number expressed by the next two figures of the root. Now,  $(63400 + n)^2 = 63400^2 + 2 \times 63400 \times n + n^2 = 4023456789$ . The square of 63400 has already been taken from the number; hence,

$8896789 = 2 \times 63400 \times n + n^2$ ; whence  $\frac{8896789}{126800} = n + \frac{n^2}{126800}$ . Since  $n$  contains only two figures, its square can not contain more than four figures, hence  $\frac{n^2}{126800}$  is less than  $\frac{1}{10}$ . Therefore, if we divide 8896789 by 126800, the quotient will be the value of  $n$  to within  $\frac{1}{10}$ .

**Scholium.**—It must not be inferred that the true remainder is 92789, for we have taken from the number only  $(63400)^2 + 2 \times 63400 \times 30$ . The true remainder is  $92789 - 30^2 = 91889$ .

With 918890000 as a new dividend, and  $2 \times 63430$  as a new divisor, the next four figures of the root may be found, etc.

**General Demonstration.**—Let  $N$  represent the number whose square root is desired to  $2r - 1$  places. Let  $a$  equal the root to  $r$  places, and  $x$  the value of the remaining  $r - 1$  figures of the root.

$$\begin{aligned} \text{Then } \sqrt{N} &= a + x, \text{ nearly,} \\ \text{and } N &= a^2 + 2ax + x^2; \\ \text{whence, } N - a^2 &= 2ax + x^2, \\ \text{and } \frac{N - a^2}{2a} &= x + \frac{x^2}{2a}. \end{aligned}$$

Now, since  $x$  contains  $r - 1$  figures,  $x^2$  can not contain more than  $2r - 2$  figures; and, since  $a$  contains  $2r - 1$  figures,  $\frac{x^2}{a} < 1$ , and  $\frac{x^2}{2a} < \frac{1}{2}$ , which is, therefore, above the limit of error if  $\frac{x^2}{2a}$  is omitted. But  $N - a^2$  is the remainder after  $r$  figures of the root have been found, and  $2a$  is the next trial divisor. Therefore, the principle is established.

7. Find  $\sqrt{7}$ ,  $\sqrt{10}$ , and  $\sqrt[4]{34}$  to 5 places.

		Form.
		40'23'45'67'89 (63430
		36
123...	423	
	369	
1264..	5445	
	5056	
1268..	38967'89	
	3804	
	92789	

**225.** If  $r$  figures of a cube root have been obtained by the usual method,  $r - 2$  more figures may be obtained by simply dividing the remainder by the next trial divisor.

Let it be required to extract the cube root of

2324376432543476.

Form.

	2'3 2'4'3 7'6'4'3 2'5'4'3'4'7'6 (132465
	1
3 . . . . .	1324
9 . . . . .	
9 . . . . .	
399 . . . . .	1197
507 . . . . .	127376
78 . . . . .	
4 . . . . .	
51484 . . . . .	102968
52272 . . . . .	24408432
1584 . . . . .	
16 . . . . .	
5243056 . . . .	20972224
5258928' . . . .	) 343620854'3476 (65
	31553568
	28085174
	26294640

**Solution:** We find the first four figures of the root by the usual method. They represent 132400. The remainder is 3436208543476. Let  $n$  represent the number expressed by the next two figures of the root. Now,  $(132400 + n)^3 = 132400^3 + 3 \times 132400^2 \times n + 3 \times 132400 \times n^2 + n^3 = 2324376432543476$ . The cube of 132400 has already been taken from the number, hence the remainder  $3436208543476 = 3 \times 132400^2 \times n + 3 \times 132400 \times n^2 + n^3$ ; whence,  $\frac{3436208543476}{3 \times 132400^2} = n + \frac{n^2}{132400} + \frac{n^3}{3 \times 132400^2}$ . Since  $n$  contains only two figures,  $n^2$  can not contain more than four figures, and  $n^3$  can not contain more than six figures. Therefore,  $\frac{n^2}{132400}$  is less than  $\frac{1}{10}$ , and  $\frac{n^3}{3 \times 132400^2}$  is less

than  $\frac{1}{10}$ , and their sum is less than  $\frac{1}{5}$ ; hence  $\frac{849620854376}{3 \times 132400^3}$  equals  $n$  to within  $\frac{1}{5}$ .

**General Demonstration.**—Let  $N$  represent the number whose cube root is desired to  $2r-2$  places. Let  $a$  equal the root to  $r$  places, and  $x$  the remaining part of the root, consisting of  $r-2$  places.

Then  $\sqrt[3]{N} = a + x$ , nearly,

and  $N = a^3 + 3a^2x + 3ax^2 + x^3$ ;

whence,  $N - a^3 = 3a^2x + 3ax^2 + x^3$ ,

and  $\frac{N - a^3}{3a^2} = x + \frac{x^2}{a} + \frac{x^3}{3a^2}$ .

Since  $x$  contains  $r-2$  figures,  $x^3$  can not contain more than  $2r-4$  figures; and, since  $a$  contains  $2r-2$  figures,  $\frac{x^2}{a} < \frac{1}{10}$ .

Again,  $x^3$  can not contain more than  $3r-6$  figures, and  $a^2$  can not contain less than  $4r-3$  figures; therefore,  $\frac{x^2}{a} < \frac{1}{10}$ , and  $\frac{x^3}{3a^2} < \frac{1}{30}$ . Therefore,  $\frac{x^2}{a} + \frac{x^3}{3a^2} < \frac{1}{2}$ , which is therefore above the limit of error, if  $\frac{x^2}{a} + \frac{x^3}{3a^2}$  is omitted. But  $N - a^3$  is the remainder after  $r$  figures of the root have been found, and  $3a^2$  is the next trial divisor. Therefore, the principle is established.

8. Find  $\sqrt[3]{9}$  and  $\sqrt[3]{99}$ .

9. Find  $\sqrt{03}$ ,  $\sqrt{\frac{5}{9}}$ , and  $\sqrt{\frac{3}{37}}$  to 7 places.

10. Extract the cube root of 2, 10, and  $\frac{2}{99}$  to six figures.

### Miscellaneous Examples.

#### EXERCISE 39.

1. Find the value of  $\left(\frac{x}{a+b} + \frac{y}{a-b}\right) \div \left(\frac{x}{a-b} - \frac{y}{a+b}\right)$ ,  
when  $x = 1$ ,  $y = 2$ ,  $a = 3$ ,  $b = 4$

2. Find the value of  $\left(\frac{2}{x^2} - \frac{1}{xy} - \frac{3}{y^2}\right) \div \left(\frac{3}{x^2} + \frac{1}{xy} - \frac{2}{y^2}\right)$ , when  $2y = 3x$

3. Find the value of  $ax + by$ ,

$$\text{when } x = \frac{cm - bn}{am - bs} \text{ and } y = \frac{an - cs}{am - bs}$$

4. Find the value of  $\frac{z+2h}{z-2h} - \frac{z+2k}{2k-z}$ , when  $z = \frac{4hk}{h+k}$

5. Reduce  $\frac{a^2 + (b+c)a + bc}{a^2 + (c+d)a + cd}$  to its lowest terms.

6. Reduce  $\frac{ax^{2m} - bx^{2m+1}}{a^{m+1} - a^m bx}$  to its lowest terms.

7. Simplify  $\frac{a^2 + x^2}{(a-b)(a-c)} + \frac{b^2 + x^2}{(b-a)(b-c)} + \frac{c^2 + x^2}{(c-a)(c-b)}$

8. Simplify  $\frac{bc(a^2 + x^2)}{(a-b)(a-c)} - \frac{ac(b^2 + x^2)}{(b-a)(c-b)} - \frac{ab(c^2 + x^2)}{(c-a)(b-c)}$

9. Write the square of  $\frac{a+x}{a-x} + \frac{a-x}{a+x}$  in three terms.

10. Write the square of  $\frac{x^2}{y+1} - \frac{y+1}{x^2}$  in three terms.

11. Find the value of  $\left(\frac{x}{y} + \frac{y}{x}\right)^3 + \left(\frac{x}{y} - \frac{y}{x}\right)^3$

12. Find the value of  $\left(\frac{x+y}{x-y} + \frac{x-y}{x+y}\right) \left(\frac{x+y}{x-y} - \frac{x-y}{x+y}\right)$

13. Find the value of  $\left(\frac{x+1}{y} - 1\right) \left(\frac{x+1}{y} + 1\right)$

14. Find the value of  $\left(\frac{x^2}{y^2} + 7\right) \left(\frac{x^2}{y^2} - 3\right)$

15. Find the value of  $\left(\frac{a+x}{ax} + \frac{b}{3}\right) \left(\frac{a+x}{ax} + \frac{2b}{3}\right)$

16. Expand  $\left(x^2 + 1 + \frac{1}{x^2}\right)^3$  and  $\left(x^2 - 1 + \frac{1}{x^2}\right)^3$

17. Expand  $\left(x + 1 - \frac{1}{x}\right)^3$  and  $\left(x - 1 + \frac{1}{x}\right)^3$

18. Simplify  $\left(\frac{x}{y} + \frac{y}{x}\right)^4 + \left(\frac{x}{y} - \frac{y}{x}\right)^4$

19. Simplify  $\left(\frac{x^2}{y^2} + 1\right)^5 - \left(\frac{x^2}{y^2} - 1\right)^5$

20. Expand  $\left(1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}\right)^3$

21. Expand  $\left(x^2 + x + \frac{1}{x} + \frac{1}{x^2}\right)^3$

22. Expand  $\left(x - \frac{1}{x} + 1\right)^4$       23. Factor  $\frac{a^2 b}{x} + \frac{a b^2}{y} + \frac{a b c}{z}$

24. Simplify  $\frac{1 - \frac{2ab}{(a+b)^2}}{1 + \frac{2ab}{(a-b)^2}} \times \left(\frac{1 + \frac{b}{a}}{1 - \frac{b}{a}}\right)^3$

25. Simplify  $\frac{\frac{(2x+1)^3 + (2x-1)^3}{(1-4x^2) - (1-2x)^2}}{\frac{(1+2x)^2 + (4x^2-1)}{(1-2x)^2 - (2x+1)^2}}$

26. Factor  $\frac{a^2}{b^2} + \frac{4a}{b} + 4$  and  $\frac{4a^3}{b^3} - \frac{12a}{b} + 9$

27. Factor  $\frac{a^2}{x^2} - \frac{x^2}{a^2}$  and  $\frac{(a+x)^2}{(a-x)^2} - \frac{(a-x)^2}{(a+x)^2}$

28. Factor  $a^4 - \frac{x^4}{a^4}$  and  $\frac{x^3}{y^3} - a^3$

29. Factor  $\frac{a^3}{b^3} + 1$  and  $\frac{a^3}{b^3} - 1$

30. Factor  $\left(\frac{x}{y}\right)^5 + z^5$  and  $\left(\frac{x}{y}\right)^5 - z^5$

31. Factor  $x^2 + 1 + \frac{1}{x^2}$  and  $x^3 - 2 + \frac{1}{x^3}$

32. Factor  $\frac{x^2}{y^2} + \frac{5x}{y} + 6$  and  $\frac{4x^2}{y^2} - \frac{4x}{y} - 15$

33. Factor  $a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2$

34. Show that

$$\begin{aligned} & \left(\frac{a+b+c}{2}\right) \left(\frac{a+b-c}{2}\right) \left(\frac{a-b+c}{2}\right) \left(\frac{c-a+b}{2}\right) \\ &= \frac{1}{2} s \left(\frac{1}{2} s - c\right) \left(\frac{1}{2} s - b\right) \left(\frac{1}{2} s - a\right) \end{aligned}$$

when  $s = a + b + c$

35. Extract the square root of  $x^2 - 2x + 3 - \frac{2}{x} + \frac{1}{x^2}$

36. Extract the square root of

$$\frac{a^2}{x^2} + \frac{2ab}{x} + b^2 + \frac{2a}{b} + 2x + \frac{x^2}{b^2}$$

37. Extract the square root of

$$\frac{9a^4}{4} + 15a^2 - \frac{9a}{4b} + 25 - \frac{15}{2ab} + \frac{9}{16a^2b^2}$$

38. Extract the cube root of

$$x^3 + 6x^2 + 15x + 20 + \frac{15}{x} + \frac{6}{x^2} + \frac{1}{x^3}$$

39. Extract the cube root of

$$\begin{aligned} & \frac{8x^3}{y^3} + \frac{8y^3}{x^3} + x^3y^3 + \frac{24x}{y} + \frac{12x^3}{y} + \frac{24y}{x} + \frac{12y^3}{x} \\ &+ 6x^3y + 6xy^3 + 24xy \end{aligned}$$

40. Which is the greater,  $\frac{a}{b}$  or  $\frac{a+m}{b+m}$ , when  $a > b$ ?  
When  $a < b$ ?

41. Which is the greater,  $\frac{a}{b}$  or  $\frac{a-m}{b-m}$ , when  $a > b$ ?  
When  $a < b$ ?

42. What principles are deducible from Examples 40 and 41?

## CHAPTER III.

### *SIMPLE EQUATIONS.*

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#### General Definitions.

**226.** An expression of equality between two equal quantities is an *Equation*.

**Illustration.**—If  $x = 2$ ,  $5x + 4 = 3x + 8$  is an equation.

**227.** The quantities placed equal to each other in an equation are the *members* of the equation.

**228.** An equation between two numerical quantities is an *Arithmetical Equation*; as,  $3 \times 4 = 12$ .

**229.** An equation between two quantities, one or both of which are literal, is an *Algebraic Equation*; as,  $2x + 3 = 5$ , or  $5x + 4 = 3x + 8$ .

**230.** An algebraic equation may have both known and unknown quantities.

**231.** An equation whose known quantities are all numerical is a *Numerical Equation*; as,  $3x - 7 = 4x + 3$ .

**232.** An equation, one or more of whose known quantities are literal, is a *Literal Equation*; as,  $ax = b$ .

**233.** An equation whose members are equal for any value that may be assigned to the unknown quantity is an *Identical Equation*; as,  $3 \times 4x = 12x$ .

**234.** An equation whose members are equal only for particular values of the unknown quantity is an *Equation*

of Condition; as,  $5x - 3 = 2x + 3$ , which is true only when  $x = 2$ .

**235.** Equations may contain one, two, or more unknown quantities.

**236.** The degree of any term of an equation is determined by the number of unknown literal prime factors it contains.

**Illustration.**—In  $axy + bx^2 + ax + by = c$ ,  $axy$  and  $bx^2$  are of the second degree, and  $ax$  and  $by$  of the first degree.

**237.** A term in an equation that does not contain an unknown quantity is an *absolute* term.

**238.** The *degree* of an equation is the degree of its highest term.

**239.** An equation of the first degree is a *simple* equation; one of the second degree a *quadratic* equation; one of the third degree a *cubic* equation; and one of the fourth degree a *bi-quadratic* equation.

**240.** The degree of an equation is determined after the equation is reduced to its *normal form*.

**241.** An equation is reduced to its normal form when its terms are made integral, and are arranged according to the descending powers of the unknown quantity.

**242.** An equation of the first degree between one or two unknown quantities is sometimes called a *Linear Equation*. This is more particularly the case when the unknown quantities are supposed to vary in value.

**243.** Any value of an unknown quantity that will satisfy an equation (that is, make its members equal) is a *root* of the equation.

## Transformation of Equations.

**244.** The process of changing the form of an equation without destroying the equality of its members is *transformation*.

**245.** An equation may be transformed :

1. By adding the same or equal quantities to both members [Ax. 2].
2. By subtracting the same or equal quantities from both members [Ax. 3].
3. By multiplying both members by the same or equal quantities [Ax. 4].
4. By dividing both members by the same or equal quantities [Ax. 5].
5. By raising both members to the same power [Ax. 6].
6. By extracting the same root of both members [Ax. 7].

**246.** If we take the equation  $ax - b = cx + d$ , and add  $b$  to both members and subtract  $cx$  from both members, we obtain  $ax - cx = d + b$ . Comparing this equation with the first, we observe that  $-b$  and  $cx$  have been transposed from one member to the other with their signs changed. Therefore,

*Prin. 1.—Any term of an equation may be transposed from one member to the other if its sign be changed.*

**247.** It is evident that, if both members of an equation containing fractional terms be multiplied by a common multiple of the denominators of those terms, the denominators will disappear by cancellation, and the equality of the members will not be destroyed [Ax. 4]. Therefore,

*Prin. 2.—Any equation containing fractional terms may be cleared of fractions by multiplying both members by any common multiple of the denominators of the fractions.*

## Simple Equations of One Unknown Quantity.

## Definitions and Principles.

**248.** Every simple equation of one unknown quantity may be reduced to the form of  $ax = b$ , in which  $a$  and  $b$  are integral and  $a$  is positive.

For, if it contains fractional terms, it may be cleared of fractions [P. 2]. If it contains unknown terms in the second member, or known ones in the first, they may be transposed [P. 1]. All the terms containing  $x$  may be collected into one term either by addition or by factoring, and all the known terms into one by addition or by inclosing them in parenthesis. If the first member is negative, the equation may be divided by  $-1$  [Ax. 5]. The equation will then be of the form of  $ax = b$ .

**Illustration.**—

Reduce  $\frac{ax}{b} + c = \frac{bx}{a} + \frac{c}{d} + m$  to its simplest form.

**Solution :**

Clearing of fractions by multiplying both members by  $abd$ ,  
 $a^2dx + abcd = b^2dx + abc + abdm$

Transposing  $abcd$  and  $b^2dx$ ,

$$a^2dx - b^2dx = abc + abdm - abcd$$

Factoring the first member, and inclosing the second in parenthesis,  
 $(a^2d - b^2d)x = (abc + abdm - abcd)$

Putting  $a$  for  $a^2d - b^2d$  and  $b$  for  $abc + abdm - abcd$ ,  
 $ax = b$ .

**249.**  $ax = b$  is called the general or normal form of a simple equation of one unknown quantity. If both members be divided by  $a$ ,  $x = \frac{b}{a}$ , which has one value and only one in any particular equation. Therefore,

**Prin. 3.**—Every simple equation of one unknown quantity has one root and only one.

**250.** The process of finding a root of an equation is called *solving* or *reducing* the equation.

**251.** A root of an equation is said to be *verified*, if, when substituted for the unknown quantity, it renders the two members of the equation identical.

### Reduction of Simple Equations.

#### Illustrations.—

1. Solve  $3x - 5 = 4x + 3 + x$ , and verify the result.

**Solution:** Given  $3x - 5 = 4x + 3 + x$  (A)

Transpose,  $3x - 4x - x = 3 + 5$  (1)

Unite terms,  $-2x = 8$  (2)

Divide by  $-2$ ,  $x = -4$

**Verification:** Substitute  $-4$  for  $x$  in equation (A),

$$-12 - 5 = -16 + 3 - 4$$

Unite terms,  $-17 = -17$ .

2. Solve  $\frac{3x}{2} - \frac{5}{3} = \frac{4x}{5} + \frac{7}{2}$ , and verify the result.

**Solution:** Clear of fractions,  $45x - 50 = 24x + 105$  (1)

Transpose,  $45x - 24x = 105 + 50$  (2)

Unite terms,  $21x = 155$  (3)

Divide by 21,  $x = 7\frac{8}{21}$

Verify by substituting  $7\frac{8}{21}$  for  $x$  in original equation,

$$\frac{66\frac{1}{2}}{6} - \frac{10}{6} = \frac{59\frac{1}{2}}{10} + \frac{35}{10}$$

Unite terms,  $9\frac{17}{42} = 9\frac{17}{42}$ .

3. Solve  $2x - \frac{3x-2}{3} = 4 - \frac{x+6}{6}$ .

**Solution:**

Clear of fractions,  $12x - 2(3x - 2) = 24 - (x + 6)$  (1)

Expand,  $12x - 6x + 4 = 24 - x - 6$  (2)

Transpose and unite terms,  $7x = 14$  (3)

$\therefore x = 2$ .

Equation (1) may be omitted if the following principle be heeded:

**252.** If a fractional term is preceded by minus, the sign of every term in the numerator must be changed when the equation is cleared of fractions. Why?

4. Solve  $\frac{4x-1}{3} - \frac{2}{5} = \frac{x-1}{3} + \frac{3}{5}$ .

**Solution :** Transpose,  $\frac{4x-1}{3} - \frac{x-1}{3} = \frac{3}{5} + \frac{2}{5}$

Unite terms,  $x = 1$ . Therefore,

**253.** It is sometimes preferable to transpose and unite terms before clearing of fractions. A good rule is, "Unite terms whenever convenient, and thus avoid long equations."

5. Solve  $\frac{4x+5}{6} + \frac{5}{12} = \frac{2x-3}{3} - \frac{x}{2}$ .

**Solution :** Separate first and third fractions into partial fractions,

$$\frac{2x}{3} + \frac{5}{6} + \frac{5}{12} = \frac{2x}{3} - 1 - \frac{x}{2} \quad (1)$$

Transpose and unite terms,  $\frac{1}{2}x = -\frac{27}{12}$

$\therefore x = -4\frac{1}{2}$ . Therefore,

**254.** Separate fractions with binomial or trinomial numerators into partial fractions whenever the transposition and combination of terms are thereby facilitated.

6. Solve  $\frac{6x+7}{15} - \frac{2x-2}{7x-6} = \frac{2x+1}{5}$ .

**Solution :**

Multiply by 15,  $6x+7 - \frac{30x-30}{7x-6} = 6x+3 \quad (1)$

Transpose and unite terms,  $4 = \frac{30x-30}{7x-6} \quad (2)$

Clear of fractions,  $28x-24 = 30x-30 \quad (3)$

Transpose and unite terms,  $-2x = -6$

$\therefore x = 3$ . Therefore,

**255.** If an equation contains both monomial and binomial denominators, clear it first of monomial denominators and simplify, then of binomial denominators and simplify.

7. Reduce  $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 1$ .

**Solution:** Clear of fractions,

$$bcx + acx + abx = abc \quad (1)$$

Factor,  $(bc + ac + ab)x = abc \quad (2)$

Divide,  $x = \frac{abc}{bc + ac + ab}.$

#### EXERCISE 40.

Solve :

1.  $\frac{1}{2}(x+1) + \frac{1}{3}(x+2) = \frac{1}{4}(x+3)$

2.  $\frac{3x-7}{5} + \frac{2x-6}{3} = \frac{8x+6}{3} - \frac{2x-3}{5}$

3.  $\frac{1}{x+1} + \frac{1}{x-1} = \frac{1}{x^2-1}$

4.  $\frac{3}{x+2} - \frac{5}{x-2} = \frac{6}{x^2-4}$

5.  $\frac{10}{x+4} + \frac{6}{x+5} = \frac{10}{x^2+9x+20}$

6.  $\frac{12}{x+3} - \frac{8}{x-4} = \frac{10}{x^2-x-12}$

7.  $\frac{3}{2x+3} + \frac{5}{3x-2} = \frac{7}{6x^2+5x-6}$

8.  $\frac{5x+6}{9} = \frac{3x-2}{5x+6} + \frac{10x-5}{18}$

9.  $\frac{x}{5} - \frac{3x-7}{2x+4} = \frac{4x+6}{20} + 2\frac{1}{2}$

10.  $\frac{7x-9}{x-3} = \frac{9x+4}{36} - \frac{6x-5}{24}$

11.  $\frac{x}{2x+1} + \frac{1}{3x(2x+3)} = \frac{x+1}{2x+3}$

12.  $8 \div \left\{ \frac{a(a-x)^2}{a^3 - ax^2} + 1 \right\} = 4$

$$13. 3x + \frac{x}{b} = x - \frac{x-a}{c}$$

$$16. \frac{ad-bc}{cd+d^2x} + \frac{bc}{d} = a$$

$$14. \frac{a^2x-x^3}{a} \times \frac{3a}{2ax-2x^3} = a$$

$$17. \frac{a}{c} - \frac{(ad-bc)x}{c(c+dx)} = b$$

$$15. \frac{x^2+(a+b)x+ab}{x^2+(b+c)x+bc} = 3$$

$$18. \frac{x^3-b^2x}{x^2+2bx+b^2} = x$$

$$19. 5x-9 + \frac{13}{2x+3} = \frac{10x^3}{2x+3} - \frac{3\frac{1}{2}x+4\frac{1}{2}}{x+1\frac{1}{2}}$$

$$20. \frac{x}{(x-1)^2} - \frac{1}{(x+1)^2} - \frac{x(x^2+3)}{(x^2-1)^2} = \frac{1}{x^2+x+1}$$

$$21. \frac{1}{x^2-7x+12} + \frac{2}{x^2-4x+3} - \frac{3}{x^2-5x+4} = \frac{1}{4} - x$$

$$22. \frac{x^3-2x+3}{x^3+1} + \frac{x-2}{x^2-x+1} - \frac{1}{x+1} = 0$$

$$23. \frac{x(16-x)}{x^2-4} + \frac{2x+3}{2-x} - \frac{2-3x}{x+2} = 6$$

$$24. \left(x + \frac{2x}{x-a}\right) - \left(x - \frac{2x}{x-a}\right) = a$$

$$25. \frac{3}{x+a} - \frac{4}{x+b} = \frac{5}{x^2+(a+b)x+ab}$$

$$26. \frac{.3}{.01-.02x} - \frac{.7}{.01+.02x} - \frac{.2-x}{\frac{1}{100}(4x^2-1)} = 0$$

$$27. \frac{x-1+\frac{6}{x-6}}{x-2+\frac{3}{x-6}} = 5$$

$$28. \frac{3}{1+\frac{3}{1+\frac{3}{1-x}}} = 10$$

### Examples Involving Simple Equations of One Unknown Quantity.

#### EXERCISE 41.

1. What number is that whose double is as much above 60 as its treble is below 100 ?

**Suggestion.**—Let  $x$  = the number, then will  $2x - 60 = 100 - 3x$ .

2. Divide 140 into two parts that are to each other as 3 to 4.

**Suggestion.**—Let  $3x$  and  $4x$  = the parts, then will  $3x + 4x = 140$ .

3. The sum of two numbers is 48, and  $\frac{2}{3}$  of the greater equals  $\frac{3}{4}$  of the less. Find the numbers.

4. Divide 75 into two parts, such that 3 times the greater exceeds 7 times the less by 15.

5. Divide 92 into four parts, such that the first may exceed the second by 10, the third by 18, and the fourth by 24.

6. Divide 116 into four parts, such that the first increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, shall all be equal.

**Suggestion.**—Let  $x - 5$ ,  $x + 4$ ,  $\frac{x}{3}$ , and  $2x$  = the required parts.

7. A grain-dealer purchased 27,000 bushels of corn, wheat, and oats. If the quantities in the order named were to each other as  $\frac{2}{3}$ ,  $\frac{3}{4}$ , and  $\frac{5}{6}$ , how many bushels of each kind did he buy ?

8. A man gave \$5 to each of a number of poor persons, and had \$100 remaining; had he given each \$8, he would have had only \$10 remaining. How many were there ?

9. Mr. Jones paid \$150 an acre for his farm, and had \$5000 over; had he paid \$180 an acre, he would have lacked \$1000 to pay for it. How many acres were in the farm ?

10. A's age is 4 times B's, and the sum of their ages is 30 years. In how many years will A be only twice as old as B?

11. The sum of A's and B's ages is 50 years. In how many years will the sum be 7 times as great as it was 20 years ago?

12. A man has 5 hours at his disposal; how far may he row down a river at the rate of 5 miles an hour, that he may return at the rate of 3 miles an hour?

13. A steamboat, whose rate in still water is 18 miles an hour, went down a stream whose current is 2 miles an hour, and returned in 6 hours from the time it started. How far did it go down the river?

14. A can do a piece of work in 5 days, and B can do it in 6 days. In what time can they together do it?

15. A can do a piece of work in  $a$  days, B in  $b$  days, and C in  $c$  days. In what time can they together do it?

16. A and B together can do a piece of work in 5 days, and A alone can do it in 8 days. In what time can B alone do it?

17. A cistern can be filled by two pipes in 5 hours and 6 hours respectively, and emptied by a third pipe in 4 hours. In what time will it be filled, if the three pipes run together?

18. A man bought a cow, a colt, and a horse. The cow cost \$40; the colt as much as the cow and half as much as the horse; and the horse  $\frac{3}{2}$  as much as the colt and the cow together. What did he pay for each and for all?

19. A and B own 440 shares of railroad stock, and  $\frac{1}{2}$  of what A owns equals  $\frac{1}{3}$  of what B owns. How many shares must A buy from B, that  $\frac{2}{3}$  of what B owns may equal  $\frac{1}{6}$  of what A owns?

20. What is the time of day, if  $\frac{2}{3}$  of the time to noon equals the times past midnight?

21. A farmer had three fields of corn. On the first grew 200 bushels; on the second as much as on the first plus  $\frac{3}{20}$  as much as on the third; and on the third,  $1\frac{1}{19}$  times as much as on the other two. How much grew on each field, and how much on all?

22. A drover bought a number of cows for \$6250; 50 of them died, after which he sold  $\frac{3}{8}$  of the remainder at cost, and received \$3000. How many did he buy?

23. What time of day is it when  $\frac{2}{3}$  of the time past midnight equals  $\frac{5}{6}$  of the time to midnight?

24. At what time between 10 and 11 o'clock are the hands of a watch—

1. Together? 2. At right angles? 3. Opposite each other?

25. A is 400 rods behind B, and travels 5 rods while B travels 3 rods. How far must each go, that A may be 200 rods ahead of B?

26. A and B start at the same time to walk in opposite directions around a circular pond 1 mile in circumference. A walks 3 rods as often as B walks 4 rods, and they meet in 5 minutes. How many yards per minute does each walk?

27. A fast train, 456 feet long, and a slow train, 600 feet long, pass each other in 12 seconds. What is the rate of each train per hour, if the speed of the fast train is double that of the slow one?

28. If a passenger train 400 feet long pass a freight train 840 feet long, running in the same direction, in  $27\frac{3}{11}$  seconds, what is the rate of each train, if the former runs 4 times as fast as the latter?

29. The distance from P to R is 63 miles. A down train leaves P at 9.30 A. M., and arrives at R at 11.30. An up train leaves R at 9.45 A. M., and arrives at P at 11.30. Where do they meet?

30. The fore-wheel of a wagon is 4 yards in circumference, and the hind-wheel 5 yards. How far must the wagon go that the fore-wheel may make 88 revolutions more than the hind-wheel?

31. A is 60 of his own steps before B, and takes 2 steps while B takes 3; but 5 of A's steps are equal to 3 of B's. How many steps must each take before B overtakes A?

**Suggestion.**—Let  $x$  = the number of steps B takes; then, since A takes 2 while B takes 3,  $\frac{2}{3}x$  = the number A takes. Let  $a$  = the length of one of A's steps, then  $\frac{5}{3}a$  = the length of one of B's steps; whence  $(60 + \frac{2}{3}x)a = \frac{5}{3}ax$ , or  $60 + \frac{2}{3}x = \frac{5}{3}x$ .

32. A and B race on bicycles. A gives B a start of 50 revolutions. A's bicycle makes 3 revolutions while B's makes 4; but 2 revolutions of the first are equal to 3 of the second. In how many revolutions will A overtake B?

33. A is worth \$4000 and B \$6000; A invests his money 5 times while B invests his 4 times, and makes 8% on each investment, while B makes only 5%. How many investments must each make before A is worth as much as B, the gains being saved, but not reinvested?

34. A certain principal will in 5 years at 6 per cent amount to \$100 more than the same principal will amount to in 7 years at 4 per cent. What is the principal?

35. If an article had cost 10% less, the gain would be 15% more. What is the gain per cent?

36. A and B in partnership gain \$1650. A owns  $\frac{2}{5}$  of the stock, plus \$300, and gains \$750. Required the whole stock and share of each.

37. A certain article of consumption is subject to a duty of 6 shillings per cwt.; in consequence of a reduction in the duty, the consumption increases one half, but the revenue falls one third. Find the duty per cwt. after the reduction.

38. In a mixture of silver and copper, consisting of 200 ounces, there are 12 ounces of copper. How much silver must be added that there may be 2 ounces of copper to 50 ounces of silver?

39. It is between 10 and 11 o'clock. In 6 minutes the minute-hand of a clock will be exactly opposite the place where the hour-hand was 3 minutes ago. Find the time.

40. A goes from P to Q, a distance of 21 miles, and immediately returns, traveling 4 miles an hour. B, who travels 3 miles an hour, starts at Q for P at the same time that A starts from P, and returns from P immediately. How far is it between the two points of meeting?

41. A can dig a ditch in one half the time that B can; B can dig it in  $\frac{2}{3}$  of the time that C can; they together can dig it in 6 days. How long will it take each alone to dig it?

### Simultaneous Equations of the First Degree, containing Two or more Unknown Quantities.

#### Definitions and Principles.

256. Every equation of the first degree containing two unknown quantities may be reduced to the form of  $ax + by = c$ , in which  $a$ ,  $b$ , and  $c$  are integral and  $a$  positive. Similarly, one of three unknown quantities may be reduced to the form of  $ax + by + cz = d$ .

(For method of proof, see Art. 248.)

257. Equations that express different relations between unknown quantities are *Independent* equations. Independent equations can not be reduced to the same form.

Thus,  $3x + 2y = 10$  and  $2x + 3y = 5$  are independent equations; but  $x + 2y = 6$  and  $3x + 6y = 18$  are not. Why?

**258.** Equations that are satisfied by the *same values* of the unknown quantities are *Simultaneous* equations. They may or may not be independent.

Thus,  $3x + 2y = 10$  and  $2x + 3y = 5$  are simultaneous, since both are satisfied by  $x = 4$  and  $y = -1$ .

**259.** A single equation of two or more unknown quantities may be satisfied by any number of values of the unknown quantities, and is, therefore, said to be *Indeterminate*.

Thus,  $x + 2y = 7$  is satisfied when  $x = 1$  and  $y = 3$ ; when  $x = 2$  and  $y = 2\frac{1}{2}$ ; when  $x = 3$  and  $y = 2$ ; when  $x = 4$  and  $y = 1\frac{1}{2}$ , etc.

**260.** Two independent simultaneous equations of the first degree between two unknown quantities can be satisfied by only one pair of values of the unknown quantities.

Thus,  $2x + 3y = 8$  and  $3x + 2y = 7$  are both satisfied only when  $x = 1$  and  $y = 2$ .

**261.** To find the values of the unknown quantities in independent simultaneous equations, we must deduce from these equations a single equation containing only one unknown quantity. This will require in general as many such equations as there are unknown quantities involved.

**262.** The process of deducing from two or more independent simultaneous equations of two or more unknown quantities, a less number of such equations containing a less number of unknown quantities, is *Elimination*.

**Note.**—Elimination is the process of getting rid of one or more unknown quantities.

**263.** There are three easy methods of elimination :

1. By substitution. 2. By comparison. 3. By addition or subtraction.

**Illustration.**—Solve  $\begin{cases} 4x - 3y = 6 & (A) \\ 3x + 4y = 17 & (B) \end{cases}$

**Solution 1:** By substitution.

Transpose  $3y$  in (A) and divide by 4,

$$x = \frac{6 + 3y}{4} \quad (1)$$

Substitute the value of  $x$  in (B),

$$3\left(\frac{6 + 3y}{4}\right) + 4y = 17 \quad (2)$$

$$18 + 9y + 16y = 68 \quad (3)$$

$$25y = 50 \quad (4)$$

$$y = 2 \quad (5)$$

Substitute the value of  $y$  in (1),

$$x = \frac{6 + 6}{4} = 3.$$


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**Solution 2:** By comparison.

Transpose  $-3y$  in (A) and  $+4y$  in (B),

$$4x = 6 + 3y \quad (1)$$

$$3x = 17 - 4y \quad (2)$$

Divide (1) by 4 and (2) by 3,

$$x = \frac{6 + 3y}{4} \quad (3)$$

$$x = \frac{17 - 4y}{3} \quad (4)$$

$$\therefore \frac{6 + 3y}{4} = \frac{17 - 4y}{3} \quad (5)$$

$$\text{Solving (5), } y = 2 \quad (6)$$

Substitute the value of  $y$  in (3), and reduce,

$$x = 3.$$


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**Solution 3:** By addition or subtraction.

Multiply (A) by 4 and (B) by 3,

$$16x - 12y = 24 \quad (1)$$

$$9x + 12y = 51 \quad (2)$$

$$\text{Add (2) to (1), } 25x = 75 \quad (3)$$

$$x = 3.$$

Multiply (A) by 3 and (B) by 4,

$$12x - 9y = 18 \quad (4)$$

$$12x + 16y = 68 \quad (5)$$

Subtract (4) from (5),

$$25y = 50 \quad (6)$$

$$y = 2.$$

### Reduction of Simultaneous Equations containing Two Unknown Quantities.

**Illustration.**—Solve :

$$2x - \frac{y+3}{4} = 7 + \frac{3y-2x}{5} \quad \left. \vphantom{\frac{3y-2x}{5}} \right\} \quad (A)$$

$$4y + \frac{x-2}{3} = 26\frac{1}{2} - \frac{2y+1}{2} \quad \left. \vphantom{\frac{2y+1}{2}} \right\} \quad (B)$$

Clear (A) and (B) of fractions,

$$40x - 5y - 15 = 140 + 12y - 8x \quad (1)$$

$$24y + 2x - 4 = 159 - 6y - 3 \quad (2)$$

Transpose and collect the terms in (1) and (2),

$$48x - 17y = 155 \quad (3)$$

$$2x + 30y = 160 \quad (4)$$

Multiply (4) by 24, and bring down (3),

$$48x + 720y = 3840 \quad (5)$$

$$48x - 17y = 155 \quad (3)$$

Subtract (3) from (5),  $737y = 3685 \quad (6)$

$$y = 5$$

Substitute the value of  $y$  in (4),

$$2x + 150 = 160 \quad (7)$$

$$2x = 10 \quad (8)$$

$$x = 5.$$

**264.** It is sometimes more convenient to eliminate one of the unknown quantities before clearing of fractions.

Solve :  $\frac{3}{2x} + \frac{5}{3y} = \frac{5}{6} \quad \left. \vphantom{\frac{5}{6}} \right\} \quad (A)$

$$\frac{4}{x} - \frac{3}{2y} = \frac{31}{30} \quad \left. \vphantom{\frac{31}{30}} \right\} \quad (B)$$

Reduce the corresponding terms in (A) and (B) to a common denominator,

$$\frac{3}{2x} + \frac{10}{6y} = \frac{25}{30} \quad (1)$$

$$\frac{8}{2x} - \frac{9}{6y} = \frac{31}{30} \quad (2)$$

Multiply (1) by 8 and (2) by 3,

$$\frac{24}{2x} + \frac{80}{6y} = \frac{200}{30} \quad (3)$$

$$\frac{24}{2x} - \frac{27}{6y} = \frac{93}{30} \quad (4)$$

Subtract (4) from (3),  $\frac{107}{6y} = \frac{107}{30}$  (5)

$$\frac{1}{6y} = \frac{1}{30} \quad (6)$$

$$6y = 30 \quad (7)$$

$$y = 5 \quad (8)$$

Substituting the value of  $y$  in (B),

$$\frac{4}{x} - \frac{3}{10} = \frac{31}{30} \quad (9)$$

$$\frac{4}{x} = \frac{40}{30} \quad (10)$$

$$\frac{1}{x} = \frac{1}{3} \quad (11)$$

$$x = 3.$$

Solve :  $ax + by = c$  } (A)

$mx + ny = p$  } (B)

Multiply (A) by  $m$  and (B) by  $a$ ,

$$amx + bmy = cm \quad (1)$$

$$amx + any = ap \quad (2)$$

Subtract (2) from (1),

$$(bm - an)y = cm - ap \quad (3)$$

$$y = \frac{cm - ap}{bm - an} \quad (4)$$

Multiply (A) by  $n$  and (B) by  $b$ ,

$$anx + bny = cn \quad (5)$$

$$bmx + bny = bp \quad (6)$$

Subtract (6) from (5),

$$(an - bm)x = cn - bp \quad (7)$$

$$x = \frac{cn - bp}{an - bm} \quad (8)$$

#### EXERCISE 42.

Solve :

$$1. \left\{ \begin{aligned} \frac{x+2}{5} - \frac{y-3}{4} &= \frac{2+y}{5} - \frac{1}{5} \\ \frac{7x+6}{3} &= \frac{8y-3}{5} + \frac{37}{15} \end{aligned} \right\}$$

$$2. \left\{ \begin{aligned} .125x + .75y &= .625 \\ .33\frac{1}{3}x + .875y &= 1.125 \end{aligned} \right\}$$

$$3. \left\{ \begin{array}{l} \frac{x+y}{2} - 4 = 0 \\ x - y + \frac{2x-3}{1\frac{1}{2}} + 2\frac{2}{3} = 0 \end{array} \right\}$$

$$4. \left\{ \begin{array}{l} \frac{\frac{1}{2}x - 2}{2\frac{1}{2}} = \frac{\frac{1}{3}y - \frac{2}{3}}{3\frac{1}{3}} \\ \frac{2-6x}{7} = \frac{11-3y}{2\frac{1}{2}} \end{array} \right\}$$

$$5. \left\{ \begin{array}{l} \frac{4y}{5+y} = \frac{5y}{12+x} \\ \frac{2x}{5} + y = 7 \end{array} \right\}$$

$$6. \left\{ \begin{array}{l} \frac{2}{x} - \frac{3}{y} = -\frac{3}{5} \\ \frac{5}{2x} + \frac{7}{3y} = 6\frac{11}{20} \end{array} \right\}$$

$$7. \left\{ \begin{array}{l} \frac{m}{x} + \frac{n}{y} = 1 \\ \frac{n}{x} + \frac{m}{y} = 1 \end{array} \right\}$$

$$8. \left\{ \begin{array}{l} \frac{3x-2y}{3x+2y} - \frac{6}{5} = \frac{1-x}{x-1} \\ 3x+2y = 15 \end{array} \right\}$$

$$9. \left\{ \begin{array}{l} \frac{15}{4x} + \frac{10}{3y} = \frac{35}{12} \\ \frac{18}{5x} - \frac{14}{9y} = \frac{19}{45} \end{array} \right\}$$

$$10. \left\{ \begin{array}{l} \frac{x}{a} + \frac{y}{b} = c \\ \frac{x}{m} + \frac{y}{n} = r \end{array} \right\}$$

$$11. \left\{ \begin{array}{l} 1.4x + .32y = 3.76 \\ .28x + 9.6y = 29.36 \end{array} \right\}$$

$$12. \left\{ \begin{array}{l} \frac{ax}{b} - \frac{cy}{d} = m \\ \frac{mx}{n} + \frac{ry}{s} = r \end{array} \right\}$$

$$13. \left\{ \begin{array}{l} \frac{a}{x+y} - \frac{b}{x-y} = 0 \\ ax - by = c \end{array} \right\}$$

$$14. \left\{ \begin{array}{l} \frac{3}{x} + \frac{3}{xy} + \frac{5}{y} = 39 \\ \frac{2}{x} + \frac{2}{xy} - \frac{7}{y} = -5 \end{array} \right\}$$

$$15. \left\{ \begin{array}{l} \frac{a}{bx} + \frac{c}{ay} = m \\ \frac{b}{ax} - \frac{d}{by} = n \end{array} \right\}$$

$$16. \left\{ \begin{array}{l} \frac{5}{6} \{x - 3y + \frac{1}{2}(x + y) - 2x\} = -17\frac{1}{2} \\ \frac{x - y - 3}{6} = \frac{2x - 3y}{3} - \frac{3}{2} \end{array} \right\}$$

$$17. \left\{ \begin{array}{l} \frac{5}{2x} + \frac{3}{5y} = \frac{1}{10} + \frac{3}{x} \\ \frac{3}{x} - \frac{7}{3y} = \frac{5}{2y} - \frac{91}{90} \end{array} \right\}$$

$$18. \left\{ \begin{array}{l} \frac{a+b}{a-b}x = \frac{a-b}{a+b}y \\ ax - by = c \end{array} \right\}$$

$$19. \left\{ \begin{array}{l} \frac{a^2x^2 - b^2y^2}{ax + by} = r \\ rx - sy = c \end{array} \right\}$$

$$20. \left\{ \begin{array}{l} x - \frac{b(n-y)}{a} = \frac{c}{a} + m \\ \frac{m}{x} = \frac{n}{y} \end{array} \right\}$$

$$21. \left\{ \begin{array}{l} \frac{x-a}{b} + \frac{y-b}{c} = d \\ \frac{ax+c}{m} - \frac{by-d}{n} = m \end{array} \right\}$$

$$22. \left\{ \begin{array}{l} \frac{x}{m} + \frac{y}{n} = p \\ \frac{m}{a-x} + \frac{n}{b+y} = 0 \end{array} \right\}$$

$$23. \left\{ \begin{array}{l} \frac{x}{x+y} = \frac{x+y}{x} \\ ax + by = c \end{array} \right\}$$

$$24. \left\{ \begin{array}{l} (a+b)x + (a-b)y = c \\ (a-b)x + (a+b)y = d \end{array} \right\}$$

$$25. \left\{ \begin{array}{l} \frac{a-b}{x} - \frac{a+b}{y} = c \\ \frac{a+b}{x} - \frac{a-b}{y} = d \end{array} \right\}$$

$$26. \left\{ \begin{array}{l} \frac{a-b}{x+y} + \frac{a+b}{x-y} = 1 \\ \frac{a+b}{x+y} - \frac{a-b}{x-y} = 0 \end{array} \right\}$$

$$27. \begin{aligned} (a+b)(x+y) + (a-b)(x-y) &= 2a^2 + 2b^2 \\ (a-b)(x+y) - (a+b)(x-y) &= 2b^2 - 2a^2 \end{aligned}$$

### Reduction of Simultaneous Equations containing Three or more Unknown Quantities.

**Illustration.**—

$$\begin{array}{lcl} \text{Solve :} & \left. \begin{array}{l} 4x - 3y - 2z = 19 \\ 3x - 4y + 6z = 28 \\ 2x + 3y - 4z = 23 \end{array} \right\} & \begin{array}{l} \text{(A)} \\ \text{(B)} \\ \text{(C)} \end{array} \end{array}$$

Multiply (C) by 2 and bring down (A),

$$4x + 6y - 8z = 46 \quad (1)$$

$$4x - 3y - 2z = 19 \quad (\text{A})$$

Subtract (A) from (1),  $9y - 6z = 27$  (2)

$$\text{or} \quad 3y - 2z = 9 \quad (3)$$

Multiply (B) by 2 and (C) by 3,

$$6x - 8y + 12z = 56 \quad (4)$$

$$6x + 9y - 12z = 69 \quad (5)$$

Subtract (4) from (5),

$$17y - 24z = 13 \quad (6)$$

Multiply (3) by 12 and bring down (6),

$$36y - 24z = 108 \quad (7)$$

$$17y - 24z = 13 \quad (6)$$

Subtract (6) from (7),  $19y = 95$  (8)

$$y = 5$$

Substitute the value of  $y$  in (3) and reduce,

$$z = 3$$

Substitute the values of  $y$  and  $z$  in (A) and reduce,

$$x = 10.$$

**Note.**—Labor might have been saved in the above solution by selecting  $y$  first for elimination, and combining equations (A) and (C), then 3 times (B) and 4 times (A). Practice only will enable the student to become expert in the elimination of unknown quantities and the reduction of equations.

#### EXERCISE 43.

Solve :

$$\begin{array}{ll} 1. \left. \begin{array}{l} x + 2y + z = 12 \\ 2x - 3y + 4z = 11 \\ x + 3y - 2z = 3 \end{array} \right\} & 3. \left. \begin{array}{l} 3y + 4z - 7x = 29 \\ 8y + 6z + 8x = 120 \\ 7y - 3z + 6x = 36 \end{array} \right\} \\ 2. \left. \begin{array}{l} 2x - 3y + 4z = -3 \\ 3x - 4y + 6z = -2 \\ 4x + 2y - 5z = 21 \end{array} \right\} & 4. \left. \begin{array}{l} 5x - 3y + 2z = -11 \\ 4z + 3x - 7y = 1 \\ 2y - 3z + 5x = -16 \end{array} \right\} \end{array}$$

$$\begin{aligned} 5. \quad & \left. \begin{aligned} 2x - 4y + 6z &= 18 \\ 3y - 2x + 4z &= 7 \\ 3z - 2y + 4x &= 39 \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 12. \quad & \left. \begin{aligned} x - y + z &= 10 \\ 3x - 8y + 10z &= 50 \\ 5x + 2y - 3z &= 40 \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 6. \quad & \left. \begin{aligned} x + y + z &= a + b + c \\ ax + by + cz &= a^2 + b^2 + c^2 \\ ax - by - cz &= a^2 - b^2 - c^2 \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 13. \quad & \left. \begin{aligned} x + y &= a \\ x + z &= b \\ y + z &= c \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 7. \quad & \left. \begin{aligned} ax + by + cz &= 3abc \\ ax - by + cz &= abc \\ by + ax - cz &= abc \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 14. \quad & \left. \begin{aligned} x + y - z &= 7 \\ x - y + z &= 5 \\ y - x + z &= 3 \end{aligned} \right\} \end{aligned}$$

**Suggestion.**—Take first the sum of the equations in 13 and 14.

$$\begin{aligned} 8. \quad & \left. \begin{aligned} x + y + z &= 12 \\ x + y + u &= 13 \\ x + z + u &= 14 \\ y + z + u &= 15 \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 15. \quad & \left. \begin{aligned} x + y &= 21 \\ y + z &= 23 \\ z + u &= 25 \\ u + t &= 27 \\ x + t &= 24 \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 9. \quad & \left. \begin{aligned} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 9 \\ \frac{2}{x} - \frac{3}{y} + \frac{4}{z} &= 11 \\ \frac{5}{x} + \frac{2}{y} - \frac{3}{z} &= 4 \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 16. \quad & \left. \begin{aligned} x &= \frac{y + z + a}{b} \\ y &= \frac{x - z - b}{a} \\ z &= \frac{y + x - c}{b} \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 10. \quad & \left. \begin{aligned} \frac{x+y}{3} - \frac{x-z}{4} &= 8\frac{5}{6} \\ \frac{x+z}{5} + \frac{y-z}{6} &= 4\frac{8}{15} \\ \frac{x-z}{6} + \frac{y+z}{8} &= 2\frac{1}{2} \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 17. \quad & \left. \begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{2}{3} \\ \frac{1}{x} + \frac{1}{z} &= \frac{3}{4} \\ \frac{1}{y} + \frac{1}{z} &= \frac{5}{6} \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned} 11. \quad & \left. \begin{aligned} \frac{a}{x} + \frac{b}{y} + \frac{c}{z} &= a^2 + b^2 + c^2 \\ \frac{b}{x} - \frac{c}{y} + \frac{a}{z} &= ab - bc + ac \\ \frac{c}{x} + \frac{a}{y} + \frac{b}{z} &= ac + ab + bc \end{aligned} \right\} \end{aligned}$$

$$\begin{array}{lcl}
 18. \quad \left. \begin{array}{l} 5x - 3y + 2z = 25 \\ 3x + 7y - 2z = -19 \\ 4x + 3y - 2z = 2 \end{array} \right\} & 19. \quad \left. \begin{array}{l} x - y - z = 3 \\ 2x + y + 2z = 61 \\ 3x - 4y + 6z = 94 \end{array} \right\}
 \end{array}$$

$$20. \quad \left. \begin{array}{l} \frac{x}{3} + \frac{y}{4} + \frac{a}{2} + \frac{1}{2}(x - y + b) = 0 \\ \frac{x}{4} - \frac{z}{3} - \frac{b}{2} - \frac{1}{3}(x + z - c) = 0 \\ \frac{y}{2} + \frac{z}{3} - \frac{c}{2} + \frac{1}{4}(y - z + a) = 0 \end{array} \right\}$$

### Examples leading to Simultaneous Equations.

**Illustrations.**—1. Three times the sum of two numbers equals 15 times their difference, and, if 6 be added to the less, it will equal the greater. What are the numbers?

**Solution:** Let  $x$  = the greater,  
and  $y$  = the less.

Then, since 3 times their sum equals 15 times their difference,

$$3(x + y) = 15(x - y) \quad (A);$$

And, since the less increased by 6 equals the greater,

$$y + 6 = x \quad (B);$$

Solving these equations, we have

$$x = 18, \text{ the greater number,}$$

$$\text{and } y = 12, \text{ the less number.}$$

2. The sum of A's, B's, and C's ages is 90 years; the sum of A's and C's is twice B's, and C's is  $\frac{4}{5}$  of the sum of A's and B's. What are their ages?

**Solution:** Let  $x$  = A's age,  
 $y$  = B's age,  
and  $z$  = C's age.

Since the sum of their ages is 90 years,

$$x + y + z = 90 \quad (A);$$

Since the sum of A's and C's is twice B's,

$$x + z = 2y \quad (B);$$

And, since C's is  $\frac{4}{5}$  of the sum of A's and B's,

$$z = \frac{4}{5}(x + y) \quad (C),$$

Solving these equations,  $x = 20$ ,  $y = 30$ , and  $z = 40$ .

## EXERCISE 44.

1. A and B have a capital of \$5000; twice A's share diminished by \$1200 is  $\frac{3}{4}$  of B's share. How much has each invested?

2. A man has two horses, and a saddle worth \$20; the poorer horse, with the saddle, is worth  $1\frac{1}{12}$  as much as the better horse without the saddle, and the better horse, with the saddle, is worth  $1\frac{3}{10}$  times as much as the poorer horse without the saddle. What is the value of each horse?

3. A man has a drove of 100 animals, consisting of horses, sheep, and cows; twice the number of horses, added to three times the number of cows, equals the number of sheep increased by 50; and the number of sheep diminished by 40 equals  $\frac{1}{2}$  of the number of horses and cows. How many of each kind has he?

4. A merchant has three kinds of tea; if he mixes 40 pounds of the first kind, 50 pounds of the second, and 60 pounds of the third, the value will be \$188; if he mixes 60 pounds of the first, 40 pounds of the second, and 50 pounds of the third, the value will be \$184; and if he mixes 50 pounds of the first, 60 pounds of the second, and 40 pounds of the third, the average price will be \$1.22 a pound. What is the price of each kind of tea?

5. A number consists of two digits whose sum is 12, and, if 36 be added to the number, its digits will be interchanged. What is the number?

**Suggestion.**—Let  $x$  = the tens' digit and  $y$  the units'; then, since the sum of the digits is 12,  $x + y = 12$ ; and, since 10 units equals 1 ten,  $10x + y$  will equal the number, and  $10y + x$  the number with the digits inverted; whence  $10x + y + 36 = 10y + x$ .

6. A number consists of three digits whose sum is 9; if 90 be added to the number, the hundreds' and tens' digits will be interchanged; but, if only 9 be added to the number, the tens' and units' digits will be interchanged. What is the number?

7. A's money, plus  $\frac{1}{2}$  of the sum of B's and C's, is \$4200; B's, plus  $\frac{1}{3}$  of the sum of A's and C's, is \$4000; and C's, plus  $\frac{1}{4}$  of the sum of A's and B's, is \$3600. What is the fortune of each?

8. A number consists of two digits whose sum, divided by the units' digit, is 5, and, if 27 be subtracted from the number, its digits will be interchanged. What is the number?

9. A number consists of three digits whose sum is twice the units' digit; the difference between the hundreds' and units' digits equals the tens' digit; and, if 72 be added to the number, the tens' and units' digits will change places. What is the number?

10. If 1 be added to the numerator of a certain fraction, its value will be  $\frac{2}{3}$ , but, if 6 be added to the denominator, its value will be  $\frac{1}{3}$ . What is the fraction?

11. A and B can do a piece of work in 20 days, A and C in 30 days, and B and C in 40 days. How long would it take each alone to do it?

**Suggestion.**—Let  $x$  = A's time,  $y$  = B's time, and  $z$  = C's time; then  $\frac{1}{x}$ ,  $\frac{1}{y}$ , and  $\frac{1}{z}$  will equal the part each can do in one day; whence  $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$ ;  $\frac{1}{x} + \frac{1}{z} = \frac{1}{30}$ ; and  $\frac{1}{y} + \frac{1}{z} = \frac{1}{40}$ .

12. Three pipes running together will fill a cistern in 10 hours; if the first runs 3 hours, the second 5 hours, and the third 21 hours, they will fill  $\frac{2}{3}$  of it; if the second runs 15 hours, it and the third will fill the remainder in 10 hours. How long would it take each pipe alone to fill the cistern?

13. If a general draws up his regiment, putting 20 men more in rank than in file, he has 75 men over; if he diminishes the number in rank by 4 and increases the number in file by 5, he lacks 30 men to complete the column. How many men has he in the regiment?

14. Two pipes, A and B, lead to a cistern, and one pipe, C, from the cistern; if A and B flow, the cistern will be filled in 10 hours; if A and C flow, it will be filled in 40 hours; and if B and C flow, it will be filled in 80 hours. In what time can A and B each fill it, and C empty it?

15. The length of a rectangle exceeds its width by 6 yards; if the length be increased by 2 yards and the width by 3 yards, the area will be increased by 114 square yards. What are its dimensions?

16. A has a rectangular field whose length exceeds its width by 20 rods; if its length were increased by 10 rods and its width diminished by 8 rods, its area would be diminished by 160 square rods. What are its dimensions?

17. Two men, 4 feet apart, walk side by side around a circular rink. How far does each walk, if the sum of their distances is one mile?

**Suggestion.**—The circumference of a circle  $= 2\pi r$ , and  $\pi = 3.1416$ .

18. The sum of the radii of two circles is 9 feet, and the circumference of one exceeds the circumference of the other by 6.2832 feet. What are the radius and circumference of each?

19. The amount of a certain principal at 5% for a certain time is \$1575, and the amount for the same time at 8% is \$1620. Required the principal and time.

20. The amount of a certain principal at  $r\%$  for a certain time is \$ $a$ , and the amount for the same time at  $r'\%$  is \$ $b$ . Required the principal and time.

21. The amount of a certain principal for 5 years, at a certain rate per cent, is \$2340, and for 8 years at the same rate is \$2664. Required the principal and rate.

22. The amount of a certain principal for  $m$  years, at a certain rate, is \$ $a$ , and for  $n$  years at the same rate \$ $b$ . Required the principal and rate.

23. A, B, and C have invested \$3000; A receives annually 6% on his investment, B 7%, and C 8%, and the sum of their incomes is \$219. If A received 7%, B 8%, and C 6%, the sum of their incomes would be \$204. How much has each invested?

24. A has invested a certain sum at a certain rate, B has invested \$1500 more at one per cent better than A, and receives an income \$110 greater; and C has invested \$1000 more than B at one per cent better, and receives an income \$105 greater. Find the capital of each and his rate of income.

25. A man invested \$2610 in 5% and 6% bonds, paying 95 for the former and 105 for the latter; his annual income from both investments was \$144. How much did he invest in each kind of bonds?

26. Two couriers, A and B, were one mile apart and approached each other; when they met, it was observed that A had traveled 280 feet farther than B; had A's speed been 104 feet per minute greater, and B's 160 feet per minute greater, each would have traveled one half a mile to meet. What was the speed of each per minute?

27. A is 500 yards behind B, and overtakes him in 50 minutes; were A's speed doubled and B's halved, A would overtake B in 10 minutes. What is the speed of each per minute?

28. Two posts, M and N, are 100 yards apart; B and C, whose rates of travel are equal, start at N at the same time that A starts at M; B travels in the direction of M and meets A in 2 minutes; C travels in the opposite direction and is overtaken by A in 10 minutes. How many yards does each travel per minute?

29. A train 300 feet long passes a train 360 feet long in  $56\frac{1}{4}$  seconds when they run in the same direction; but when they run in opposite directions it passes in  $6\frac{1}{4}$  seconds. What are the rates of the trains in miles per hour?

30. A invested a certain sum in 6's at 96, B invested \$840 more than A in 5's at 94, and C invested \$2560 more than A in 4's at 92; B's annual income was \$20 less than C's. How much did each invest, and what was the income of each?

31. In a mile race, A gives B a start of 440 feet, and beats him  $1\frac{1}{2}$  minutes. At a second trial, A gives B a start of 75 seconds and beats him by 660 feet. Find the rate of each in miles per hour.

32. A, B, and C have each a fortune. A spends  $\frac{1}{5}$  of his the first year, B  $\frac{1}{4}$  of his, and C  $\frac{1}{3}$  of his, when they together have \$27,000 left. The second year, A spends  $\frac{1}{5}$  of his remainder, B  $\frac{1}{4}$  of his, and C  $\frac{1}{3}$  of his, and they have remaining \$19,816 $\frac{2}{3}$ . The third year, A again spends  $\frac{1}{5}$  of his remainder, B  $\frac{1}{4}$  of his, and C  $\frac{1}{3}$  of his, and they have \$14,626.94 $\frac{4}{5}$  remaining. How much had each at first?

33. I have four tanks; if the second be filled from the first, it will leave the first one fourth full. If the third be now filled from the second, it will leave the second one fifth full. If the fourth be now filled from the third, it will leave the third one third full. The third now contains 10 gallons more than the second. How many gallons does each tank hold?

34. A, B, and C, in partnership, gain \$960; A owns  $\frac{1}{4}$  of the stock, plus \$1000, B's gain is \$240, and C's \$420. Required each one's share of the stock.

35. A banker has two kinds of coin; it takes  $a$  pieces of the first to make a dollar, and  $b$  of the second to make the same sum. How many pieces of each kind must be taken that  $c$  pieces will make a dollar?

36. A mass of tin and lead weighing 240 pounds loses 28 pounds when weighed in water. How many pounds of each metal are there in the mass, if 74 pounds of tin lose 10 pounds, and 115 pounds of lead lose 10 pounds?

## CHAPTER IV.

### *THEORY OF EXPONENTS AND RADICALS.*

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#### I. Fractional Exponents.

##### Definitions and Principles.

265. We learned [120, P. 1] that dividing the exponent of a factor by the index of a root extracts the root of the factor. If this principle be now so generalized as to include the case in which the exponent of the factor is not divisible by the index of the root, its application will give rise to quantities with fractional exponents. Thus,

$$\sqrt[n]{a^2} = a^{\frac{2}{n}}; \sqrt[n]{a^m} = a^{\frac{m}{n}}; \text{ etc.}$$

266. A fractional exponent, from the nature of its origin, denotes *a root of a power*, the numerator being the exponent of the power, and the denominator the index of the root.

267. Since  $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \sqrt[n]{a \times a \times a \times \dots \text{ to } m \text{ factors}}$   
 $= \sqrt[n]{a} \times \sqrt[n]{a} \times \sqrt[n]{a} \times \dots \text{ to } m \text{ factors}$  [121, P. 2]  $= (\sqrt[n]{a})^m$ ,  
it follows that *a fractional exponent may also be interpreted as a power of a root.*

**Note.**— $\sqrt[n]{a^m}$  and  $(\sqrt[n]{a})^m$  are always equal in numerical value for any values of  $m$  and  $n$ , but they may differ in sign. Thus, if  $m = 4$  and  $n = 2$ ,  $\sqrt[n]{a^m} = \sqrt{a^4} = \pm a^2$ ; but  $(\sqrt[n]{a})^m = (\sqrt{a})^4 = +a^2$ , but not  $-a^2$ .

**268.** It is evident that the  $n$ th power of the  $n$ th root of a quantity equals the quantity itself. Therefore,

$$(a^{\frac{1}{n}})^n = a^n.$$

**269.** If we let  $a^{\frac{1}{n}} = r$ ,  
 then will  $(a^{\frac{1}{n}})^n = r^n$ ,  
 or  $a^n = r^n$ ,  
 and  $(a^n)^p = (r^n)^p$ ,  
 or  $a^{np} = r^{np}$ ;  
 whence  $\sqrt[n]{a^{np}} = \sqrt[n]{r^{np}}$ ,  
 or  $a^{\frac{np}{n}} = r^p$ .

$\therefore a^{\frac{1}{n}}$  and  $a^{\frac{p}{np}}$  are equivalent. Therefore,

**Prin. 1.**—*Multiplying or dividing both terms of a fractional exponent by the same quantity does not alter its value.*

**270.**  $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}} \times a^{\frac{qr}{qs}}$  [269, P. 1]  
 $= \sqrt[qs]{a^{ps} \times a^{qr}}$  [121, p. 2]  $= \sqrt[qs]{a^{ps+qr}}$  [86, P. 4]  
 $= a^{\frac{ps+qr}{qs}}$  [120, P. 1]  $= a^{\frac{p}{q} + \frac{r}{s}}$ . Therefore,

**Prin. 2.**—*The exponent of a factor in the product equals the sum of the exponents of the same factor in the multiplicand and multiplier, when the exponents are positive fractions.*

**271.** Let  $a^{\frac{p}{q}} \div a^{\frac{r}{s}} = x$ ;  
 then,  $xa^{\frac{r}{s}} = a^{\frac{p}{q}}$ ,  
 and  $(xa^{\frac{r}{s}})^q = (a^{\frac{p}{q}})^q$ ;  
 or,  $xa^{\frac{r}{s}} \times xa^{\frac{r}{s}} \times \dots$  to  $qs$  factors,  
 $= a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \dots$  to  $qs$  factors,  
 or  $x^q a^r = a^p$ ;  
 whence,  $x^q = a^p \div a^r = a^{p-r}$ ,  
 and  $x = a^{\frac{p-r}{q}} = a^{\frac{p}{q} - \frac{r}{s}}$ .  
 $\therefore a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q} - \frac{r}{s}}$ . Hence,

**Prin. 3.**—*The exponent of a factor in the quotient equals the exponent of the same factor in the dividend minus the exponent of that factor in the divisor when the exponents are positive fractions.*

## II. Negative Exponents.

**272.** We learned [100 and 271, P. 3] that the exponent of a factor in the quotient equals the exponent of the same factor in the dividend minus the exponent of that factor in the divisor, when the exponents are positive integers or positive fractions. If this principle be accepted as true when the exponent of the divisor exceeds that of the dividend, *negative exponents* will arise from its application.

Thus,  $a^3 \div a^5 = a^{-2}$ ;  $a^m \div a^n = a^{m-n}$ , in which  $m - n$  is negative when  $n > m$ .

**273.**  $a^m \div a^n = a^{m-n}$  for any positive rational values of  $m$  and  $n$  [100 and 271, P. 3].

$$\text{But } a^m \div a^n = \frac{a^m}{a^n} = \frac{a^m \div a^m}{a^n \div a^m} = \frac{1}{a^{n-m}}.$$

$$\therefore a^{m-n} = \frac{1}{a^{n-m}}.$$

If  $m > n$ ,  $m - n$  is positive and  $n - m$  negative.

If  $m < n$ ,  $m - n$  is negative and  $n - m$  positive.

Therefore,

**Prin. 1.**—*Any quantity with a positive or a negative exponent equals the reciprocal of the quantity with the sign of the exponent changed.*

$$\text{274. } \frac{a^{-n}}{b^{-n}} = a^{-n} \times \frac{1}{b^{-n}} = \frac{1}{a^n} \times b^n = \frac{b^n}{a^n};$$

$$\text{also, } \frac{a^n}{b^n} = a^n \times \frac{1}{b^n} = \frac{1}{a^{-n}} \times b^{-n} = \frac{b^{-n}}{a^{-n}}.$$

Therefore,

**Prin. 2.**—A factor may be transferred from either term of a fraction to the other, if the sign of its exponent be changed.

### General Principles of Exponents.

**275.** Let  $m$  and  $n$  represent two exponents, positive or negative, integral or fractional.

1. When  $m$  and  $n$  are positive integers :

$$a^m \times a^n = a^{m+n} \text{ [86, P. 4].}$$

2. When  $m = \frac{p}{q}$  and  $n = \frac{r}{s}$  :

$$a^m \times a^n = a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q} + \frac{r}{s}} \text{ [270, P. 2]} = a^{m+n}.$$

3. When  $m = -x$  and  $n = -y$  and  $x$  and  $y$  are integral or fractional :

$$\begin{aligned} a^m \times a^n &= a^{-x} \times a^{-y} = \frac{1}{a^x} \times \frac{1}{a^y} \text{ [273, P. 1]} = \frac{1}{a^{x+y}} \\ &= a^{-x-y} = a^{m+n}. \end{aligned} \text{ Therefore,}$$

**Prin. 1.**  $a^m \times a^n = a^{m+n}$  for any rational values of  $m$  and  $n$ .

**276.** Let  $m$  and  $n$  be any rational exponents ; then,

1. When  $m$  and  $n$  are positive integers :

$$a^m \div a^n = a^{m-n} \text{ [100, P. 3].}$$

2. When  $m = \frac{p}{q}$  and  $n = \frac{r}{s}$  :

$$a^m \div a^n = a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q} - \frac{r}{s}} \text{ [271, P. 3]} = a^{m-n}.$$

3. When  $m = -x$  and  $n = -y$  and  $x$  and  $y$  are integral or fractional :

$$\begin{aligned} a^m \div a^n &= a^{-x} \div a^{-y} = \frac{1}{a^x} \div \frac{1}{a^y} = \frac{1}{a^x} \times a^y = \frac{a^y}{a^x} = \\ &= a^{y-x} = a^{-x-(-y)} = a^{m-n}. \end{aligned} \text{ Therefore,}$$

**Prin. 2.**  $a^m \div a^n = a^{m-n}$  for any rational values of  $m$  and  $n$ .

**277.** Let  $m$  and  $n$  be any rational exponents ; then,

1. When  $n$  is a positive integer :

$$(a^m)^n = a^{mn} \text{ [109, P. 1].}$$

2. When  $n = \frac{p}{q}$  :  $(a^m)^n = (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} =$

$$a^{\frac{mp}{q}} = a^{m \times \frac{p}{q}} = a^{mn}.$$

3. When  $n = -x$  and  $x$  is integral or fractional :

$$(a^m)^n = (a^m)^{-x} = \frac{1}{(a^m)^x} = \frac{1}{a^{mx}} = a^{-mx} =$$

$$a^{m \times (-x)} = a^{mn}. \text{ Therefore,}$$

**Prin. 3.**  $(a^m)^n = a^{mn}$  for any rational values of  $m$  and  $n$ .

**278.** Let  $n =$  any rational exponent ; then,

1. When  $n$  is a positive integer :

$$(a \times b)^n = a^n \times b^n \text{ [110, P. 2].}$$

2. When  $n = \frac{p}{q}$  :  $(a \times b)^n = (a \times b)^{\frac{p}{q}} = \sqrt[q]{(a \times b)^p} =$

$$\sqrt[q]{a^p b^p} = a^{\frac{p}{q}} \times b^{\frac{p}{q}} \text{ [121, P. 2]} = a^n \times b^n.$$

3. When  $n = -x$  and  $x$  is integral or fractional :

$$(a \times b)^n = (a \times b)^{-x} = \frac{1}{(a \times b)^x} = \frac{1}{a^x \times b^x} =$$

$$a^{-x} \times b^{-x} = a^n \times b^n. \text{ Therefore,}$$

**Prin. 4.**  $(a \times b)^n$  and  $a^n \times b^n$  are equivalent for any rational value of  $n$ .

**Corollary.**—Let  $b = c \times d$ , then  $b^n = c^n \times d^n$ , and  $(a \times c \times d)^n = a^n \times c^n \times d^n$ ; etc.

**279.** Let  $n$  represent any rational exponent ; then,

$$\left(\frac{a}{b}\right)^n = (a \times b^{-1})^n = a^n \times b^{-n} = \frac{a^n}{b^n}. \text{ Therefore,}$$

**Prin. 5.**  $\left(\frac{a}{b}\right)^n$  and  $\frac{a^n}{b^n}$  are equivalent for any rational value of  $n$ .

## EXERCISE 4B.

Simplify :

1.  $a^{\frac{1}{2}} \times a^{\frac{1}{2}}$ ;  $a^{\frac{3}{2}} \times a^{\frac{1}{2}}$ ;  $a^{-4} \times a^{-5}$ ;  $a^{-\frac{3}{2}} \times a^{-\frac{5}{2}}$ ;  $a^{-m} \times a^n$

2.  $x^{\frac{3}{2}} \times x^{-\frac{3}{2}}$ ;  $x^{\frac{1}{m}} \times x^{\frac{1}{n}}$ ;  $x^{-1} \times x^{\frac{1}{n}}$ ;  $x^{-\frac{1}{m}} \times x^{-\frac{1}{n}}$ ;  
 $x^{-s} \times x^{\frac{1}{s}}$

3.  $2a^2b^{\frac{1}{2}} \times 3a^{-5}b^{\frac{1}{2}}$ ;  $6a^{-3}x^{\frac{5}{2}} \times (-2a^5x^{\frac{3}{2}}y)$ ;  
 $-7^2a^3b^{-2}c^{-1} \times 7^3a^{-8}b^{-3}c^{-\frac{1}{2}}$

4.  $x^3 \div x^{-3}$ ;  $x^{\frac{5}{2}} \div x^{\frac{3}{2}}$ ;  $x^{\frac{1}{m}} \div x^{\frac{1}{n}}$ ;  $x^{\frac{m}{n}} \div x^{\frac{n}{m}}$ ;  $1 \div x^{-\frac{5}{2}}$

5.  $(x^{-2}y^3z) \div (x^{-5}y^4z^{-3})$ ;  $(9a^{-3}b^{\frac{3}{2}}c^{-1}) \div (3a^5b^{-\frac{1}{2}}c^4)$ ;  
 $(15a^{\frac{3}{2}}b^{\frac{3}{2}}c^{\frac{5}{2}}) \div (5a^{-\frac{1}{2}}b^{-1}c^2)$

6.  $(a^{-2})^3$ ;  $(a^{\frac{5}{2}})^4$ ;  $(x^{-4})^{\frac{3}{2}}$ ;  $(x^{\frac{3}{2}})^{\frac{5}{2}}$ ;  $(x^{-\frac{3}{2}})^{-\frac{3}{2}}$ ;  $(x^{-m})^n$ ;  
 $(x^{\frac{m}{n}})^{-\frac{n}{m}}$

7.  $(3a^{-2}b^3)^3$ ;  $(a^{-5}b^{\frac{3}{2}}c^{\frac{3}{2}})^{-2}$ ;  $(4^{-1}b^{-1}c^{-1})^{-1}$ ;  
 $\left(\frac{2}{3}x^m y^{\frac{1}{m}} z^{\frac{m}{n}}\right)^3$ ;  $(8a^3b^{-\frac{3}{2}}c)^{\frac{3}{2}}$

8.  $4^3 \times 3^3$ ;  $2^4 \times 2^3 \times 2^2$ ;  $3^{\frac{3}{2}} \times 3^{\frac{3}{2}} \times 3^{\frac{5}{2}}$ ;  $8^{\frac{3}{2}} \times 27^{\frac{3}{2}}$ ;  
 $4^{-\frac{1}{2}} \times 8^{-\frac{1}{2}} \times 16^{-\frac{1}{2}}$

9.  $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$ ;  $(a^{\frac{3}{2}} - b^{\frac{3}{2}})^2$ ;  $(a^{-\frac{1}{2}} - b^{-\frac{1}{2}})^2$ ;  
 $(a^{-\frac{3}{2}} + b^{\frac{3}{2}})(a^{-\frac{3}{2}} - b^{\frac{3}{2}})$

10.  $(a^{\frac{3}{2}} + x^{\frac{5}{2}})(a^{\frac{3}{2}} - x^{\frac{5}{2}})$ ;  $(x^2 + x^{-2})(x^2 - x^{-2})$ ;  
 $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^3$ ;  $(a^{-4} - b^{-3})^3$

11.  $(x^{\frac{3}{2}} + 7)(x^{\frac{3}{2}} + 5)$ ;  $(x^{-3} + 1)(x^{-3} - 4)$ ;  
 $(x^{-m} + 2a)(x^{-m} - 3a)$

12.  $(x^{\frac{3}{2}} + y^{\frac{3}{2}})(x^{\frac{1}{2}} - y^{\frac{3}{2}})$ ;  $(x^{-\frac{3}{2}} + 1)(3x^{\frac{3}{2}} + 5)$ ;  
 $(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y)(x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y)$

13.  $(x^{-2} + y^{-2})^4$ ;  $\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{3}{4}y^{\frac{5}{2}}\right)^4$ ;  $(x^{\frac{1}{m}} + y^{\frac{1}{n}})^5$ ;  
 $(2x^{-1} - 3x^2)^5$

$$14. \frac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}; \frac{a-b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}; \frac{a-b}{a^{\frac{1}{4}}+b^{\frac{1}{4}}}; \frac{a+b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}; \frac{a-b}{a^{\frac{1}{5}}-b^{\frac{1}{5}}}$$

**Suggestion.**—See 136, 138, and 137.

$$15. \frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}; \frac{a^{\frac{4}{3}}-b^{\frac{4}{3}}}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}; \frac{a^{-\frac{2}{3}}-b^{-\frac{2}{3}}}{a^{-\frac{1}{3}}-b^{-\frac{1}{3}}}; \frac{8a^{\frac{3m}{2}}+1}{2a^{\frac{m}{2}}+1}$$

16. Resolve into two binomial factors :

$$a-b; a^{\frac{1}{2}}-b^{\frac{1}{2}}; a^{-1}-b^{-1}; x^{-2}-y^{-2}; x^3-y^3$$

17. Resolve into one binomial and one trinomial factor :

$$a-b; a+b; a^2-b^2; a^2+b^2; a^{-2}-b^{-2}; a^{\frac{1}{2}}+b^{\frac{1}{2}}$$

$$18. \text{Factor } x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y; x-4x^{\frac{1}{2}}y^{\frac{1}{2}}+4y; \\ 4x^{\frac{3}{2}}-12x^{\frac{1}{2}}y^{\frac{1}{2}}+9y$$

$$19. \text{Factor } x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y; 4x^2+2xy+y^2; \\ x^{-2}+x^{-1}y^{-1}+y^{-2}$$

$$20. \text{Factor } ax^{\frac{3}{2}}+by^{\frac{3}{2}}+bx^{\frac{3}{2}}+ay^{\frac{3}{2}}; a^{-5}x+b^{-10}x$$

21. Reduce to lowest terms :

$$\frac{x-y}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}; \frac{x^{\frac{3}{2}}+y^{\frac{3}{2}}}{x-y}; \frac{x^{\frac{3}{2}}+y^{\frac{3}{2}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}; \frac{x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}}{x^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{1}{3}}}$$

22. Express in fractional exponents :

$$\sqrt{4a^3b^{-2}c}; \sqrt[3]{8a^{-2}b^{\frac{2}{3}}c^{-1}}; \sqrt[3]{2a^{\frac{2}{3}}b^{\frac{1}{3}}c^5}; \sqrt[4]{x^{-2}y^{\frac{1}{2}}z^{\frac{3}{4}}}$$

$$23. \text{Clear } \frac{a^{-2}b^3c^3}{x^{-5}y^2} \text{ of negative exponents.}$$

**Suggestion.**—Multiply both terms by  $a^2x^5$ .

24. Clear of negative exponents :

$$\frac{a^{-3}b^2c}{x^{-\frac{2}{3}}y^{-5}z^2}; \frac{a^{-\frac{2}{3}}b^{-1}}{x^{-1}y^{-\frac{2}{3}}}; \frac{1}{a^{-3}b^{-2}}; \frac{x^{-5}y^{-3}z^{\frac{3}{2}}}{x^2y^{-2}z^2}; \frac{x^{-2}z^3}{y^{-1}z^{-1}}$$

25. Clear of negative exponents :

$$\frac{x^{-2}+y^{-2}}{z}; \frac{x^{-3}+y^2}{x^{-2}y^2}; \frac{1}{x^{-\frac{1}{2}}+y^{-\frac{1}{2}}}; \frac{x-y}{x^{-1}-y^{-1}}; \frac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}}$$

## EXERCISE 4B.

Simplify :

1.  $a^{\frac{1}{2}} \times a^{\frac{1}{2}}$ ;  $a^{\frac{3}{2}} \times a^{\frac{1}{2}}$ ;  $a^{-4} \times a^{-5}$ ;  $a^{-\frac{3}{2}} \times a^{-\frac{5}{2}}$ ;  $a^{-m} \times a^n$

2.  $x^{\frac{3}{2}} \times x^{-\frac{1}{2}}$ ;  $x^{\frac{1}{m}} \times x^{\frac{1}{n}}$ ;  $x^{-1} \times x^{\frac{1}{n}}$ ;  $x^{-\frac{1}{m}} \times x^{-\frac{1}{n}}$ ;  
 $x^{-s} \times x^{\frac{1}{s}}$

3.  $2a^2b^{\frac{1}{2}} \times 3a^{-5}b^{\frac{1}{2}}$ ;  $6a^{-3}x^{\frac{1}{2}} \times (-2a^5x^{\frac{1}{2}}y)$ ;  
 $-7^2a^3b^{-2}c^{-1} \times 7^3a^{-8}b^{-3}c^{-\frac{1}{2}}$

4.  $x^3 \div x^{-3}$ ;  $x^{\frac{1}{2}} \div x^{\frac{3}{2}}$ ;  $x^{\frac{1}{m}} \div x^{\frac{1}{n}}$ ;  $x^{\frac{m}{n}} \div x^{\frac{n}{m}}$ ;  $1 \div x^{-\frac{1}{2}}$

5.  $(x^{-2}y^3z) \div (x^{-5}y^6z^{-3})$ ;  $(9a^{-3}b^{\frac{1}{2}}c^{-1}) \div (3a^5b^{-\frac{1}{2}}c^4)$ ;  
 $(15a^{\frac{3}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}) \div (5a^{-\frac{1}{2}}b^{-1}c^2)$

6.  $(a^{-2})^3$ ;  $(a^{\frac{1}{2}})^4$ ;  $(x^{-4})^{\frac{1}{2}}$ ;  $(x^{\frac{1}{2}})^{\frac{1}{2}}$ ;  $(x^{-\frac{1}{2}})^{-\frac{1}{2}}$ ;  $(x^{-m})^n$ ;  
 $(x^{\frac{m}{n}})^{-\frac{n}{m}}$

7.  $(3a^{-2}b^3)^3$ ;  $(a^{-5}b^{\frac{1}{2}}c^{\frac{1}{2}})^{-2}$ ;  $(4^{-1}b^{-1}c^{-1})^{-1}$ ;  
 $\left(\frac{2}{3}x^m y^{\frac{1}{2}} z^{\frac{m}{2}}\right)^3$ ;  $(8a^3b^{-\frac{1}{2}}c)^{\frac{1}{2}}$

8.  $4^3 \times 3^3$ ;  $2^4 \times 2^3 \times 2^2$ ;  $3^{\frac{1}{2}} \times 3^{\frac{1}{2}} \times 3^{\frac{1}{2}}$ ;  $8^{\frac{1}{2}} \times 27^{\frac{1}{2}}$ ;  
 $4^{-\frac{1}{2}} \times 8^{-\frac{1}{2}} \times 16^{-\frac{1}{2}}$

9.  $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$ ;  $(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2$ ;  $(a^{-\frac{1}{2}} - b^{-\frac{1}{2}})^2$ ;  
 $(a^{-\frac{1}{2}} + b^{\frac{1}{2}})(a^{-\frac{1}{2}} - b^{\frac{1}{2}})$

10.  $(a^{\frac{1}{2}} + x^{\frac{1}{2}})(a^{\frac{1}{2}} - x^{\frac{1}{2}})$ ;  $(x^3 + x^{-2})(x^3 - x^{-2})$ ;  
 $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^3$ ;  $(a^{-4} - b^{-3})^3$

11.  $(x^{\frac{1}{2}} + 7)(x^{\frac{1}{2}} + 5)$ ;  $(x^{-3} + 1)(x^{-3} - 4)$ ;  
 $(x^{-m} + 2a)(x^{-m} - 3a)$

12.  $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})$ ;  $(x^{-\frac{1}{2}} + 1)(3x^{\frac{1}{2}} + 5)$ ;  
 $(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y)(x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y)$

13.  $(x^{-2} + y^{-2})^4$ ;  $\left(\frac{2}{3}x^{\frac{1}{2}} - \frac{3}{4}y^{\frac{1}{2}}\right)^4$ ;  $(x^{\frac{1}{m}} + y^{\frac{1}{n}})^5$ ;  
 $(2x^{-1} - 3x^2)^6$

$$14. \frac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}; \frac{a-b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}; \frac{a-b}{a^{\frac{1}{4}}+b^{\frac{1}{4}}}; \frac{a+b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}; \frac{a-b}{a^{\frac{1}{5}}-b^{\frac{1}{5}}}$$

**Suggestion.**—See 136, 138, and 137.

$$15. \frac{a^{\frac{3}{4}}+b^{\frac{3}{4}}}{a^{\frac{1}{4}}+b^{\frac{1}{4}}}; \frac{a^{\frac{4}{5}}-b^{\frac{4}{5}}}{a^{\frac{1}{5}}-b^{\frac{1}{5}}}; \frac{a^{-\frac{3}{2}}-b^{-\frac{3}{2}}}{a^{-\frac{1}{2}}-b^{-\frac{1}{2}}}; \frac{8a^{\frac{3m}{2}}+1}{2a^{\frac{m}{2}}+1}$$

16. Resolve into two binomial factors :

$$a-b; a^{\frac{1}{2}}-b^{\frac{1}{2}}; a^{-1}-b^{-1}; x^{-2}-y^{-2}; x^3-y^3$$

17. Resolve into one binomial and one trinomial factor :

$$a-b; a+b; a^2-b^2; a^3+b^3; a^{-2}-b^{-2}; a^{\frac{2}{3}}+b^{\frac{2}{3}}$$

$$18. \text{Factor } x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y; x-4x^{\frac{1}{2}}y^{\frac{1}{2}}+4y; \\ 4x^{\frac{3}{2}}-12x^{\frac{1}{2}}y^{\frac{1}{2}}+9y$$

$$19. \text{Factor } x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y; 4x^2+2xy+y^2; \\ x^{-2}+x^{-1}y^{-1}+y^{-2}$$

$$20. \text{Factor } ax^{\frac{2}{3}}+by^{\frac{2}{3}}+bx^{\frac{1}{3}}+ay^{\frac{1}{3}}; a^{-5}x+b^{-10}x$$

21. Reduce to lowest terms :

$$\frac{x-y}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}; \frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x-y}; \frac{x^{\frac{3}{4}}+y^{\frac{3}{4}}}{x^{\frac{1}{4}}-y^{\frac{1}{4}}}; \frac{x^{\frac{4}{5}}+x^{\frac{3}{5}}y^{\frac{1}{5}}+y^{\frac{4}{5}}}{x^{\frac{1}{5}}-x^{\frac{1}{5}}y^{\frac{1}{5}}+y^{\frac{1}{5}}}$$

22. Express in fractional exponents :

$$\sqrt{4a^3b^{-2}c}; \sqrt[3]{8a^{-2}b^{\frac{2}{3}}c^{-1}}; \sqrt[3]{2a^{\frac{2}{3}}b^{\frac{1}{3}}c^{\frac{2}{3}}}; \sqrt[4]{x^{-2}y^{\frac{1}{2}}z^{\frac{3}{4}}}$$

$$23. \text{Clear } \frac{a^{-2}b^3c^3}{x^{-6}y^2} \text{ of negative exponents.}$$

**Suggestion.**—Multiply both terms by  $a^2x^6$ .

24. Clear of negative exponents :

$$\frac{a^{-3}b^2c}{x^{-\frac{2}{3}}y^{-5}z^2}; \frac{a^{-\frac{2}{3}}b^{-1}}{x^{-1}y^{-\frac{2}{3}}}; \frac{1}{a^{-3}b^{-2}}; \frac{x^{-5}y^{-3}z^{\frac{2}{3}}}{x^2y^{-2}z^2}; \frac{x^{-2}z^3}{y^{-1}z^{-1}}$$

25. Clear of negative exponents :

$$\frac{x^{-2}+y^{-2}}{z}; \frac{x^{-3}+y^2}{x^{-2}y^2}; \frac{1}{x^{-\frac{1}{2}}+y^{-\frac{1}{2}}}; \frac{x-y}{x^{-1}-y^{-1}}; \frac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}}$$

26. Express in the integral form  $\frac{4x^2y^{-3}}{a^4b^{-3}}$

**Suggestion.**—Multiply both terms by  $a^{-4}b^3$ .

27. Express in the integral form :

$$\frac{a^2x}{b^2c^{-2}}; \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{x^{-2}y^4}; \frac{x^3y^{-\frac{1}{2}}}{m^{\frac{1}{2}}n^{-\frac{3}{2}}}; \frac{6xy}{3^{-2}x^{-4}z^{\frac{5}{2}}}; \frac{-x^{-1}y^{-1}}{x^{-3}y^{-2}}$$

28. Express under radical sign,  $a^{\frac{3}{2}}b^{-\frac{5}{2}}c^{\frac{7}{2}}$

**Suggestion.**— $a^{\frac{3}{2}}b^{-\frac{5}{2}}c^{\frac{7}{2}} = a^{\frac{3}{2}}b^{-1\frac{1}{2}}c^{\frac{7}{2}} = (a^3b^{-10}c^7)^{\frac{1}{2}} = \sqrt[12]{a^6b^{-10}c^7}$ .

29. Express under radical sign :

$$a^{\frac{3}{2}}b^{\frac{3}{2}}c^{\frac{1}{2}}; a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{-\frac{3}{2}}; (a-b)^{\frac{1}{2}}(a+b)^{\frac{1}{2}}; x^{-1}y^{-\frac{1}{2}}z^{-2}$$

30. Express the reciprocal of  $\frac{x^{-2}-y^{-2}}{x^{-1}y^{-1}}$ ,  $\frac{x^{-2}y^{-3}}{a^{-3}y^{-\frac{1}{2}}}$ ,

and  $\sqrt{\frac{a-b}{a+b}}$ , in positive exponents.

31. Simplify  $\{(a^{-2})^{-\frac{3}{2}}\}^{-\frac{2}{3}} \times \{(-a^{-2})^{-\frac{1}{2}}\}^{-\frac{2}{3}} \div (a^{-\frac{1}{2}})^{-\frac{3}{2}}$

32. Simplify  $(a-b)^2(a+b)^2$ ;  $(2a+3b)^{\frac{1}{2}}(2a-3b)^{\frac{1}{2}}$ ;  
 $(a+b)^{-\frac{1}{2}}(a-b)^{-\frac{1}{2}}(a^2+b^2)^{-\frac{1}{2}}$ ;  $(a^2-x^2)^{\frac{3}{2}}(a^2+x^2)^{\frac{3}{2}}$

33. Simplify :

$$\frac{(x^2-y^2)^2}{(x+y)^2}; \frac{(x^3+y^3)^3}{(x+y)^3}; \frac{(a^3-b^3)^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}}; \frac{(x^m-y^m)^{-\frac{3}{2}}}{(x^m+y^m)^{-\frac{3}{2}}}$$

34. Place the coefficients of the following quantities within the parentheses :

$$a^6(a+b)^2; 8b^3(a-b)^3; 4a^2(ab+b^2)^{\frac{3}{2}}; a^{-6}(x^2+y^2)^{-\frac{1}{2}}$$

35. Extract the square root of  $1-x$  to four terms.

36. Extract the square root of :

$$x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y+2x^{\frac{1}{2}}z^{\frac{1}{2}}+2y^{\frac{1}{2}}z^{\frac{1}{2}}+z$$

37. Extract the cube root of  $1-x^{-2}$  to three terms.

38. Extract the cube root of  $x^3-3x^{\frac{3}{2}}y^{\frac{1}{2}}+3x^{\frac{1}{2}}y^{\frac{3}{2}}-y^3$   
 $+3x^{\frac{3}{2}}z^{\frac{1}{2}}-6x^{\frac{3}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}+3x^{\frac{1}{2}}y^{\frac{3}{2}}z^{\frac{1}{2}}-3y^{\frac{3}{2}}z^{\frac{1}{2}}+z^3$

## Radicals.

## Definitions and Principles.

**280.** A *Radical* is an indicated root of a quantity.

**281.** A radical in which the indicated root may be exactly obtained is a *rational radical*; as,  $\sqrt{4x^2y^4}$ .

**282.** A radical in which the indicated root can not be exactly obtained is an *irrational radical*, or a *Surd*; as,  $\sqrt{2x}$ .

**283.** An *imaginary surd* is an indicated even root of a negative quantity.

**284.** A *pure surd* is one that contains no rational factors; as,  $\sqrt{2a}$ .

**285.** A *mixed surd* is one that contains both rational and irrational factors; as,  $\sqrt{4a^2b} = 2a\sqrt{b}$ .

**286.** A *real quantity* is one that is not imaginary; as, 12,  $\sqrt{49}$ ,  $\sqrt[3]{15}$ .

**287.** The degree of a radical is the degree of the indicated root. Thus,  $\sqrt[5]{a}$  is of the fifth degree.

**288.** A radical is in its simplest form when it is in the lowest degree to which it can be reduced, and contains no factor whose indicated root can be exactly obtained.

**289.** Radicals that contain the same surd factors when reduced to simplest form are *similar*.

**290.**  $\sqrt[n]{\frac{a}{b}} = \sqrt[n]{\frac{a \times b^{n-1}}{b \times b^{n-1}}} = \sqrt[n]{\frac{a b^{n-1}}{b^n}} = \sqrt[n]{\frac{1}{b^n} \times a b^{n-1}} = \frac{1}{b} \sqrt[n]{a b^{n-1}}$  [121, P. 2], which is in its simplest form.

Therefore,

**Prin. 1.**—No fractional radical is in its simplest form.

291.  $\sqrt[n]{a \times b \times c \times \dots} = \sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c} \times \dots$   
 [121, P. 2]. Therefore,

**Prin. 2.**—Any root of the product of two or more quantities and the product of the like roots of those quantities are equivalent.

292.  $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$  [217, P. 3]. Therefore,

**Prin. 3.**—Any root of the quotient of two quantities and the quotient of the like roots of those quantities are equivalent.

293.  $\sqrt[n]{a^n} = a$  [120, P. 1]. Therefore,

**Prin. 4.**—Any quantity equals any root of an equal power of that quantity.

## Calculus of Radicals.

### Illustrations.—

1. Simplify  $\sqrt[3]{40}$ ,  $\sqrt[4]{\frac{2}{3}}$ , and  $\sqrt[4]{27a^3b^6}$ .

**Solutions:** 1.  $\sqrt[3]{40} = \sqrt[3]{8 \times 5} = \sqrt[3]{8} \times \sqrt[3]{5} = 2\sqrt[3]{5}$ .

2.  $\sqrt[4]{\frac{2}{3}} = \sqrt[4]{\frac{2 \times 3^3}{3 \times 3^3}} = \sqrt[4]{\frac{54}{27}} = \sqrt[4]{\frac{1}{3^3}} \times \sqrt[4]{54} = \pm \frac{1}{3} \sqrt[4]{54}$ .

3.  $\sqrt[4]{27a^3b^6} = \sqrt[4]{3^3a^3b^6} = \sqrt[4]{3a^3b^6} = \sqrt[4]{b^3} \times \sqrt[4]{3a} = \pm b\sqrt[4]{3a}$ .

2. Place the coefficient of  $\frac{2}{3}\sqrt[3]{5}$  under the radical sign.

**Solution:**

$\frac{2}{3}\sqrt[3]{5} = \sqrt[3]{\left(\frac{2}{3}\right)^3} \times \sqrt[3]{5} = \sqrt[3]{\frac{8}{27}} \times \sqrt[3]{5} = \sqrt[3]{\frac{8}{27} \times 5} = \sqrt[3]{\frac{40}{27}}$ .

3. Reduce  $\sqrt{3}$ ,  $\sqrt[3]{4}$ ,  $\sqrt[4]{3}$ , and  $\sqrt[5]{2}$  to the same degree.

**Solution:**  $\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{6}{12}} = \sqrt[12]{3^6} = \sqrt[12]{729}$

$\sqrt[3]{4} = 4^{\frac{1}{3}} = 4^{\frac{4}{12}} = \sqrt[12]{4^4} = \sqrt[12]{256}$

$\sqrt[4]{3} = 3^{\frac{1}{4}} = 3^{\frac{3}{12}} = \sqrt[12]{3^3} = \sqrt[12]{27}$

$\sqrt[5]{2} = 2^{\frac{1}{5}} = 2^{\frac{2}{10}} = \sqrt[10]{2^2} = \sqrt[10]{4}$

4. Find the value of  $\sqrt{128} + \sqrt{\frac{1}{8}} - \sqrt{\frac{9}{32}}$ .

**Solution :**  $\sqrt{128} = \sqrt{64} \times \sqrt{2} = 8\sqrt{2}$   
 $\sqrt{\frac{1}{8}} = \sqrt{\frac{2}{16}} = \sqrt{\frac{1}{16}} \times \sqrt{2} = \frac{1}{4}\sqrt{2}$   
 $-\sqrt{\frac{9}{32}} = -\sqrt{\frac{18}{64}} = -\sqrt{\frac{9}{64}} \times \sqrt{2} = -\frac{3}{8}\sqrt{2}$   
 $\therefore \sqrt{128} + \sqrt{\frac{1}{8}} - \sqrt{\frac{9}{32}} = 8\sqrt{2} + \frac{1}{4}\sqrt{2} - \frac{3}{8}\sqrt{2} = \frac{63}{8}\sqrt{2}.$

5. Find the value of  $4\sqrt[3]{6} \times 2\sqrt[3]{4}$  and  $2\sqrt{3x} \times 4\sqrt[3]{3x^2}$ .

**Solutions :** 1.  $4\sqrt[3]{6} \times 2\sqrt[3]{4} = 4 \times 2 \times \sqrt[3]{6} \times \sqrt[3]{4} = 8\sqrt[3]{6 \times 4}$   
 $= 8\sqrt[3]{24} = 8\sqrt[3]{8 \times 3} = 8 \times 2\sqrt[3]{3} = 16\sqrt[3]{3}.$   
 2.  $2\sqrt{3x} \times 4\sqrt[3]{3x^2} = 2 \times 4 \times (3x)^{\frac{1}{2}} \times (3x^2)^{\frac{1}{3}} = 8 \times (3x)^{\frac{1}{2}} \times (3x^2)^{\frac{1}{3}}$   
 $= 8 \times \sqrt[6]{27x^3} \times \sqrt[6]{9x^4} = 8\sqrt[6]{243x^7}$   
 $= 8\sqrt[6]{x^6} \times \sqrt[6]{243x} = \pm 8x\sqrt[6]{243x}.$

6. Find the value of  $4\sqrt[3]{6} \div 2\sqrt[3]{4}$  and  $6\sqrt{5} \div 5\sqrt[3]{2}$ .

**Solutions :** 1.  $4\sqrt[3]{6} \div 2\sqrt[3]{4} = 2\sqrt[3]{6} \div \sqrt[3]{4} = 2\sqrt[3]{\frac{6}{4}} = 2\sqrt[3]{\frac{3}{2}}$   
 $= 2\sqrt[3]{\frac{1}{8}} \times \sqrt[3]{12} = 2 \times \frac{1}{2} \times \sqrt[3]{12} = \sqrt[3]{12}.$   
 2.  $6\sqrt{5} \div 5\sqrt[3]{2} = \frac{6}{5}(5)^{\frac{1}{2}} \div (2)^{\frac{1}{3}} = \frac{6}{5}(5)^{\frac{1}{2}} \times (2)^{\frac{2}{3}} = \frac{6}{5}\sqrt[6]{125} \times \sqrt[6]{4}$   
 $= \frac{6}{5}\sqrt[6]{\frac{125}{4}} = \frac{6}{5}\sqrt[6]{\frac{2000}{64}} = \frac{6}{5}\sqrt[6]{\frac{1}{64}} \times \sqrt[6]{2000} = \pm \frac{3}{5}\sqrt[6]{2000}.$

7. Find the value of  $(5\sqrt[3]{3a^2})^2$ ,  $(3\sqrt{5})^4$ , and  $(a\sqrt[3]{2x})^3$ .

**Solutions :** 1.  $(5\sqrt[3]{3a^2})^2 = \{5(3a^2)^{\frac{1}{3}}\}^2 = 5^2 \times (3a^2)^{\frac{2}{3}} = 25\sqrt[3]{9a^4}$   
 $= 25\sqrt[3]{a^3} \times \sqrt[3]{9a} = 25a\sqrt[3]{9a}.$   
 2.  $(3\sqrt{5})^4 = 3^4 \times (5^{\frac{1}{2}})^4 = 81 \times 5^2 = 81 \times 25 = 2025.$   
 3.  $(a\sqrt[3]{2x})^3 = a^3 \times \{(2x)^{\frac{1}{3}}\}^3 = a^3 \times (2x)^1 = a^3\sqrt{2x}.$

8. Find the value of  $\sqrt[3]{\sqrt{a}}$ ,  $\sqrt[3]{3\sqrt{2a}}$ , and  $\sqrt[4]{4a^2x^2}$ .

**Solutions :** 1.  $\sqrt[3]{\sqrt{a}} = (a^{\frac{1}{2}})^{\frac{1}{3}} = a^{\frac{1}{6}} = \sqrt[6]{a}.$   
 2.  $\sqrt[3]{3\sqrt{2a}} = \{3(2a)^{\frac{1}{2}}\}^{\frac{1}{3}} = 3^{\frac{1}{3}} \times (2a)^{\frac{1}{6}} = 3^{\frac{2}{3}} \times (2a)^{\frac{1}{6}}$   
 $= 9^{\frac{1}{3}} \times (2a)^{\frac{1}{6}} = (18a)^{\frac{1}{6}} = \sqrt[6]{18a}.$   
 3.  $\sqrt[4]{4a^2x^2} = \sqrt[4]{4a^2x^2} = \sqrt[4]{4a^2x^2} = \sqrt[4]{2a^2x^2}.$

9. Rationalize the denominators of :

$$\frac{2}{\sqrt{3}}, \frac{3}{\sqrt[3]{4}}, \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}, \text{ and } \frac{3}{\sqrt{2}+\sqrt{3}+\sqrt{5}}.$$

Solutions: 1.  $\frac{2}{\sqrt{3}} = \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{2\sqrt{3}}{3} = \frac{2}{3}\sqrt{3}.$

2.  $\frac{3}{\sqrt[3]{4}} = \frac{3\sqrt[3]{4^2}}{\sqrt[3]{4^3}} = \frac{3\sqrt[3]{16}}{4} = \frac{3}{4}\sqrt[3]{16}.$

3.  $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{(\sqrt{3}-\sqrt{2})^2}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = \frac{3-2\sqrt{6}+2}{3-2} = 5-2\sqrt{6}.$

4.  $\frac{3}{\sqrt{2}+\sqrt{3}+\sqrt{5}} = \frac{3(\sqrt{2}+\sqrt{3}-\sqrt{5})}{\{(\sqrt{2}+\sqrt{3})+\sqrt{5}\}\{(\sqrt{2}+\sqrt{3})-\sqrt{5}\}} = \frac{3(\sqrt{2}+\sqrt{3}-\sqrt{5})}{(\sqrt{2}+\sqrt{3})^2-5} = \frac{3(\sqrt{2}+\sqrt{3}-\sqrt{5})}{2\sqrt{6}} = \frac{3(\sqrt{2}+\sqrt{3}-\sqrt{5})\sqrt{6}}{2\sqrt{6} \times \sqrt{6}} = \frac{3(\sqrt{12}+\sqrt{18}-\sqrt{30})}{12} = \frac{1}{4}(2\sqrt{3}+3\sqrt{2}-\sqrt{30}) = \frac{1}{2}\sqrt{3} + \frac{3}{4}\sqrt{2} - \frac{1}{4}\sqrt{30}.$

#### EXERCISE 46.

Simplify :

1.  $\sqrt[3]{54}$

2.  $\sqrt[4]{48a^5b^6c^3}$

3.  $\sqrt[3]{\frac{16}{25}}$

4.  $\sqrt{a^3-2a^2b+ab^2}$

7.  $x^{-1}\sqrt{x^ny^{n-1}z^{n+1}}$

5.  $\sqrt[3]{a^4(a-b)^5}$

8.  $\sqrt[2n]{a^{3n}b^{2n+1}c^{4n+1}}$

6.  $\sqrt[2]{a^{2n}b^3c^{n+1}}$

9.  $x^{n+1}\sqrt{x^{2n+2}y^{3n+3}z^{n+2}}$

10.  $\sqrt[6]{256}, \sqrt[9]{64a^{15}b^6c^3}, \sqrt[2n]{a^4b^8c^{2n}}$

11.  $\sqrt[8]{\frac{16}{81}a^4b^8}, \sqrt[3m]{\frac{a^{2m}}{b^{3m}}}, \sqrt[2r]{\frac{a^r b^r}{c^{2r}d^{2r}}}$

12.  $\sqrt{\frac{a+b}{a-b}}, \sqrt[3]{\frac{a-b}{(a+b)^2}}, \sqrt[3]{\frac{a^2+ab+b^2}{(a-b)^2}}$

Place the coefficients under the radical signs :

$$13. x \sqrt[3]{ax}, \frac{5}{8} \sqrt[3]{\frac{8}{9}}, (a+x) \sqrt{a+x}, (a-x)^{-1} \sqrt{a^2-x^2}$$

$$14. \frac{x+1}{x-1} \sqrt{\frac{x-1}{x+1}}, \frac{1}{ab} \sqrt[3]{a^2b^2-a^3b^3}, \frac{1-x}{1+x} \sqrt[3]{\frac{(1+x)^2}{(1-x)^2}}$$

Reduce the following to the same degree :

$$15. \sqrt{\frac{2}{3}}, \sqrt[3]{\frac{3}{4}}, \text{ and } \sqrt[4]{\frac{5}{6}}$$

$$16. \sqrt{ax}, \sqrt[5]{a^2x}, \text{ and } \sqrt[10]{ax^2}$$

$$17. \sqrt{3a^n}, \sqrt[4]{2a^{n+1}}, \text{ and } \sqrt[3]{4a^{n-1}}$$

$$18. \sqrt[3]{2a}, \sqrt[2]{2a}, \text{ and } \sqrt[3]{2a}$$

Determine whether the following radicals are similar :

$$19. \sqrt{8a^3}, \sqrt[3]{18ab^2}, \text{ and } \sqrt{32a^5b^4}$$

$$20. \sqrt[3]{-\frac{1}{4}}, \sqrt[3]{\frac{16}{27}}, \text{ and } \sqrt[3]{\frac{27}{32}}$$

$$21. \sqrt{\frac{1}{a+b}} \text{ and } \sqrt{\frac{a+b}{(a-b)^2}}$$

$$22. \sqrt{\frac{(x-1)^2}{x+1}} \text{ and } \sqrt{\frac{(x+1)^2}{(x-1)^2}}$$

$$23. \sqrt[3]{25}, \sqrt[3]{40}, \text{ and } \sqrt[3]{-135}$$

$$24. \sqrt{\frac{1}{2}}, \sqrt[3]{\frac{2}{3}}, \text{ and } \sqrt[4]{\frac{3}{4}}$$

Find the value of :

$$25. \sqrt{ab} + \sqrt{\frac{4}{9}ab} + \sqrt{\frac{4}{25}ab}$$

$$26. \sqrt[3]{3a^4} + 2\sqrt[3]{3a^4} - 2a\sqrt[3]{24a}$$

$$27. a\sqrt[3]{24a} + 2\sqrt[3]{3ab^3} - 2c\sqrt[3]{3a}$$

$$28. 2\sqrt[3]{ab} - 3\sqrt[3]{a^2b^2} + \sqrt[3]{a^3b^3}$$

$$29. a \sqrt{\frac{a}{b^3}} + b \sqrt{\frac{1}{ab}} - 2 \sqrt{\frac{a}{b}}$$

$$30. \sqrt{\frac{1}{2}} + \frac{1}{3} \sqrt[4]{4} - \frac{1}{2} \sqrt[6]{\frac{1}{8}}$$

31.

$$\left( \sqrt{\frac{3}{4}} + \sqrt{1\frac{7}{18}} \right) - \left( \sqrt{\frac{3}{16}} - \sqrt{\frac{8}{9}} \right) + \left( 2\sqrt{3} - \sqrt{\frac{1}{2}} \right)$$

$$32. (\sqrt{12} + 3\sqrt[3]{5} - \sqrt[4]{2}) - \left( \sqrt{\frac{27}{16}} - \frac{1}{3}\sqrt[3]{40} + \sqrt[4]{32} \right)$$

$$33. \sqrt{\frac{x^2y + 2xy^2 + y^3}{(x-y)^2}} - \sqrt{\frac{x^2y - 2xy^2 + y^3}{(x+y)^2}} - \sqrt{\frac{16x^2y^3}{(x+\frac{1}{2}y)^2(x-y)^2}}$$

$$34. 2\sqrt[3]{5} \times 3\sqrt[3]{3} \times \sqrt[3]{6}$$

$$35. 2\sqrt{2} \times 2\sqrt[3]{2} \times 2\sqrt[4]{2}$$

$$36. \sqrt{\frac{1}{2}} \times 2\sqrt{\frac{2}{3}} \times 3\sqrt{\frac{6}{7}}$$

$$37. 5\sqrt[3]{9} \div \frac{1}{2}\sqrt[3]{3}$$

$$38. \sqrt{\frac{1}{2}} \times \sqrt[3]{\frac{1}{3}} \times \sqrt[4]{\frac{1}{4}}$$

$$39. 2\sqrt{2} \div \sqrt[3]{\frac{1}{2}}$$

$$40. \sqrt{\frac{2}{3}} \times \sqrt{\frac{3}{4}} \div \sqrt{\frac{5}{6}}$$

$$41. \frac{2}{3}\sqrt[3]{\frac{2}{3}} \div \frac{1}{2}\sqrt[4]{\frac{1}{2}}$$

$$42. (x\sqrt{y} + x^2\sqrt{xy} + \sqrt{x}) \times \sqrt{xy}$$

$$43. (\sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2}) \div \sqrt[12]{2}$$

$$44. (\sqrt{x} + \sqrt{y} + \sqrt{z})(\sqrt{x} - \sqrt{y} + \sqrt{z})$$

$$45. (\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2})(\sqrt[3]{x} - \sqrt[3]{y})$$

$$46. (x + y) \div (\sqrt[3]{x} + \sqrt[3]{y})$$

$$47. (x + \sqrt{xy} + y) \div (\sqrt{x} + \sqrt[4]{xy} + \sqrt{y})$$

$$48. (\sqrt{x} - \sqrt{y}) \div (\sqrt[4]{x} + \sqrt[4]{y})$$

$$49. (x\sqrt[3]{x} + \sqrt[3]{x^2} + 1) \div (\sqrt[3]{x^2} + \sqrt[3]{x} + 1)$$

$$50. \left(\frac{a}{b} \sqrt[4]{\frac{a}{b}}\right)^8, (mn\sqrt{mn})^6, (\sqrt{ax} \div a^2x^3)^4$$

$$51. (3\sqrt[4]{3ab})^3, (xy\sqrt[4]{xy})^4, \sqrt[4]{3b}\sqrt[4]{3b}$$

$$52. \sqrt{16}\sqrt[3]{16x^4y^3}, \sqrt[3]{\sqrt[2]{a^2b^2}}, \sqrt[3]{3ab}\sqrt[3]{3ab}$$

$$53. (\sqrt{a+b} - \sqrt{a-b})^2 \qquad 54. (a\sqrt{a} + b\sqrt{b})^4$$

$$55. (\sqrt{x-y} + \sqrt{x+y})^3 \qquad 56. \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{3}{4}}\right)^4$$

$$57. \sqrt{(x-2)\sqrt{x^3} + \sqrt[3]{x^3}}$$

$$58. \sqrt{(4x+12\sqrt{xy}+9y)}$$

$$59. \sqrt[3]{(x-3\sqrt{x^2y}+3\sqrt[3]{xy^2}-y)}$$

$$60. \sqrt{(x^2+2x\sqrt{xy}+y^2+3xy+2y\sqrt{xy})}$$

$$61. \sqrt[3]{(x+3x^{\frac{2}{3}}y^{\frac{1}{3}}+3x^{\frac{1}{3}}y^{\frac{2}{3}}+y+3x^{\frac{2}{3}}z^{\frac{1}{3}}+6x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}+3y^{\frac{2}{3}}z^{\frac{1}{3}}+3x^{\frac{1}{3}}z^{\frac{2}{3}}+3y^{\frac{1}{3}}z^{\frac{2}{3}}+z)}$$

Rationalize the denominators of :

$$62. \frac{\sqrt{x-2y}}{\sqrt{x+2y}}, \frac{3+\sqrt{2}}{1-\sqrt{2}}$$

$$63. \frac{2}{\sqrt[3]{5}}, \frac{\sqrt{2}}{\sqrt[3]{3}}$$

$$64. \frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{2}+3\sqrt{3}}, \frac{\sqrt{5}-2\sqrt{3}}{\sqrt{5}+2\sqrt{3}}$$

$$65. \frac{a-\sqrt{a^2+x^2}}{a+\sqrt{a^2+x^2}}$$

$$66. \frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$$

$$67. \frac{\sqrt{2}}{\sqrt{3}-\sqrt{5}+\sqrt{6}}$$

$$68. \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$$

$$69. \text{ Show that } \left\{ \frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} \right\}^{\frac{1}{2}} + \left\{ \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}} \right\}^{\frac{1}{2}} = \frac{2(2-x)}{x^2} (\sqrt{x} - \sqrt{x-x^2})$$

## Imaginary Quantities.

## Definitions and Principles.

**294.** An expression having one or more terms containing imaginary surds is an *Imaginary Quantity*.

**295.** A term containing an imaginary surd is an *Imaginary Term*.

**296.** A quantity containing no other than imaginary terms is a *Pure Imaginary Quantity*; as,  $a\sqrt{-b}$ , or  $\sqrt{-a} + \sqrt{-b}$ .

**297.** A quantity containing both real and imaginary terms is a *Complex Imaginary Quantity*; as,  $a \pm \sqrt{-b}$ , or  $\sqrt{a} \pm \sqrt{-b}$ .

$$\begin{aligned} \text{298. } \sqrt{-b^2} &= \sqrt{b^2 \times (-1)} = \sqrt{b^2} \times \sqrt{-1} = \\ &= \pm b\sqrt{-1}; \text{ also, } \sqrt{-b} = \sqrt{b \times (-1)} = \\ &= \sqrt{b} \times \sqrt{-1} = \pm b^{\frac{1}{2}}\sqrt{-1}. \text{ Therefore,} \end{aligned}$$

**Prin. 1.**—Every imaginary term of the second degree may be reduced to the form of  $\pm n\sqrt{-1}$ , in which  $n$  is real.

**299.** The factor  $\sqrt{-1}$  is called the *Imaginary Unit*, and for convenience is often represented by the letter  $i$ . Thus,  $a\sqrt{-1} = ai$ , and  $a + b\sqrt{-1} = a + bi$ .

**300.** The algebraic sum of a real and an imaginary term is called a *Complex Number*; as,  $a \pm b\sqrt{-1}$ .

**301.** Two complex imaginary quantities of the form of  $a + b\sqrt{-1}$  and  $a - b\sqrt{-1}$ , or  $a + bi$  and  $a - bi$ , are called a *Conjugate Pair*.

**302.** Since  $2n$  represents an even number and  $2n - 1$  an odd number for any integral values of  $n$ ,  $\sqrt[2n]{-1}$  is imaginary and  $\sqrt[2n-1]{-1}$  is real.

**303.** If, in the two forms

$$\begin{array}{ll}
 (\sqrt{-1})^{2n} & \text{and } (\sqrt{-1})^{2n-1}, \text{ we let} \\
 n=1, (\sqrt{-1})^2 = -1 & \text{and } (\sqrt{-1})^1 = \sqrt{-1} \\
 n=2, (\sqrt{-1})^4 = +1 & \text{and } (\sqrt{-1})^3 = -\sqrt{-1} \\
 n=3, (\sqrt{-1})^6 = -1 & \text{and } (\sqrt{-1})^5 = \sqrt{-1} \\
 n=4, (\sqrt{-1})^8 = +1 & \text{and } (\sqrt{-1})^7 = -\sqrt{-1} \\
 n=5, (\sqrt{-1})^{10} = -1 & \text{and } (\sqrt{-1})^9 = \sqrt{-1} \\
 n=6, (\sqrt{-1})^{12} = +1 & \text{and } (\sqrt{-1})^{11} = -\sqrt{-1} \\
 n=7, (\sqrt{-1})^{14} = -1 & \text{and } (\sqrt{-1})^{13} = \sqrt{-1} \\
 \text{etc.,} & \text{etc.,} \quad \text{etc.,} \quad \text{etc.}
 \end{array}$$

Therefore,

**Prin. 2.**—The even powers of  $\sqrt{-1}$  are real,  $+1$  when the exponent of the power is divisible by 4, and  $-1$  when divisible by 2, but not by 4.

**Prin. 3.**—The odd powers of  $\sqrt{-1}$  are imaginary,  $+\sqrt{-1}$  when the exponent of the power is one more than a multiple of 4, and  $-\sqrt{-1}$  when it is one less than a multiple of 4.

**304.**  $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 - (b\sqrt{-1})^2 = a^2 + b^2$ . Therefore,

**Prin. 4.**—The product of a conjugate pair of imaginaries is real.

**Illustrative Examples.**—1. Find the sum and the difference of  $3 + \sqrt{-4}$  and  $5 - \sqrt{-9}$ .

$$\begin{array}{rcl}
 \text{Solution:} & 3 + \sqrt{-4} & = 3 + 2\sqrt{-1} \\
 & 5 - \sqrt{-9} & = 5 - 3\sqrt{-1} \\
 \text{Sum} & = & 8 - \sqrt{-1} \\
 \text{Difference} & = & -2 + 5\sqrt{-1}
 \end{array}$$

2. Find the product of  $2\sqrt{-2}$  and  $3\sqrt{-3}$ .

$$\begin{array}{l}
 \text{Solution:} \\
 2\sqrt{-2} \times 3\sqrt{-3} = 2 \times \sqrt{2} \times \sqrt{-1} \times 3 \times \sqrt{3} \times \sqrt{-1} = \\
 2 \times 3 \times \sqrt{2} \times \sqrt{3} \times (\sqrt{-1})^2 = 6 \times \sqrt{6} \times (-1) = -6\sqrt{6}.
 \end{array}$$

3. Find the product of  $5 - 2\sqrt{-2}$  and  $6 + 3\sqrt{-5}$ .

$$\begin{aligned} \text{Solution: } 5 - 2\sqrt{-2} &= 5 - 2\sqrt{2} \times i \\ 6 + 3\sqrt{-5} &= 6 + 3\sqrt{5} \times i \\ \hline &30 - 12\sqrt{2} \times i \\ &\quad + 15\sqrt{5} \times i - 6\sqrt{10} \times i^2 \\ \hline &30 + (15\sqrt{5} - 12\sqrt{2})i + 6\sqrt{10} = \\ &30 + 15\sqrt{-5} - 12\sqrt{-2} + 6\sqrt{10}. \end{aligned}$$

4. Divide  $5\sqrt{-2}$  by  $2\sqrt{-3}$ .

$$\begin{aligned} \text{Solution: } 5\sqrt{-2} \div 2\sqrt{-3} &= (5\sqrt{2} \times i) \div (2\sqrt{3} \times i) = \\ 5\sqrt{2} \div 2\sqrt{3} &= \frac{5}{2} \sqrt{\frac{2}{3}} = \frac{5}{6} \sqrt{6}. \end{aligned}$$

5. Find the value of  $(2\sqrt{-2})^5$ .

$$\begin{aligned} \text{Solution: } (2\sqrt{-2})^5 &= (2\sqrt{2} \times i)^5 = 2^5 \times (\sqrt{2})^5 \times i^5 = \\ 32 \times 4\sqrt{2} \times \sqrt{-1} &= 128\sqrt{-2}. \end{aligned}$$

#### EXERCISE 47.

Find the value of :

- $\sqrt{-16} + \sqrt{-9} - \sqrt{-25}$
- $\sqrt{-a^4} - \sqrt{-9a^4} + \sqrt{-16b^2}$
- $2\sqrt{-x^2} - 3\sqrt{-y^2} + 5\sqrt{-4z^2}$
- $\sqrt{-2} \times \sqrt{-8}; \sqrt{-10} \times \sqrt{-4}; 3\sqrt{-2} \times 4\sqrt{-2}$
- $(2 + \sqrt{-3})(2 - \sqrt{-3}); (\sqrt{-2} + \sqrt{-3})(\sqrt{-2} - \sqrt{-3})$
- $(\sqrt{-2} + \sqrt{-3} - \sqrt{-4}) \times \sqrt{3}; (\sqrt{-6} - \sqrt{-4} + \sqrt{-5}) \times 2\sqrt{-3}$
- $(2\sqrt{-2} + 3\sqrt{-3} + 4\sqrt{-5})(3\sqrt{-2} - 4\sqrt{-3})$
- $4\sqrt{-8} \div 3\sqrt{-2}; 5\sqrt{-ab} \div 2\sqrt{-a^2}$
- $(7\sqrt{-4} + 3\sqrt{-8} - 4\sqrt{-10}) \div 2\sqrt{-2}$
- $(8\sqrt{-8} + 4\sqrt{2} - 6\sqrt{-3}) \div 2\sqrt{3}$

11.  $(-2\sqrt{2} + 2\sqrt{3} + \sqrt{10} - \sqrt{15}) \div (\sqrt{-2} - \sqrt{-3})$   
 12.  $(\sqrt{-2})^3$ ;  $(2\sqrt{-4})^4$ ;  $(-3\sqrt{-3})^2$ ;  $\{x^2\sqrt{-(1-x)}\}^4$   
 13.  $(1 + \sqrt{-1})^2$ ;  $(2 - \sqrt{-2})^3$ ;  $(\sqrt{-2} - \sqrt{-3})^2$ ;  
 $\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{-1}\right)^4$   
 14.  $\frac{1}{\sqrt{-2}}$ ;  $\frac{2}{\sqrt{-3}}$ ;  $\frac{1}{1 - \sqrt{-1}}$ ;  $\frac{\sqrt{-2} - \sqrt{-3}}{\sqrt{-2} + \sqrt{-3}}$

## Binomial Surds.

### Definitions and Principles.

**305.** A binomial surd is a binomial in which one or both terms are surds; as,  $a \pm \sqrt{b}$  or  $\sqrt{a} \pm \sqrt{b}$ .

**Note.**—The discussion under this head will be limited to binomial surds of the second degree.

**306.** The general form of a binomial surd is  $\sqrt{a} \pm \sqrt{b}$ , in which the coefficients of the terms have been placed under the radical signs, and  $\sqrt{a}$  may be rational or irrational.

**307.**  $(\sqrt{a} \pm \sqrt{b})^2 = a \pm 2\sqrt{ab} + b = (a + b) \pm 2\sqrt{ab}$ .  
Therefore,

**Prin. 1.**—The square of a binomial surd is a binomial surd with a rational term.

**Prin. 2.**—If a binomial surd is a perfect square, and the coefficient of the irrational term is reduced to 2, the quantity under the radical sign is composed of two factors whose sum is the rational term.

**308.** The general form of a binomial surd that is a perfect square is  $p \pm 2\sqrt{q}$ , in which  $q$  is the product of two factors whose sum is  $p$  [P. 2].

**309.** Let  $p = m + n$  and  $q = mn$ ;  
 then  $p \pm 2\sqrt{q} = (m + n) \pm 2\sqrt{mn} = \square$  [307, P. 2],  
 $p^2 = (m + n)^2 = m^2 + 2mn + n^2$ ,  
 $(2\sqrt{q})^2 = (2\sqrt{mn})^2 = 4mn$ ,  
 $p^2 - (2\sqrt{q})^2 = m^2 - 2mn + n^2 = (m - n)^2$ .

Therefore,

**Prin. 3.**—If a binomial surd is a perfect square, the difference of the squares of its terms is a perfect square.

**Note.**—This principle will enable us to determine whether a binomial surd is a perfect square. Thus,  $11 + 6\sqrt{2}$  is a perfect square, since  $11^2 - (6\sqrt{2})^2 = 121 - 72 = 49$ , a square.

**310.** Let  $p \pm 2\sqrt{q} = (m + n) \pm 2\sqrt{mn} = \square$ ,  
 then  $\sqrt{p \pm 2\sqrt{q}} = \sqrt{m + n \pm 2\sqrt{mn}} = \sqrt{m} \pm \sqrt{n}$ .  
 Therefore,

**Prin. 4.**—The square root of a binomial surd of the form of  $p \pm 2\sqrt{q}$  is the sum or difference of the square roots of two quantities whose product is  $q$  and whose sum is  $p$ .

**Note.**—This principle will enable us to find the square root of a binomial surd by inspection when the factors of  $q$  are readily seen. Thus,  $11 + 6\sqrt{2} = 11 + 2\sqrt{18}$ . Now,  $18 = 9 \times 2$  and  $9 + 2 = 11$ . Therefore,  $\sqrt{11 + 2\sqrt{18}} = \sqrt{9} + \sqrt{2} = 3 + \sqrt{2}$ .

**311.** Let  $p \pm 2\sqrt{q} = (m + n) \pm 2\sqrt{mn} = \square$ ;  
 then  $(m - n)^2 = p^2 - (2\sqrt{q})^2 = p^2 - 4q$  [309].  
 $\therefore m - n = \sqrt{p^2 - 4q}$ ;  
 but,  $m + n = p$ ;  
 adding,  $2m = p + \sqrt{p^2 - 4q}$ ; (A)  
 subtracting,  $2n = p - \sqrt{p^2 - 4q}$ . (B)

By means of (A) and (B) the values of  $m$  and  $n$  may easily be obtained when they can not be seen by inspection.

**Illustration.**—Find the square root of

$$\frac{5}{6} - \frac{1}{3}\sqrt{6}, \text{ or } \sqrt{\frac{5}{6} - \frac{1}{3}\sqrt{6}}.$$

**Solution :**

$$\sqrt{\frac{5}{6} - \frac{1}{3}\sqrt{6}} = \sqrt{\frac{5}{6} - 2\sqrt{\frac{1}{6}}}. \text{ Here } p = \frac{5}{6} \text{ and } q = \frac{1}{6};$$

$$\therefore 2m = \frac{5}{6} + \sqrt{\frac{1}{36}} = 1, \text{ and } 2n = \frac{5}{6} - \sqrt{\frac{1}{36}} = \frac{2}{3}$$

$$m = \frac{1}{2}, \text{ and } n = \frac{1}{3}.$$

$$\therefore \sqrt{\frac{5}{6} - \frac{1}{3}\sqrt{6}} = \sqrt{m} - \sqrt{n} = \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{3}} = \frac{1}{2}\sqrt{2} - \frac{1}{3}\sqrt{3}.$$

#### EXERCISE 48.

Extract the square root of :

- |  |                                   |
|--|-----------------------------------|
| 1. $7 + 4\sqrt{3}$   | 13. $35 - 12\sqrt{6}$             |
| 2. $11 - 6\sqrt{2}$  | 14. $3 - \sqrt{2}$                |
| 3. $13 - 4\sqrt{3}$  | 15. $a^2 + 2x\sqrt{a^2 - x^2}$    |
| 4. $5 + 2\sqrt{6}$   | 16. $(2x - y) - 2\sqrt{x^2 - xy}$ |
| 5. $16 - 2\sqrt{15}$   | 17. $2x + 2\sqrt{x^2 - a^2}$      |
| 6. $12 - 2\sqrt{35}$   | 18. $0 + 2\sqrt{-1}$              |
| 7. $29 + 12\sqrt{5}$   | 19. $-5 - 2\sqrt{6}$              |
| 8. $103 + 42\sqrt{6}$  | 20. $2x^2 + 2\sqrt{x^4 - x^2y^2}$ |
| 9. $54 - 36\sqrt{2}$   | 21. $(x^2 + x) + 2x\sqrt{x}$      |
| 10. $x + y - 2\sqrt{xy}$   | 22. $95 + 20\sqrt{-5}$            |
| 11. $\frac{7}{10} + \frac{1}{5}\sqrt{10}$  | 23. $\frac{17}{12} - \sqrt{2}$    |
| 12. $\sqrt{xy} - \sqrt{\frac{x}{y}}$   | 24. $-\frac{1}{50}\sqrt{-1}$      |
| 25. $(x^2 + 2xy + y^2) + 2(x + y)\sqrt{xy}$  |                                   |
| 26. Simplify $\frac{\sqrt{5 + 2\sqrt{6}} - \sqrt{5 - 2\sqrt{6}}}{\sqrt{5 + 2\sqrt{6}} + \sqrt{5 - 2\sqrt{6}}}$ |                                   |

## Reduction of Simple Radical Equations.

**Illustration.**—Solve  $\sqrt{x+a} = 3 + \sqrt{x}$ .**Solution:** Square both members,

$$x+a = 9 + 6\sqrt{x} + x \quad (1)$$

$$\text{Transpose,} \quad -6\sqrt{x} = 9-a$$

$$\text{Square,} \quad 36x = 18 - 18a + a^2$$

$$\therefore \quad x = \frac{1}{36}(18 - 18a + a^2).$$

## EXERCISE 49.

Solve :

1.  $2 + \sqrt{x} = 3 - \sqrt{x}$

14.  $\frac{x-4}{\sqrt{x}+2} = 5$

2.  $\sqrt{x} + \sqrt{a} = \sqrt{x-a}$

15.  $\frac{x-9}{\sqrt{x}-3} = 2$

3.  $\sqrt{2+\sqrt{3-\sqrt{x}}} = \sqrt{5}$

4.  $\sqrt{3x+2a} = \sqrt{3a}\sqrt{2a}$

16.  $\frac{ax-b}{\sqrt{ax}+\sqrt{b}} = a$

5.  $\sqrt{x+10} - \sqrt{x-10} = 2$

6.  $\sqrt{\frac{x}{a}} + \sqrt{ax} = a$

17.  $\frac{x-c}{\sqrt{x}+\sqrt{c}} = 2\sqrt{x}$

7.  $\sqrt{a+x} + \sqrt{\frac{a}{x}} = \sqrt{x}$

18.  $\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}} = 2$

8.  $\sqrt{1+\sqrt{4+\sqrt{3x}}} = 3$

19.  $\frac{\sqrt{ax}-b}{\sqrt{ax}+b} = c$

9.  $\sqrt{a+\sqrt{b+\sqrt{x}}} = \sqrt{a}$

10.  $\sqrt[3]{1+\sqrt[3]{1+x}} = 2$

20.  $\frac{a+\sqrt{a+x}}{a-\sqrt{a+x}} = -b$

11.  $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} = m$

21.  $\frac{\sqrt{2x}-\sqrt{5x}}{\sqrt{3x}+\sqrt{2x}} = x$

12.  $\frac{\sqrt{x}+\sqrt{ax}}{\sqrt{x}-\sqrt{ax}} = (1-a)x$

22.  $\frac{\sqrt{2x}+\sqrt{3x}}{\sqrt{2x}-\sqrt{3x}} = \sqrt{x}$

13.  $\sqrt{x} + \frac{1}{\sqrt{x}} = \sqrt{a} + \frac{1}{\sqrt{a}}$

$$23. \frac{1}{\sqrt{x+1}} + \sqrt{x+1} = 2\sqrt{x+1}$$

$$24. \sqrt[3]{2x + \sqrt{2x-6}} = \sqrt[3]{2x+2}$$

$$25. \frac{\sqrt{ax} + \sqrt{bx}}{\sqrt{ax} - \sqrt{bx}} = \sqrt{ax}$$

$$26. \frac{x + \sqrt{x}}{x - \sqrt{x}} = a + a\sqrt{x}$$

### Supplementary Theorems.

**312. Definition.**—Any surd of the second degree is called a *Quadratic Surd*.

**313. Theorem I.**—No two dissimilar quadratic surds can be equal.

**Demonstration.**—Let  $\sqrt{a}$  and  $\sqrt{b}$  be two dissimilar quadratic surds. If possible, let the  $\sqrt{a} = \sqrt{b}$ ; then, by squaring,  $a = b$ , which can not be, if  $\sqrt{a}$  and  $\sqrt{b}$  are dissimilar.

**314. Theo. II.**—The product and the quotient of two similar quadratic surds are rational.

**Demonstration.**—Any two similar quadratic surds can be reduced to the form of  $a\sqrt{b}$  and  $c\sqrt{b}$ . Now,

$$1. a\sqrt{b} \times c\sqrt{b} = a \times b^{\frac{1}{2}} \times c \times b^{\frac{1}{2}} = acb, \text{ a rational product.}$$

$$2. a\sqrt{b} \div c\sqrt{b} = ab^{\frac{1}{2}} \div cb^{\frac{1}{2}} = \frac{a}{c}, \text{ a rational quotient.}$$

**315. Theo. III.**—If the product or the quotient of two quadratic surds is rational, the surds are similar.

**Demonstration.**—Let  $\sqrt{a} \times \sqrt{b} = c$ , then  $\sqrt{a} \times \sqrt{b} \times \sqrt{b} = c\sqrt{b}$ , or  $b\sqrt{a} = c\sqrt{b}$ , which is possible only when  $\sqrt{a}$  and  $\sqrt{b}$  are similar.

Also, if we let  $\sqrt{a} \div \sqrt{b} = c$ , then  $c\sqrt{b} = \sqrt{a}$ , which is possible only when  $\sqrt{a}$  and  $\sqrt{b}$  are similar.

**316. Theo. IV.**—The product or the quotient of two dissimilar quadratic surds is a quadratic surd.

**Demonstration.**—Let  $\sqrt{a}$  and  $\sqrt{b}$  be two dissimilar quadratic surds. Now,  $a$  and  $b$  have each one or more factors of the first degree, else would  $\sqrt{a}$  and  $\sqrt{b}$  not be surds, and  $b$  must have at least one factor different from the factors of  $a$ , else would  $\sqrt{a}$  and  $\sqrt{b}$  not be dissimilar; therefore,  $\sqrt{ab}$  must have at least one factor of the first degree under the radical sign, which makes it a surd. Likewise,  $\sqrt{\frac{a}{b}}$ , or  $\frac{1}{b} \sqrt{ab}$  must be a surd.

**317. Theo. V.**—*The sum or the difference of two dissimilar quadratic surds can not be rational, nor can it be a single surd.*

**Demonstration.**—Let  $\sqrt{a}$  and  $\sqrt{b}$  be two dissimilar quadratic surds, and, if possible, let  $\sqrt{a} \pm \sqrt{b} = \sqrt{c}$ , in which  $\sqrt{c}$  may be rational or irrational. Squaring,  $a \pm 2\sqrt{ab} + b = c$ , or  $\pm 2\sqrt{ab} = c - a - b$ . But  $2\sqrt{ab}$  is irrational [Theo. 4], and can not, therefore, equal the rational quantity  $c - a - b$ .

**318. Definition.**—Two binomial surds of the form of  $a + \sqrt{b}$  and  $a - \sqrt{b}$  are called *conjugate quadratic surds*.

**319. Theo. VI.**—*The sum and the product of two conjugate quadratic surds are rational.*

**Demonstration.**—1.  $(a + \sqrt{b}) + (a - \sqrt{b}) = 2a$ , a rational sum.

2.  $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - (\sqrt{b})^2 = a^2 - b$ , a rational product.

**320. Theo. VII.**—*A quadratic surd can not be the sum or the difference of a rational and a surd quantity.*

**Demonstration.**—

If  $\sqrt{a}$  could be  $c \pm \sqrt{d}$ ,  $a$  would equal  $c^2 \pm 2c\sqrt{d} + d$ ; or,  $\pm 2c\sqrt{d}$  would equal  $a - c^2 - d$ , which is impossible, since a surd can not equal a rational quantity.

**321. Theo. VIII.**—*If  $a + \sqrt{b} = x + \sqrt{y}$  ( $\sqrt{b}$  and  $\sqrt{y}$  being irrational), then will  $a = x$  and  $\sqrt{b} = \sqrt{y}$ .*

**Demonstration.**—By transposing,  $\sqrt{b} - \sqrt{y} = x - a$ . This is only possible when  $\sqrt{b} = \sqrt{y}$  and  $x = a$ , else we would have the difference of two dissimilar quadratic surds a rational quantity, which is contrary to Theo. V.

**322. Theo. IX.**—If the sum and the product of  $a + \sqrt{b}$  and  $c + \sqrt{d}$  are both rational, then  $a = c$  and  $\sqrt{b} + \sqrt{d} = 0$ .

**Demonstration.**—1. If the sum of  $a + \sqrt{b}$  and  $c + \sqrt{d}$  is rational, then is  $a + c + \sqrt{b} + \sqrt{d}$  rational; but this is only possible when  $\sqrt{b} + \sqrt{d}$  is 0 [Theo. V].

2. If  $\sqrt{b} + \sqrt{d} = 0$ ,  $\sqrt{b} = -\sqrt{d}$ , and the product of  $a + \sqrt{b}$  and  $c + \sqrt{d}$ , or  $ac + a\sqrt{d} + c\sqrt{b} + \sqrt{b}d = ac - (a - c)\sqrt{b} - b$ , which can only be rational when  $(a - c)\sqrt{b}$  is rational, or when  $a = c$ .

### Miscellaneous Examples.

#### EXERCISE 50.

1. Simplify  $18\sqrt{\frac{5}{72}}$ ;  $\sqrt{\frac{x^3y^2}{4a^2b}}$ ;  $\sqrt[4]{48x^3y^6}$ ;  $\left(\frac{3}{4}\right)^{\frac{1}{2}}$
  2. Simplify  $(64a^3)^{\frac{1}{2}}$ ;  $\sqrt[4]{25x^4y^2}$ ;  $\{(a^2)^{-\frac{1}{2}}\}^{\frac{1}{2+1}}$
  3. Simplify  $6\sqrt[3]{4x^2} + \sqrt[3]{2x} + 2\sqrt[3]{8x^3} - 4\sqrt[3]{16x}$
  4. Simplify  $\sqrt[3]{\frac{27x^4y^4}{2b}} - \sqrt[3]{\frac{xy^4}{54b}}$
  5. From  $\sqrt{\frac{x^2y + 2xy^2 + y^3}{x^2 - 2xy + y^2}}$  take  $\sqrt{\frac{x^2y - 2xy^2 + y^3}{x^2 + 2xy + y^2}}$
  6. Show that  $2\sqrt[3]{3} \times \sqrt[3]{72} = 8\sqrt{6} \div \sqrt[3]{64}$
  7. Multiply  $a - \sqrt{ab} + b$  by  $a^{\frac{1}{2}} + b^{\frac{1}{2}}$
  8. Divide  $\frac{1}{2}\sqrt{\frac{1}{2}}$  by  $\sqrt{2} + 3\sqrt{\frac{1}{2}}$ ; and  $(64)^{\frac{1}{2}}$  by 2
  9. Which is the greater,  $4\sqrt{5}$  or  $7\sqrt[3]{3}$ ?
- Suggestion.**—Place coefficients under radical signs and reduce to a common index.
10. Arrange in order of magnitude  $3\sqrt{6}$ ,  $4\sqrt{5}$ , and  $5\sqrt{3}$
  11. Arrange in order of magnitude  $\sqrt{3}$ ,  $\sqrt[3]{5}$ , and  $\sqrt[4]{6}$

12. Arrange in order of magnitude

$$\frac{2}{3} \sqrt{\frac{2}{3}}, \frac{1}{3} \sqrt{\frac{4}{5}}, \text{ and } \frac{2}{5} \sqrt{\frac{3}{4}}$$

13. Simplify  $\frac{3\sqrt{5}}{4\sqrt{6}} \times \frac{7\sqrt{24}}{6\sqrt{20}} \div \frac{21\sqrt{75}}{6\sqrt{18}}$

14. Simplify  $\sqrt{\frac{xy^3}{z^2}} + \frac{1}{2z} \sqrt{x^3y - 4x^2y^2 + 4xy^3}$

15. Multiply  $4\sqrt{5+3\sqrt{2}}$  by  $6\sqrt{2}$ ;  $\sqrt{x-\sqrt{x}}$  by  $\sqrt{x+\sqrt{x}}$

16. Simplify  $(\sqrt[3]{\sqrt{a}-\sqrt{b}})^6$ ;  $\left(\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}}\right)^6$ ;  
 $\left(\frac{1}{2} - \sqrt{\frac{1}{2}}\right)^2 \times \left(\frac{1}{2} + \sqrt{\frac{1}{2}}\right)^2$

17. Simplify

$$\sqrt[3]{(a+b)\sqrt{a+b}}; \sqrt[3]{(3x^2-2y^2)^{\frac{2}{3}}}; \sqrt[3]{100\sqrt{27a^6b^9}}$$

18. Rationalize the denominator of  $\frac{\sqrt{x-2y} + \sqrt{x+2y}}{\sqrt{x+2y} - \sqrt{x-2y}}$

19. Simplify  $\frac{x+2\sqrt{x+y}}{x-2\sqrt{x+y}} + \frac{x-2\sqrt{x+y}}{x+2\sqrt{x+y}}$

20. Simplify  $\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} + \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}}$

21. Simplify  $\frac{1+\sqrt{-1}}{1-\sqrt{-1}} + \frac{1-\sqrt{-1}}{1+\sqrt{-1}}$

22. Expand  $(1 + \sqrt{-1})^5$ ;  $\left(\frac{3}{2} - \frac{3}{2}\sqrt{-3}\right)^4$ ;  
 $(-\sqrt{-1} - 2\sqrt{-2})^5$

23. Simplify  $\sqrt{7+2\sqrt{12}}$ ;  $\sqrt{19-4\sqrt{22}}$ ;  $\sqrt{-5+2\sqrt{6}}$

24. Divide  $x-y$  by  $\sqrt[5]{x} - \sqrt[5]{y}$ ;  $x+y$  by  $\sqrt[5]{x} + \sqrt[5]{y}$

25. Divide  $x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$  by  $x^{\frac{1}{4}} - x^{\frac{1}{8}}y^{\frac{1}{8}} + y^{\frac{1}{4}}$

26. Resolve into two binomial factors,

$$x - 2\sqrt{xy} + y; \quad x + 2\sqrt{xy} + y; \quad x^{\frac{1}{2}} - y^{\frac{1}{2}};$$

$$x + 8\sqrt{x} + 15; \quad x - 2\sqrt{x} - 15$$

27. Reduce  $\frac{x-y}{\sqrt{x}+\sqrt{y}}$ ;  $\frac{x+y}{\sqrt[3]{x}+\sqrt[3]{y}}$ ; and  $\frac{x+5\sqrt{x}+6}{x-\sqrt{x}-6}$   
to lowest terms.

28. Reduce  $\frac{1}{a-b}$ ,  $\frac{1}{\sqrt{a}-\sqrt{b}}$ , and  $\frac{1}{\sqrt{a}+\sqrt{b}}$   
to similar fractions.

29. Find the value of

$$\frac{1}{2x} + \frac{1}{2\sqrt{2x} + \sqrt{2x}} + \frac{1}{2\sqrt{2x} - \sqrt{2x}}$$

30. Of what number are  $5 + \sqrt{-2}$ ,  $5 - \sqrt{-2}$ ,  $3 + \sqrt{-1}$ ,  
and  $3 - \sqrt{-1}$  the factors?

31. Place the coefficients of the following expressions  
within the parentheses:  $x(a-x)^{\frac{1}{2}}$ ;  $x^2(x^{-2}+x)^{-\frac{3}{2}}$ ;  
 $x^{-\frac{3}{2}}(a+bx)^{-\frac{1}{2}}$ ;  $x^{\frac{m}{n}}(a-x)^{-\frac{p}{q}}$

32. Place the monomial factors in the following expressions  
before the parentheses:  $(ax-bx^2)^3$ ;  $(x^2+x)^{\frac{1}{2}}$ ;  
 $(ax^3-bx^2)^{-\frac{1}{2}}$ ;  $(x^{4n}-x^{2n})^{\frac{m}{n}}$ ;  $(x^{-5r}+x^{-10r})^{-\frac{p}{r}}$

33. Solve  $\frac{\sqrt{ax} + \sqrt{b}}{\sqrt{ax} - \sqrt{b}} = \frac{\sqrt{a} + \sqrt{b}}{\sqrt{b}}$ .

34. Solve  $\begin{cases} \sqrt{a-x} - \sqrt{y-x} = \sqrt{y} \\ \sqrt{b-x} + \sqrt{y-x} = \sqrt{y} \end{cases}$

35. Solve  $\begin{cases} \frac{1}{\sqrt{x-3}} - \frac{2}{\sqrt{y-2}} = \frac{1}{6} \\ \sqrt{y-2} = \frac{5}{2} \sqrt{x-3} \end{cases}$

36. Solve  $\frac{1}{\sqrt{x-p} - \sqrt{p}} + \frac{1}{\sqrt{x-p} + \sqrt{p}} = \sqrt{x-p}$

## CHAPTER V.

### *EQUATIONS OF THE SECOND DEGREE, INDETERMINATE EQUATIONS, AND INEQUALITIES.*

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#### Quadratic Equations of One Unknown Quantity.

##### Definitions.

**323.** A quadratic equation containing only the *second power* of an unknown quantity is a *pure*, or *incomplete*, quadratic equation ; as,  $3x^2 + 7 = 4x^2 + 2$ .

**324.** A quadratic equation containing both the second and first powers of an unknown quantity is an *affected*, or *complete*, quadratic equation ; as,  $3x^2 - 4x = 7$ .

**325.** Every pure quadratic equation may be reduced to the form of  $ax^2 = b$ , in which  $a$  and  $b$  are integral and  $a$  is positive.

(For method of proof, see Art. 248.)

**326.** Every affected quadratic equation may be reduced to the form of  $ax^2 + bx = c$ , in which  $a$ ,  $b$ , and  $c$  are integral and  $a$  is positive.

(For method of proof, see Art. 248.)

**327.** If we divide both members of the equation  $ax^2 + bx = c$  by  $a$ , we obtain

$$x^2 + \frac{b}{a}x = \frac{c}{a}.$$

If we now represent  $\frac{b}{a}$  by  $p$ , and  $\frac{c}{a}$  by  $q$ , we obtain

$$x^2 + px = q. \text{ Therefore,}$$

*Every affected quadratic equation may be reduced to the form of  $x^2 + px = q$ , in which  $p$  and  $q$  may be integral or fractional, positive or negative.*

**328.** The general form  $x^2 + px = q$  may evidently assume any one of the four following special forms :

$$1. x^2 + px = +q$$

$$3. x^2 + px = -q$$

$$2. x^2 - px = +q$$

$$4. x^2 - px = -q$$

### Solution of Quadratics.

**329.** To solve a pure quadratic equation, it is only necessary to reduce it to the normal form,  $ax^2 = b$ , then divide by the coefficient of  $x^2$  and extract the square root of both members. Thus,

Given  $\frac{x^2}{a} - \frac{3ax^2}{c} = b$ , to find the values of  $x$ .

Clearing of fractions,  $cx^2 - 3a^2x^2 = abc$

Factoring,  $(c - 3a^2)x^2 = abc$

Dividing,  $x^2 = \frac{abc}{c - 3a^2}$

Extracting  $\sqrt{\phantom{x}}$ ,  $x = \pm \sqrt{\frac{abc}{c - 3a^2}}$

**330.** Sometimes affected quadratics may be put in the form of pure quadratics with compound terms. Thus,

Given  $x^2 + 2ax + a^2 + 8b^2 = 3x^2 + 6ax + 3a^2$  to find the values of  $x$ .

Factor,  $(x + a)^2 + 8b^2 = 3(x + a)^2 \quad (1)$

Transpose, etc.,  $2(x + a)^2 = 8b^2$

Divide by 2,  $(x + a)^2 = 4b^2$

Extract  $\sqrt{\phantom{x}}$ ,  $x + a = \pm 2b$

Transpose,  $x = -a \pm 2b$

**331.** To solve a complete quadratic equation :

1. Reduce it to the form of  $ax^2 + bx = c$ .
2. Multiply or divide both members by any quantity that will render the first term a perfect square.
3. Add to both members such a quantity as will make the first member the square of a binomial.
4. Extract the square root of both members, thereby reducing the equation to two simple equations.
5. Solve the simple equations.

**Illustration.**—Solve  $8x^2 + 5x = 42$ .

Multiply both members by 2,

$$16x^2 + 10x = 84 \quad (1)$$

Regard  $16x^2 + 10x$  the first two terms of the square of a binomial; then  $\sqrt{16x^2}$ , or  $4x$ , is the first term of the binomial, and  $10x$  is twice the product of the two terms of the binomial; hence,  $10x$  divided by  $2 \times 4x$ , or  $\frac{5}{4}$ , is the second term of the binomial, and  $\left(\frac{5}{4}\right)^2$ , or  $\frac{25}{16}$ , is the third term of the square of the binomial; add this to both members of (1),

$$16x^2 + 10x + \left(\frac{5}{4}\right)^2 = 84 + \frac{25}{16} = \frac{1369}{16} \quad (2)$$

$$\text{Extract } \sqrt{\quad}, \quad 4x + \frac{5}{4} = \pm \frac{37}{4} \quad (3)$$

$$\text{Reduce} \quad 4x + \frac{5}{4} = + \frac{37}{4}$$

$$x = 2$$

$$\text{Reduce} \quad 4x + \frac{5}{4} = - \frac{37}{4}$$

$$x = -2\frac{5}{8}.$$

Either of these roots will satisfy the original equation :

For, put 2 for  $x$  in (1); then will  $32 + 10 = 42$ , which is correct.

Put  $-2\frac{5}{8}$ , or  $-\frac{21}{8}$  for  $x$  in (1); then will  $55\frac{1}{8} - 13\frac{1}{8} = 42$ , which is also correct.

**332.** The process of making the first member a perfect square is called "Completing the square."

## Methods of Completing the Square.

**333. The General Method.**—This is the method employed in the above example, and may be stated as follows :

*Rule.*—Multiply or divide both members of the equation by a quantity that will make the first term a square ; then divide the coefficient of the second term by twice the square root of the coefficient of the first term, and add the square of the quotient to both members.

**334. Special Methods.**—

1. *The Common Method.*—If the equation be divided through by the coefficient of  $x^2$ , thereby putting it in the form of  $x^2 + px = q$ , the quantity to be added to both members is always the square of half the coefficient of  $x$ ,  $\left(\frac{p}{2}\right)^2$ .

The objection to this method is that it often gives rise to unwieldy fractions.

2. *The Hindoo Method.*—If the equation reduced to the form  $ax^2 + bx = c$  be multiplied through by 4 times the coefficient of  $x^2$  ( $4a$ ), the quantity to be added to both members is always the square of the coefficient of  $x$  in the original equation ( $b^2$ ). This method avoids fractions, but often gives rise to very large numbers.

3. *The Eclectic Method.*—If the equation reduced to the form  $ax^2 + bx = c$  be multiplied through by the coefficient of  $x^2$  ( $a$ ), the quantity to be added to both members is always the square of half the coefficient of  $x$  in the original equation,  $\left(\frac{b}{2}\right)^2$ . This method avoids the large fractions of the common method and the large numbers of the Hindoo method. When the coefficient of the second term is even, the equation will remain integral ; when odd, the denominator of the fractions produced will always be 4.

**Illustrations.**—1. Solve  $ax^2 + bx = c$ .

Multiply by  $4a$ ,  $4a^2x^2 + 4abx = 4ac$

Complete the square by the Hindoo method,

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2$$

Extract the square root,  $2ax + b = \pm \sqrt{4ac + b^2}$

Transpose  $b$ ,  $2ax = -b \pm \sqrt{4ac + b^2}$

Divide by  $2a$ ,  $x = \frac{1}{2a}(-b \pm \sqrt{4ac + b^2})$

2. Solve  $(a - b)x^2 - (a + b)x = a$ .

Multiply by  $(a - b)$ ,

$$(a - b)^2x^2 - (a^2 - b^2)x = a^2 - ab$$

Complete the square by the eclectic method [334, 3],

$$(a - b)^2x^2 - \left(\frac{a + b}{2}\right)^2 = \frac{4a^2 - 4ab}{4} + \frac{a^2 + 2ab + b^2}{4} = \frac{1}{4}(5a^2 - 2ab + b^2)$$

Extract the square root,

$$(a - b)x - \frac{a + b}{2} = \pm \frac{1}{2} \sqrt{5a^2 - 2ab + b^2}$$

Transpose  $-\frac{a + b}{2}$ ,  $(a - b)x = \frac{a + b}{2} \pm \frac{1}{2} \sqrt{5a^2 - 2ab + b^2}$

$$(a - b)x = \frac{1}{2}(a + b \pm \sqrt{5a^2 - 2ab + b^2})$$

Divide by  $a - b$ ,  $x = \frac{1}{2(a - b)}(a + b \pm \sqrt{5a^2 - 2ab + b^2})$

#### EXERCISE 31.

Solve :

1.  $x^2 + 2ax = 3a^2$

8.  $6x^2 - 13x = 85$

2.  $3x^2 + 5x = 22$

9.  $20x^2 + 31x = -42$

3.  $ax^2 - bx = -c$

10.  $mx^2 - nx = r$

4.  $\frac{x^2 + 4}{3} = 1 - \frac{3x^2 - 47}{6}$

11.  $\frac{x^2}{(a + x)^2} = \frac{b^2}{(a - b)^2}$

5.  $\frac{6x - 5}{3} - \frac{3x^2}{6x - 5} = 5\frac{1}{3}$

12.  $\frac{x + 9}{x - 9} + \frac{x - 9}{x + 9} = 4$

6.  $\frac{x}{a} + \frac{a}{x} = \frac{x + a}{ax}$

13.  $\frac{x + a}{x - a} - \frac{x - a}{x + a} = a$

7.  $(2x - a)^2 - 8b^2 = 2(2x - a)^2$

$$14. \frac{15}{x+2} - \frac{15}{x-2} = \frac{3}{x^2-4} - 5$$

$$15. (x-a)(x-b) = \frac{3}{4}(a-b)^2$$

$$16. (x-c)(a-b) = (x-a)(x-b)$$

$$17. (p-q)x^2 + (p+q)x = -q$$

$$18. abx^2 - (a+b)(ab+1)x = -(a+b)^2$$

$$19. x^2 + ax - bx - cx = c(a-b)$$

$$20. \frac{a-x}{b-x} + \frac{a+x}{b+x} = c$$

$$23. \frac{x+2}{x-7} - \frac{x-2}{x+8} = -\frac{4}{5}$$

$$21. \frac{c}{x} + \frac{1}{x}\sqrt{c^2-x^2} = \frac{x}{c}$$

$$24. \frac{x+a}{x} + \frac{x}{x+a} = b$$

$$22. \frac{x+1}{x+3} + \frac{x+3}{x+1} = \frac{58}{21}$$

$$25. \frac{1}{3x+1} - \frac{2}{2-5x} = 2$$

$$26. x + \sqrt{x^2-a^2} = \frac{a^2}{\sqrt{x^2-a^2}}$$

$$27. \sqrt{\frac{x-a}{x+a}} + \sqrt{\frac{x+a}{x-a}} = a^2$$

$$28. \frac{\sqrt{x^2+1} - \sqrt{x^2-1}}{\sqrt{x^2+1} + \sqrt{x^2-1}} = \frac{1}{2}$$

$$29. \frac{1}{x + \sqrt{a-x^2}} + \frac{1}{x - \sqrt{a-x^2}} = 2x$$

$$30. \frac{a}{a^2-x^2} + \frac{b}{a+x} = \frac{b}{x-a} + a$$

$$31. (x-2)(x+3) + (x-3)(x+4) = (x+4)(x-5)$$

$$32. \frac{x^4+x}{x^4+x^3+1} - \frac{1}{x^3+x+1} = 1 - \frac{1}{x^3-x+1}$$

$$33. \frac{x^2-4x-12}{x^2+8x+12} = \frac{x^2-7x+12}{x^2+x-12} - \frac{1}{5}$$

$$34. \frac{1}{m+n+x} = \frac{1}{m} + \frac{1}{n} + \frac{1}{x}$$

$$35. (a+2b)^2 x^2 + 4(a+2b)x = 12$$

$$36. \frac{x^2 - a^2}{x+a} - \frac{x^2 - b^2}{x-b} = (x-a)(x+b)$$

$$37. \frac{4x^2 - a^2}{2x+a} - \frac{2x-a}{4x^2 - a^2} = (2x+a) + \frac{1}{2}x$$

$$38. \frac{x^2 + (a+b)x + ab}{x^2 + (a-b)x - ab} = a - \frac{x^2 + (a-b)x - ab}{x^2 + (a+b)x + ab}$$

$$39. (a+b+c)x^2 + (a+b-c)x = a-b+c$$

$$40. \frac{x^2+x}{x^2-x} + \frac{x^2-x}{x^2+x} = \frac{a^2+a}{a^2-a} + \frac{a^2-a}{a^2+a}$$

$$41. \frac{2x^2+x-3}{2x^2+5x+3} + \frac{9x^2-4}{3x^2+x-2} = x+5$$

$$42. \frac{x^2+5x+6}{x+4} - \frac{x^2+x-20}{x+5} = x$$

$$43. \frac{x}{x^2-4} + \frac{2x}{4x^2-9} = 2 - \frac{3}{4x^2-9} - \frac{2}{x^2-4}$$

### Solution of Equations in the Quadratic Form.

**335.** An equation has the quadratic form when it has only two terms containing the unknown quantity, and the degree of one of them is double that of the other; as,  $x^{2p} + ax^p = q$ ,  $(a+bx)^{2n} + p(a+bx)^n = q$ , or  $(x+a) + \sqrt{x+a} = b$ .

**336.** An equation in the quadratic form may be reduced to two equations of half the degree by completing the square, and extracting the square root of both members.

**Illustrations.**—1. Solve  $x^4 + 4x^2 = 32$ .

Complete the square,  $x^4 + 4x^2 + 4 = 36$

Extract the  $\sqrt{\phantom{x}}$ ,  $x^2 + 2 = \pm 6$

Transpose,  $x^2 = 4$  or  $-8$

Extract the  $\sqrt{\phantom{x}}$ ,  $x = \pm 2$  or  $\pm 2\sqrt{-2}$ .

2. Solve  $x - 2\sqrt{x} = 3$ .

Complete the square,

$$x - 2\sqrt{x} + 1 = 4$$

Extract the  $\sqrt{\phantom{x}}$ ,  $x^{\frac{1}{2}} - 1 = \pm 2$

Transpose,  $x^{\frac{1}{2}} = 3$  or  $-1$

Square,  $x = 9$  or  $1$ .

3. Solve  $x^2 + 2x + 4\sqrt{x^2 + 2x + 6} = 6$ .

Add 6,

$$(x^2 + 2x + 6) + 4\sqrt{x^2 + 2x + 6} = 12$$

Complete the square,

$$(x^2 + 2x + 6) + 4\sqrt{x^2 + 2x + 6} + 4 = 16$$

Extract the  $\sqrt{\phantom{x}}$ ,  $\sqrt{x^2 + 2x + 6} + 2 = \pm 4$

Transpose,  $\sqrt{x^2 + 2x + 6} = 2$  or  $-6$

Square,  $x^2 + 2x + 6 = 4$  or  $36$

Transpose,  $x^2 + 2x = -2$  or  $30$

Reduce,  $x = -1 \pm \sqrt{-1}$  or  $-1 \pm \sqrt{31}$ .

#### EXERCISE 52.

Solve :

1.  $x^4 - 6x^2 = 27$

7.  $3x^{-\frac{1}{2}} - 5x^{-\frac{3}{2}} = -\frac{17}{16}$

2.  $x^6 + 2x^3 = 48$

8.  $\sqrt[3]{x^3} + 3\sqrt[3]{x} = 18$

3.  $2x + 3x^{\frac{1}{2}} = 14$

9.  $x\sqrt{x} + \sqrt[4]{x^3} = 72$

4.  $x^3 + 3x^{\frac{3}{2}} = 40$

10.  $x^{2a} + px^a = q$

5.  $4x^{\frac{2}{3}} - 2x^{\frac{1}{3}} = 12$

11.  $(x+2)^2 + 4(x+2) = 45$

6.  $x^{-2} + 4x^{-1} = 5$

12.  $x + 4 + \sqrt{x+4} = 20$

13.  $(x^2 + x)^2 - 3(x^2 + x) = 108$

14.  $\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) = 17\frac{7}{9}$

$$15. 2x^2 + 6 + 3\sqrt{2x^2 + 6} = 10$$

$$16. x^2 + x - 4\sqrt{x^2 + x + 4} = -4$$

$$17. 3x^2 - 2\sqrt{3x^2 + 2x - 7} = 66 - 2x$$

$$18. 4x + \frac{4}{x} - 8\sqrt{x + \frac{1}{x}} = -4$$

$$19. x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) = 9\frac{1}{4}$$

$$20. (a + bx)^{2n} + p(a + bx)^n = q$$

### Solution of Quadratic Equations by Substitution.

337. In the general equation  $x^2 + px = q$ ,

$$x = \begin{cases} 1. -\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q} \\ 2. -\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q} \end{cases}$$

in which  $p^2$  is essentially positive.  $p$  and  $q$  may be either positive or negative.

338. Equations reduced to the form  $x^2 + px = q$  may be readily solved by substitution in the above formulas.

**Illustrations.**—1. Solve  $x^2 + 4x = 5$ .

Here  $p = 4$  and  $q = 5$ ; hence,

$$x = \begin{cases} 1. -2 + \sqrt{4 + 5} = 1 \\ 2. -2 - \sqrt{4 + 5} = -5 \end{cases}$$

2. Solve  $3x^2 - 4x = 15$ .

Divide by 3,  $x^2 - \frac{4}{3}x = 5$ . Here  $p = -\frac{4}{3}$  and  $q = 5$ ; hence,

$$x = \begin{cases} 1. \frac{2}{3} + \sqrt{\frac{4}{9} + 5} = 3 \\ 2. \frac{2}{3} - \sqrt{\frac{4}{9} + 5} = -1\frac{2}{3} \end{cases}$$

3. Solve  $5x^2 + 8x = -3$ .

Divide by 5,  $x^2 + \frac{8}{5}x = -\frac{3}{5}$ . Here  $p = \frac{8}{5}$  and  $q = -\frac{3}{5}$ ;  
hence,

$$x = \begin{cases} 1. -\frac{4}{5} + \sqrt{\frac{16}{25} - \frac{3}{5}} = -\frac{8}{5} \\ 2. -\frac{4}{5} - \sqrt{\frac{16}{25} - \frac{3}{5}} = -1 \end{cases}$$

## EXERCISE 58.

Solve by substitution :

- |                         |  |
|-------------------------|--|
| 1. $x^2 + 3x = 40$      | 9. $x^2 - \frac{5}{6}ax = -\frac{1}{6}a^2$ |
| 2. $3x^2 - 5x = 12$     | 10. $12x^2 + 4x = 5$                       |
| 3. $5x^2 + 15x = -10$   | 11. $x^2 - 2x = -2$                        |
| 4. $3x^2 - 9x = -6$     | 12. $x^2 - 2ax = a - a^2$                  |
| 5. $7x^2 + 6x = 8$      | 13. $4x^2 - 4x = -3$                       |
| 6. $x^2 - 4x = -1$      | 14. $4x^2 + 4x = -13$                      |
| 7. $x^2 - (a+b)x = -ab$ | 15. $4x^2 - 4ax = b^2 - a^2$               |
| 8. $x^2 - 2x = a - 1$   | 16. $16x^2 + 24x = -39$                    |

### Formation and Solution of Quadratic Equations by Inspection.

339. Since the sum of the two roots

$$-\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q} \text{ and } -\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q}$$

is  $-p$ , and their product is  $-q$ , it follows that,

*Prin. 1.*—The sum of the two roots of an equation of the form of  $x^2 + px = q$  equals the coefficient of  $x$  with the sign changed, and their product equals the known term with the sign changed.

This principle will enable us to form quadratic equations with given roots and solve them by inspection.

**Illustrations.—**

1. Form the equation whose roots are  $+4$  and  $-2$ .

**Solution:** The sum of  $+4$  and  $-2$  is  $+2$ , and their product is  $-8$ ; therefore, the equation is  $x^2 - 2x = 8$  [P. 1].

2. Solve by inspection the equation  $x^2 + 2x = 15$ .

**Solution:** The sum of the two roots is  $-2$ , and their product is  $-15$  [P. 1]. The roots are evidently  $-5$  and  $3$ .

**EXERCISE 84.**

Form the equation whose roots are :

1. 3 and 4

10.  $a$  and 0

2.  $-3$  and  $-4$

11.  $\sqrt{2}$  and  $-\sqrt{2}$

3. 3 and  $-4$

12.  $\sqrt{-1}$  and  $-\sqrt{-1}$

4. 4 and  $-3$

13.  $2 + \sqrt{3}$  and  $2 - \sqrt{3}$

5.  $2\frac{1}{2}$  and  $\frac{1}{2}$

14.  $12\frac{1}{2}$  and  $6\frac{1}{4}$

6.  $3\frac{1}{2}$  and  $-2$

15.  $1 + \sqrt{-1}$  and  $1 - \sqrt{-1}$

7.  $a$  and  $2a$

16.  $a + b$  and  $a - b$

8.  $3b$  and  $-2b$

17.  $3 + \sqrt{-2}$  and  $3 - \sqrt{-2}$

9.  $-\frac{2}{3}$  and  $-\frac{3}{4}$

18.  $\frac{2}{3} + \sqrt{\frac{2}{3}}$  and  $\frac{2}{3} - \sqrt{\frac{2}{3}}$

Solve by inspection :

19.  $x^2 - 4x = -4$

24.  $x^2 - 6ax = -9a^2$

20.  $x^2 - 3x = 18$

25.  $x^2 + (b - a)x = ab$

21.  $x^2 + 5x = 14$

26.  $x^2 - 2ax = -(a^2 - b^2)$

22.  $x^2 + 5x = -6$

27.  $x^2 - \frac{5}{2}x = -1$

23.  $x^2 + 7x = -12$

28.  $x^2 - ax = -\frac{a^2}{4}$

# Character of the Roots of the Quadratic Equation.

340. By an inspection of the two roots

$$-\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q} \text{ and } -\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q},$$

it will readily appear that,

1. If  $q$  is positive both roots are real.

For  $\frac{1}{4}p^2 + q$  is positive, whence  $\sqrt{\frac{1}{4}p^2 + q}$  is real.

2. If  $q$  is negative and numerically less than  $\frac{1}{4}p^2$  both roots are real.

For  $\frac{1}{4}p^2 + (-q)$  is positive, whence  $\sqrt{\frac{1}{4}p^2 + (-q)}$  is real.

3. If  $q$  is negative and numerically greater than  $\frac{1}{4}p^2$  both roots are imaginary.

For  $\frac{1}{4}p^2 + (-q)$  is negative, whence  $\sqrt{\frac{1}{4}p^2 + (-q)}$  is imaginary.

4. If  $q$  is negative and numerically equal to  $\frac{1}{4}p^2$  both roots are  $-\frac{1}{2}p$ .

For  $\frac{1}{4}p^2 + (-q)$  is 0, whence  $\sqrt{\frac{1}{4}p^2 + (-q)}$  is 0.

5. If  $\frac{1}{4}p^2 + q$  is a perfect square both roots are rational.

For both terms of each root will be rational.

6. If  $\frac{1}{4}p^2 + q$  is positive and not a perfect square both roots are surds.

For the second term of each root is a surd.

7. If  $p$  and  $q$  are both positive, the roots have opposite signs and the negative is numerically the greater.

For  $\sqrt{\frac{1}{4}p^2 + q} > \sqrt{\frac{1}{4}p^2}$  or  $\frac{1}{2}p$ ; therefore,

$-\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q}$  is positive [72, P. 3],

and  $-\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q}$  is negative [71, P. 2].

The negative is numerically the greater, since the arithmetical sum of two numbers is greater than their arithmetical difference.

8. If  $p$  is negative and  $q$  positive, the roots have opposite signs and the positive is numerically the greater.

For the two roots will then be

$$-\frac{1}{2}(-p) + \sqrt{\frac{1}{4}(-p)^2 + q} = \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q},$$

which is positive

$$\text{and } -\frac{1}{2}(-p) - \sqrt{\frac{1}{4}(-p)^2 + q} = \frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q},$$

which is negative.

The positive is numerically the greater, since the arithmetical sum is greater than the arithmetical difference.

9. If  $p$  is positive and  $q$  is negative and both roots are real, they are both negative.

For  $\sqrt{\frac{1}{4}p^2 + (-q)}$ , or  $\sqrt{\frac{1}{4}p^2 - q} < \sqrt{\frac{1}{4}p^2}$ , or  $\frac{1}{2}p$ ;

therefore,  $-\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$  is negative

and  $-\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$  is negative.

10. If  $p$  and  $q$  are both negative and both roots are real, they are both positive.

For the two roots will then be

$$-\frac{1}{2}(-p) + \sqrt{\frac{1}{4}(-p)^2 + (-q)} = \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q},$$

which is positive and

$$-\frac{1}{2}(-p) - \sqrt{\frac{1}{4}(-p)^2 + (-q)} = \frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q},$$

which is positive.

**341.** The above principles will enable the student to determine by inspection the character of the roots of any quadratic equation of the form of  $x^2 + px = q$ .

The following order of inquiry is recommended :

1. Are the roots real or imaginary ?
2. Are they rational or irrational ?
3. What are their signs and relative values ?

**Illustrations.**—1. Determine the character of the roots of the equation  $x^2 + 7x = -6$ .

**Solution :** Here  $p = 7$  and  $q = -6$ ; hence

$$\sqrt{\frac{1}{4}p^2 + q} = \sqrt{\frac{49}{4} - \frac{24}{4}} = \sqrt{\frac{25}{4}}$$

is real and rational. Both roots are negative [340, 9].

2. Determine the character of the roots of

$$x^2 - 4x = -8.$$

**Solution :** Here  $p = -4$  and  $q = -8$ ; hence

$$\sqrt{\frac{1}{4}p^2 + q} = \sqrt{4 - 8} = \sqrt{-4}$$

is imaginary; whence both roots are imaginary [340, 3].

#### EXERCISE 85.

Determine the character of the roots of:

1.  $x^2 + 6x = 16$

5.  $x^2 + 7x = 9$

2.  $x^2 - 8x = 9$

6.  $x^2 + 5x = -7$

3.  $x^2 + 4x = -3$

7.  $x^2 - 6x = -15$

4.  $x^2 - 6x = -5$

8.  $x^2 - 7x = 10$

- |  |                         |
|--|-------------------------|
| 9. $x^2 - 8x = -20$                                | 15. $4x^2 + 7x = -20$   |
| 10. $x^2 + ax = 2a^2$                              | 16. $5x^2 - 3x = 14$    |
| 11. $3x^2 + 7x = 2$                                | 17. $7x^2 - 5x = 0$     |
| 12. $2x^2 - 3x = 5$                                | 18. $9x^2 + 6x = -1$    |
| 13. $4x^2 - 8x = 10$                               | 19. $4x^2 + 8x = -10$   |
| 14. $2x^2 - 3x = -30$                              | 20. $64x^2 - 80x = -25$ |
| 21. $\frac{1}{2}x^2 - \frac{1}{3}x = -\frac{5}{6}$ |                         |

### Solution of Equations by Factoring.

**342.** Any trinomial of the form of  $x^2 + px + q$  or  $ax^2 + bx + c$  is called a *Quadratic Trinomial*.

Every quadratic trinomial may be factored. Thus,

$$1. \quad x^2 + px + q = x^2 + px + \frac{p^2}{4} - \frac{p^2}{4} + q =$$

$$\left(x + \frac{p}{2}\right)^2 - \left(\frac{p^2}{4} - q\right) = \left(x + \frac{p}{2}\right)^2 - \left(\sqrt{\frac{p^2}{4} - q}\right)^2 =$$

$$\left(x + \frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right)\left(x + \frac{p}{2} - \sqrt{\frac{p^2}{4} - q}\right) \quad (\text{A}).$$

$$2. \quad ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) =$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{4ac}{4a^2}\right) =$$

$$a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2\right\} =$$

$$a\left(x + \frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}}\right)\left(x + \frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}}\right) \quad (\text{B}).$$

**343.** Let it be required to solve the quadratic equations:

$$1. \quad x^2 + px + q = 0$$

$$2. \quad ax^2 + bx + c = 0$$

$$1. \quad x^2 + px + q = 0.$$

Factor (v. A),

$$\left(x + \frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right) \left(x + \frac{p}{2} - \sqrt{\frac{p^2}{4} - q}\right) = 0 \quad (1)$$

Equation (1) may evidently be satisfied by placing either factor equal to zero and in no other way; therefore,

$$x + \frac{p}{2} + \sqrt{\frac{p^2}{4} - q} = 0$$

$$\text{whence, } x = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}$$

$$\text{Also, } x + \frac{p}{2} - \sqrt{\frac{p^2}{4} - q} = 0$$

$$\text{whence, } x = -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}.$$

$$2. \quad \text{Given } ax^2 + bx + c = 0.$$

Factor (v. B),

$$a \left(x + \frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}}\right) \left(x + \frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}}\right) = 0 \quad (1)$$

$$\therefore x + \frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}} = 0$$

$$\text{whence, } x = -\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}} = -\frac{1}{2a}(b + \sqrt{b^2 - 4ac})$$

$$\text{Also, } x + \frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}} = 0$$

$$\text{whence, } x = -\frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}} = -\frac{1}{2a}(b - \sqrt{b^2 - 4ac}).$$

**Illustrations.**—1. Solve  $x^2 - 3x - 10 = 0$ .

$$\text{Factor, } (x + 2)(x - 5) = 0$$

$$\text{whence, } x + 2 = 0, \text{ or } x - 5 = 0$$

$$\therefore x = -2 \text{ or } +5.$$

$$2. \quad \text{Solve } x^3 + 3x^2 - 4x - 12 = 0.$$

$$\text{Factor, } x^2(x + 3) - 4(x + 3) = 0$$

$$\text{or, } (x^2 - 4)(x + 3) = 0$$

$$\text{or, } (x + 2)(x - 2)(x + 3) = 0$$

$$\text{whence, } x + 2 = 0, x - 2 = 0, \text{ or } x + 3 = 0$$

$$\therefore x = -2, +2, \text{ or } -3.$$

3. Solve  $x^2 - x + 1 = 0$ .

Apply formula (A) [342],  $p = -1$  and  $q = 1$

$$\therefore x^2 - x + 1 = \left(x - \frac{1}{2} + \sqrt{\frac{1}{4} - 1}\right) \left(x - \frac{1}{2} - \sqrt{\frac{1}{4} - 1}\right) = 0$$

whence,  $x - \frac{1}{2} + \sqrt{\frac{1}{4} - 1} = 0$

or,  $x = +\frac{1}{2} - \sqrt{\frac{1}{4} - 1} = +\frac{1}{2}(1 - \sqrt{-3})$

and  $x - \frac{1}{2} - \sqrt{\frac{1}{4} - 1} = 0$

or,  $x = \frac{1}{2} + \sqrt{\frac{1}{4} - 1} = \frac{1}{2}(1 + \sqrt{-3}).$

4. Solve  $3x^2 - 4x + 7 = 0$ .

Apply formula (B) [342],  $a = 3$ ,  $b = -4$ ,  $c = 7$

$$3x^2 - 4x + 7 = 3\left(x - \frac{2}{3} + \sqrt{\frac{16-84}{36}}\right)\left(x - \frac{2}{3} - \sqrt{\frac{16-84}{36}}\right) = 0$$

whence,  $x - \frac{2}{3} \pm \sqrt{-\frac{17}{9}} = 0$

$$\therefore x = \frac{2}{3} \mp \sqrt{-\frac{17}{9}} = \frac{1}{3}(2 \mp \sqrt{-17}).$$

5. Solve  $x^3 - 1 = 0$ .

Factor,  $(x^3 + 1)(x^3 - 1) = 0$

or,  $(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1) = 0$

$\therefore x \pm 1 = 0$ , or  $x = \mp 1$ ,

$$x^2 - x + 1 = 0, \text{ or } x = \frac{1}{2}(1 \pm \sqrt{-3}),$$

and  $x^2 + x + 1 = 0$ , or  $x = -\frac{1}{2}(1 \pm \sqrt{-3}).$

6. Solve  $x^4 + x^2 + 1 = 0$ .

Factor,  $(x^3 + x + 1)(x^3 - x + 1) = 0$

whence,  $x^3 + x + 1 = 0$ , or  $x = -\frac{1}{2}(1 \pm \sqrt{-3})$

and  $x^3 - x + 1 = 0$ , or  $x = \frac{1}{2}(1 \pm \sqrt{-3}).$

7. Solve  $x^4 + 8x^3 + 24x^2 + 32x + 12 = 0$ .

Find by extracting  $\sqrt{\phantom{x}}$ , that  $(x^2 + 4x + 4)^2 - 4 = 0$ .

Factor,  $(x^2 + 4x + 2)(x^2 + 4x + 6) = 0$

whence,  $x^2 + 4x + 2 = 0$ , or  $x = -2 \pm \sqrt{2}$

and  $x^2 + 4x + 6 = 0$ , or  $x = -2 \pm \sqrt{-2}.$

EXERCISE 56.

Solve :

1.  $x^4 - 4 = 0$
  2.  $x^3 - 8 = 0$
  3.  $x^3 + 8 = 0$
  4.  $x^2 + 8x + 15 = 0$
  5.  $x^2 + 3x - 28 = 0$
  6.  $x^2 - 13x = -40$
  7.  $x^4 - 5x^2 = -4$
  8.  $x^4 - 25x^2 = -144$
  9.  $6x^2 + 11x = 10$
  10.  $9x^3 - 6x^2 - 24x = 0$
  11.  $x^4 - 2 + \frac{1}{x^4} = 0$
  12.  $x^4 + 4x^2 + 16 = 0$
  13.  $x^4 + 2x^2 = -9$
  14.  $8x^3 - 27 = 0$
  15.  $16x^4 = 256$
  16.  $x^3 + 3x^2 - 9x - 27 = 0$
  17.  $x^4 + 2x^3 + x + 2 = 0$
  18.  $x^5 - 4x^3 + x^2 = 4$
  19.  $x^4 - 4x^2 - a^2x^2 = -4a^2$
  20.  $4x^3 - 2x^2 - 36x = -18$
  21.  $x^3 + 6x^2 + 12x + 8 = 0$
  22.  $x^4 + 4x^2 + \frac{4}{x^2} + \frac{1}{x^4} = -6$
  23.  $x^4 + 2x^3 - 3x^2 - 4x + 4 = 0$
  24.  $x^4 - 12x^3 + 54x^2 - 108x = -72$
  25.  $x^4 + 1$
- Suggestion.**  $x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2 = (x^2 + 1)^2 - (x\sqrt{2})^2$ .
26. Find the three cube roots of 1
  27. Find the three cube roots of  $-1$
  28. Find the four fourth roots of 1 ;  $-1$
  29. Find the five fifth roots of 1 ;  $-1$
  30. Solve  $x^5 - 1 = 0$
  31. Solve  $x^5 + 1 = 0$
  32. Solve  $x^4 - 16 = 0$
  33. Solve  $x^4 + 16 = 0$
  34. Solve  $x^5 + a^5 = 0$
  35. Solve  $x^5 - a^5 = 0$
  36. Solve  $x^4 - \frac{1}{81} = 0$
  37. Solve  $x^4 + \frac{1}{81} = 0$

38. Solve  $x^4 + 2x^3 + 3x^2 + 2x - 3 = 0$

39. Solve  $x^3 + 30x^2 + 300x + 875 = 0$

40. Solve  $8x^5 - 12x^6 + 6x^3 + 7 = 0$

41. Solve  $4x^4 + 20ax^3 + 29a^2x^2 + 10a^3x - 3a^4 = 0$

42. Solve  $27x^3 - 135ax^2 + 225a^2x - 61a^3 = 0$

### Simultaneous Equations solvable as Quadratics.

344. The general form of an equation of the second degree containing two unknown quantities is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

345. A *Homogeneous Equation* is one in which all the terms containing unknown quantities are of the same degree ; as,

$$ax^2 + bxy + cy^2 = d$$

$$x^{2n} + x^n y^n + y^{2n} = a.$$

346. A *Symmetrical Equation* is one in which the two unknown quantities may change places without affecting the equality ; as,

$$5x^2 - 10xy + 5y^2 = 12, \text{ and}$$

$$5y^2 - 10yx + 5x^2 = 12.$$

347. The solution of two simultaneous equations of the second degree frequently depends upon the solution of a bi-quadratic equation.

**Illustration.**—Let it be required to solve

$$x^2 + y = 3 \quad (\text{A})$$

$$x + y^2 = 4 \quad (\text{B})$$

Transpose (A),  $y = 3 - x^2$  (1)

Square (1),  $y^2 = 9 - 6x^2 + x^4$  (2)

Substitute (2) in (B),

$$x + 9 - 6x^2 + x^4 = 4, \text{ an equation of the fourth degree.}$$

**348.** When one equation is of the first degree and the other of the second degree they are solvable as quadratics.

**Illustration.**—Solve  $\begin{cases} x + 2y = 6 \\ x^2 + xy = 8 \end{cases}$  (A)  
(B)

Transpose equation (A),  $2y = 6 - x$  (1)

Divide by 2,  $y = \frac{6-x}{2}$  (2)

Substitute (2) in (B),  $x^2 + x\left(\frac{6-x}{2}\right) = 8$  (3)

Clear of fractions,  $2x^2 + 6x - x^2 = 16$  (4)

Collect terms,  $x^2 + 6x = 16$  (5)

Reduce,  $x = -8, \text{ or } 2$  (6)

Substitute (6) in (2),  $y = 7 \text{ or } 2$

Therefore, when  $x = -8, y = 7$ ; and when  $x = 2, y = 2$ .

**I.** Many equations of this class may be solved by more elegant methods.

**Illustration.**—Solve  $\begin{cases} x + y = 5 \\ xy = 6 \end{cases}$  (A)  
(B)

Square (A),  $x^2 + 2xy + y^2 = 25$  (1)

Multiply (B) by (4),  $4xy = 24$  (2)

Subtract (2) from (1),  $x^2 - 2xy + y^2 = 1$  (3)

Extract  $\sqrt{\phantom{x}}$ ,  $x - y = \pm 1$  (4)

Solve  $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$

$x = 3$  and  $y = 2$ .

Solve  $\begin{cases} x + y = 5 \\ x - y = -1 \end{cases}$

$x = 2$  and  $y = 3$ .

In a similar manner may be solved equations of the form of  $x - y = a$  and  $xy = b$ ;

$$x^2 \pm y^2 = a \text{ and } xy = b;$$

$$x^2 \pm y^2 = a \text{ and } x \pm y = b;$$

$$x^2 + xy + y^2 = a \text{ and } x^2 - xy + y^2 = b.$$

**II.** Some equations of a higher degree may be reduced to this class by division.

$$\text{Illustration.}—\text{Solve } \begin{cases} x^3 + y^3 = 35 \\ x + y = 5 \end{cases} \quad \begin{matrix} \text{(A)} \\ \text{(B)} \end{matrix}$$

$$\text{Divide (A) by (B),} \quad x^3 - xy + y^3 = 7 \quad (1)$$

$$\text{Square (B),} \quad x^2 + 2xy + y^2 = 25 \quad (2)$$

$$\text{Subtract (1) from (2),} \quad 3xy = 18 \quad (3)$$

$$\text{Divide by 3,} \quad xy = 6 \quad (4)$$

$$\text{Subtract (4) from (1),} \quad x^3 - 2xy + y^3 = 1 \quad (5)$$

$$\text{Extract } \sqrt{\phantom{x}}, \quad x - y = \pm 1 \quad (6)$$

$$\text{Solve } \begin{cases} x + y = 5 \\ x - y = 1 \end{cases} \quad \begin{matrix} \text{(B)} \\ \text{(6)} \end{matrix}$$

$$x = 3 \text{ and } y = 2.$$

$$\text{Solve } \begin{cases} x + y = 5 \\ x - y = -1 \end{cases} \quad \begin{matrix} \text{(B)} \\ \text{(6)} \end{matrix}$$

$$x = 2 \text{ and } y = 3.$$

In a similar manner may be solved equations of the form of

$$x^3 - y^3 = a \text{ and } x - y = b;$$

$$x^4 - y^4 = a \text{ and } x^2 \pm y^2 = b;$$

$$x^4 + x^2 y^2 + y^4 = a \text{ and } x^2 \pm xy + y^2 = b;$$

$$x^2 + xy + y^2 = a \text{ and } x \pm \sqrt{xy} + y = b.$$

III. Sometimes a common factor may be removed from both members of an equation and its degree thereby lowered.

$$\text{Illustration.}—\text{Solve } \begin{cases} x^3 - y^3 = 7(x - y)^3 \\ x + y = 6 \end{cases} \quad \begin{matrix} \text{(A)} \\ \text{(B)} \end{matrix}$$

$$\text{Divide (A) by } x - y, \quad x^2 + xy + y^2 = 7(x - y)^2 \quad (1)$$

$$\text{Expand (1),} \quad x^2 + xy + y^2 = 7x^2 - 14xy + 7y^2 \quad (2)$$

Transpose and collect terms,

$$6x^2 - 15xy + 6y^2 = 0 \quad (3)$$

$$\text{Divide by 3,} \quad 2x^2 - 5xy + 2y^2 = 0 \quad (4)$$

Square (B) and multiply by 2,

$$2x^2 + 4xy + 2y^2 = 72 \quad (5)$$

$$\text{Subtract (4) from (5),} \quad 9xy = 72 \quad (6)$$

$$\text{whence,} \quad xy = 8 \quad (7)$$

Subtract  $4 \times (7)$  from square of (B),

$$x^2 - 2xy + y^2 = 4 \quad (8)$$

$$\text{Extract } \sqrt{\phantom{x}}, \quad x - y = \pm 2 \quad (9)$$

$$\text{Solve } \begin{cases} x + y = 6 \\ x - y = 2 \end{cases} \quad \begin{matrix} \text{(B)} \\ \text{(9)} \end{matrix}$$

$$x = 4 \text{ and } y = 2.$$

$$\text{Solve } \begin{cases} x + y = 6 \\ x - y = -2 \end{cases} \quad \begin{matrix} \text{(B)} \\ \text{(9)} \end{matrix}$$

$$x = 2 \text{ and } y = 4.$$

IV. Sometimes there is a common factor in the first members of the two equations which may be removed by division.

$$\text{Illustration.—Solve } \begin{cases} x^2 - y^2 = 12 \\ xy + y^2 = 12 \end{cases} \quad \begin{matrix} \text{(A)} \\ \text{(B)} \end{matrix}$$

$$\text{Factor (A) and (B),} \quad (x + y)(x - y) = 12 \quad (1)$$

$$y(x + y) = 12 \quad (2)$$

$$\text{Divide (1) by (2),} \quad \frac{x - y}{y} = 1 \quad (8)$$

$$\text{whence,} \quad x = 2y \quad (4)$$

$$\text{Substitute (4) in (A) and reduce,} \quad y = \pm 2$$

$$\text{whence,} \quad x = \pm 4.$$

V. Sometimes one or both equations may be made to assume the quadratic form.

**Illustration.—**

$$\text{Solve } \begin{cases} x^2 + y^2 - x - y = 14 \\ xy + x + y = 14 \end{cases} \quad \begin{matrix} \text{(A)} \\ \text{(B)} \end{matrix}$$

$$\text{Add twice (B) to (A),} \quad x^2 + 2xy + y^2 + x + y = 42 \quad (1)$$

$$\text{or,} \quad (x + y)^2 + (x + y) = 42$$

$$\text{Complete the square,} \quad (x + y)^2 + (x + y) + \frac{1}{4} = \frac{169}{4} \quad (2)$$

$$\text{Extract the } \sqrt{\phantom{x}}, \quad x + y + \frac{1}{2} = \pm \frac{13}{2}$$

$$\text{whence,} \quad x + y = 6 \text{ or } -7 \quad (3)$$

$$\text{Substitute } x + y = 6 \text{ in (A) and (B),} \quad x^2 + y^2 = 20 \quad (4)$$

$$xy = 8 \quad (5)$$

$$\text{Subtract twice (5) from (4),} \quad x^2 - 2xy + y^2 = 4 \quad (6)$$

$$\text{Extract } \sqrt{\phantom{x}}, \quad x - y = \pm 2 \quad (7)$$

$$\text{Solve } \begin{cases} x + y = 6 \\ x - y = 2 \end{cases} \quad \begin{matrix} \text{(3)} \\ \text{(7)} \end{matrix}$$

$$x = 4 \text{ and } y = 2.$$

$$\text{Solve } \begin{cases} x + y = 6 \\ x - y = -2 \end{cases} \quad \begin{matrix} (3) \\ (7) \end{matrix}$$

$$x = 2 \text{ and } y = 4.$$

$$\text{Substitute } x + y = -7 \text{ in (A) and (B), } \quad \begin{matrix} x^2 + y^2 = 7 & (8) \\ xy = 21 & (9) \end{matrix}$$

$$\text{Subtract twice (9) from (8), } \quad x^2 - 2xy + y^2 = -35 \quad (10)$$

$$\text{Extract } \sqrt{\phantom{x}}, \quad x - y = \pm \sqrt{-35} \quad (11)$$

$$\text{Solve } \begin{cases} x + y = -7 \\ x - y = \sqrt{-35} \end{cases} \quad \begin{matrix} (3) \\ (11) \end{matrix}$$

$$x = \frac{1}{2}(-7 + \sqrt{-35}) \text{ and } y = \frac{1}{2}(-7 - \sqrt{-35}).$$

$$\text{Solve } \begin{cases} x + y = -7 \\ x - y = -\sqrt{-35} \end{cases} \quad \begin{matrix} (3) \\ (11) \end{matrix}$$

$$x = \frac{1}{2}(-7 - \sqrt{-35}) \text{ and } y = \frac{1}{2}(-7 + \sqrt{-35}).$$

**Note.**—It is not sufficient that the student find all the different values of  $x$  and  $y$ , but he must also properly pair them.

#### EXERCISE 57.

Solve :

$$\begin{aligned} 1. \quad x + y &= 9 \\ xy &= 20 \end{aligned}$$

$$\begin{aligned} 8. \quad x^2 + y^2 &= 90 \\ x + y &= 12 \end{aligned}$$

$$\begin{aligned} 2. \quad x - y &= 4 \\ xy &= 12 \end{aligned}$$

$$\begin{aligned} 9. \quad x^2 + y^2 &= 52 \\ x - y &= 2 \end{aligned}$$

$$\begin{aligned} 3. \quad x + 2y &= 13 \\ xy &= 15 \end{aligned}$$

$$\begin{aligned} 10. \quad x^2 - y^2 &= 60 \\ xy &= 16 \end{aligned}$$

$$\begin{aligned} 4. \quad x - 3y &= 3 \\ xy &= 36 \end{aligned}$$

$$\begin{aligned} 11. \quad x^2 - y^2 &= 60 \\ x + y &= 10 \end{aligned}$$

$$\begin{aligned} 5. \quad 3x + 2y &= 20 \\ xy &= 16 \end{aligned}$$

$$\begin{aligned} 12. \quad x^2 - y^2 &= 45 \\ x - y &= 3 \end{aligned}$$

$$\begin{aligned} 6. \quad 4x - 3y &= -1 \\ xy &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} 13. \quad 4x^2 - 9y^2 &= 112 \\ 2x + 3y &= 28 \end{aligned}$$

$$\begin{aligned} 7. \quad x^2 + y^2 &= 25 \\ xy &= 12 \end{aligned}$$

$$\begin{aligned} 14. \quad x^2 + y^2 &= 91 \\ x + y &= 7 \end{aligned}$$

15.  $x^3 - y^3 = 189$

$x - y = 3$

16.  $x^3 + y^3 = 1027$

$x^2 - xy + y^2 = 79$

17.  $x^3 - y^3 = 61$

$x^2 + xy + y^2 = 61$

18.  $x^4 + x^2 y^2 + y^4 = 481$

$x^2 + xy + y^2 = 37$

19.  $x^4 + x^2 y^2 + y^4 = 91$

$x^2 - xy + y^2 = 7$

20.  $8x^3 - 27y^3 = 271$

$2x - 3y = 1$

21.  $\frac{1}{x^2} + \frac{1}{y^2} = \frac{13}{36}$

$\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$

22.  $\frac{1}{x^2} - \frac{1}{y^2} = \frac{9}{400}$

$\frac{1}{x} - \frac{1}{y} = \frac{1}{20}$

23.  $x^2 + y^2 = a^2 + b^2$

$xy = ab$

24.  $x^2 y^2 + xy = 42$

$x + y = 5$

25.  $x^2 + xy = 35$

$xy + y^2 = 14$

26.  $x + y = 35$

$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5$

27.  $x^2 + y^2 + x + y = 50$

$xy + x + y = 29$

28.  $x^2 + y^2 + x - y = 14$

$xy + x - y = 7$

29.  $x^3 + y^3 = 35$

$x^2 y + xy^2 = 30$

30.  $x^3 - y^3 = 98$

$x^2 y - xy^2 = 30$

31.  $(x+y)^3 + (x+y) = 30$

$xy = 6$

32.  $x^3 + 3xy + 2y^3 = 150$

$2x = 3y - 5$

33.  $x^2 + 3xy + 2y^2 = 63$

$3x^2 + 8xy + 4y^2 = 171$

34.  $\frac{x^2 + y^2}{x + y} = 45$

$x + y + \sqrt{2xy} = 15$

35.  $\frac{1}{x^3} + \frac{1}{y^3} = 35$

$\frac{1}{x} + \frac{1}{y} = 5$

36.  $\frac{1}{x^3} - \frac{1}{y^3} = 61$

$\frac{1}{x} - \frac{1}{y} = 1$

37.  $\frac{1}{x} - \frac{1}{y} = 37$

$\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} = 1$

38.  $x^2 + xy + y^2 = 21$

$x - \sqrt{xy} + y = 3$

39.  $x^4 + y^4 = 20$

$x^2 - xy\sqrt{2} + y^2 = 2$

40.  $x + y = 5$

$x^{\frac{1}{2}} + y^{\frac{1}{2}} - x^{\frac{1}{2}} y^{\frac{1}{2}} \sqrt{2} = 1$

41.  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 9$

$x^{\frac{1}{3}} + y^{\frac{1}{3}} = 3$

**349.** Two simultaneous symmetrical equations may often be solved by putting

$$x = u + v$$

$$y = u - v$$

**Illustration.**—Solve  $x + y = 5$  (A)

$$x^4 + y^4 = 97 \quad (B)$$

Let  $x = u + v$  and  $y = u - v$ , then

$$x + y = 2u = 5; \text{ whence}$$

$$u = \frac{5}{2} \quad (1)$$

$$x^4 = (u + v)^4 = u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4 \quad (2)$$

$$y^4 = (u - v)^4 = u^4 - 4u^3v + 6u^2v^2 - 4uv^3 + v^4 \quad (3)$$

$$x^4 + y^4 = 2u^4 + 12u^2v^2 + 2v^4 = 97 \quad (4)$$

$$\text{Substitute (1) in (4),} \quad \frac{625}{8} + 75v^2 + 2v^4 = 97 \quad (5)$$

$$\text{Simplify,} \quad 16v^4 + 600v^2 = 151 \quad (6)$$

$$\text{Reduce,} \quad v = \pm \frac{1}{2}, \text{ or } \pm \frac{1}{2} \sqrt{-151}; \quad (7)$$

$$\text{whence } x = u + v = 3, 2, \text{ or } \frac{5}{2} \pm \frac{1}{2} \sqrt{-151} \quad (8)$$

$$\text{and } y = u - v = 2, 3, \text{ or } \frac{5}{2} \mp \frac{1}{2} \sqrt{-151} \quad (9)$$

Many equations of this class may be solved by more elegant methods.

**Illustration.**—Take equations (A) and (B), above.

Raise (A) to the 4th power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 625 \quad (1)$$

Add equation (B),

$$2x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 2y^4 = 722 \quad (2)$$

$$\text{Divide by 2,} \quad x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4 = 361 \quad (3)$$

$$\text{Extract } \sqrt{\phantom{x}}, \quad x^2 + xy + y^2 = \pm 19 \quad (4)$$

$$\text{Square (A),} \quad x^2 + 2xy + y^2 = 25 \quad (5)$$

$$\text{Subtract (4) from (5),} \quad xy = 6 \text{ or } 44 \quad (6)$$

$$\text{Solve (5) and (6),} \quad x = 3, 2, \text{ or } \frac{5}{2} \pm \frac{1}{2} \sqrt{-151} \quad (7)$$

$$y = 2, 3, \text{ or } \frac{5}{2} \mp \frac{1}{2} \sqrt{-151} \quad (8)$$

**Note.**—If the equations are symmetrical in every other respect than in sign, they may sometimes be solved by the same methods as strictly symmetrical equations.

## EXERCISE 58.

Solve :

$$\begin{aligned} 1. \quad x^2 + y^2 + x + y &= 24 \\ x^2 + xy + y^2 &= 27 \end{aligned}$$

$$\begin{aligned} 4. \quad x - y &= 3 \\ x^5 - y^5 &= 3093 \end{aligned}$$

$$\begin{aligned} 2. \quad x^4 + y^4 &= 272 \\ x + y &= 6 \end{aligned}$$

$$\begin{aligned} 5. \quad x^2y + xy^2 &= 84 \\ x + y &= 7 \end{aligned}$$

$$\begin{aligned} 3. \quad x^5 + y^5 &= 275 \\ x + y &= 5 \end{aligned}$$

$$\begin{aligned} 6. \quad x^2y + xy^2 &= 48 \\ x^2y - xy^2 &= 16 \end{aligned}$$

$$\begin{aligned} 7. \quad 3x^2 - 4xy + 3y^2 &= 7 \\ 2x^2 + 2y^2 - 3x - 3y &= 1 \end{aligned}$$

$$\begin{aligned} 8. \quad x^3 + y^3 &= 72 \\ x^2 + 2xy + y^2 + x + y &= 42 \end{aligned}$$

$$\begin{aligned} 9. \quad x^2 + xy + y^2 - x - y &= 52 \\ x + xy + y &= 29 \end{aligned}$$

$$\begin{aligned} 10. \quad 3x^2 + 2xy + 3y^2 + 2x + 2y &= 100 \\ x^2 + y^2 + xy &= 31 \end{aligned}$$

**350.** Two homogeneous equations of the second degree may be solved by putting  $y = vx$ .

**Illustration.**—Solve  $x^2 + xy + 2y^2 = 86$  (A)

$2x^2 + 2xy + y^2 = 97$  (B)

Put  $y = vx$  in (A) and (B),

$$x^2 + vx^2 + 2v^2x^2 = 86 \quad (1)$$

$$2x^2 + 2vx^2 + v^2x^2 = 97 \quad (2)$$

From (1),

$$x^2 = \frac{86}{1 + v + 2v^2} \quad (3)$$

From (2),

$$x^2 = \frac{97}{2 + 2v + v^2} \quad (4)$$

Equating (1) and (2),

$$\frac{86}{1 + v + 2v^2} = \frac{97}{2 + 2v + v^2} \quad (5)$$

Reduce,

$$v = \frac{5}{4} \text{ or } -\frac{5}{9} \quad (6)$$

Substitute (6) in (3),

$$x^2 = 16 \text{ or } 1 \quad (7)$$

whence,

$$x = \pm 4 \text{ or } \pm 1 \quad (8)$$

Substitute (6) and (8) in  $y = vx$ ,

$$y = \pm 5 \text{ or } \mp 5 \quad (9)$$

Some equations of this class may be solved by more elegant methods.

**Illustration.**—Take equations (A) and (B), above :

$$\text{Multiply (A) by 2, } 2x^2 + 2xy + 4y^2 = 172 \quad (1)$$

$$\text{Subtract, } 2x^2 + 2xy + y^2 = 97 \quad (B)$$

$$3y^2 = 75 \quad (2)$$

$$\text{whence, } y^2 = 25 \quad (3)$$

$$\text{and } y = \pm 5 \quad (4)$$

$$\text{Substitute (4) in (B), } 2x^2 \pm 10x + 25 = 97 \quad (5)$$

$$x^2 \pm 5x = 36 \quad (6)$$

$$\text{Reduce, } x = 4, -9, 9, \text{ and } -4 \quad (7)$$

Sometimes the first members have a common factor which will cancel out by division.

#### EXERCISE 59.

Solve :

$$\begin{array}{ll} 1. \quad x^2 + 3xy = 54 & 2. \quad 2x^2 - 3xy + y^2 = -1 \\ \quad 2y^2 - 5xy = -25 & \quad 3x^2 - 4xy + 2y^2 = 6 \end{array}$$

$$\begin{array}{l} 3. \quad x^2 + 5xy + 6y^2 = 180 \\ \quad x^2 + xy - 6y^2 = 0 \end{array}$$

$$\begin{array}{l} 4. \quad 6x^2 + 5xy - 6y^2 = 45 \\ \quad 2x^2 + 7xy + 6y^2 = 145 \end{array}$$

$$\begin{array}{l} 5. \quad 4x^2 + 2xy - 2y^2 = 18 \\ \quad 6x^2 + xy - 2y^2 = 24 \end{array}$$

$$\begin{array}{l} 6. \quad 2x^2 - 3xy + 5y^2 = 98 \\ \quad 3x^2 + 2xy - y^2 = 32 \end{array}$$

$$\begin{array}{l} 7. \quad 5xy + 2x^2 - 3y^2 = -248 \\ \quad 2y^2 + 5x^2 - 3xy = 175 \end{array}$$

$$\begin{array}{l} 8. \quad 9x^2 - 25y^2 = 11 \\ \quad 34xy - 15x^2 + 15y^2 = 23 \end{array}$$

$$\begin{array}{l} 9. \quad x^2 + y^2 - 3xy = -31 \\ \quad 3x^2 + 5y^2 + 10xy = 622 \end{array}$$

$$\begin{array}{l} 10. \quad 10x^2 + 29xy + 10y^2 = 580 \\ \quad 10x^2 + 21xy - 10y^2 = 0 \end{array}$$

$$11. \quad x + \sqrt{xy} + y = 13$$

$$2x - 3\sqrt{xy} - 5y = 4$$

$$12. \quad 2x - \sqrt{xy} - 3y = -4$$

$$3x + 5\sqrt{xy} + 2y = 20$$

$$13. \quad 5x + 3\sqrt{xy} + 2y = 20$$

$$2x - 2\sqrt{xy} - 6y = -12$$

$$14. \quad x^4 - x^2y^2 + y^4 = 16$$

$$2x^4 + 3x^2y^2 - 3y^4 = 32$$

$$15. \quad 2x^2 + 3xy + y^2 = 28$$

$$6x^2 + xy - y^2 = 56$$

$$16. \quad x^2 + xy - 15 = xy - y^2 - 2 = 0$$

**351.** An equation with compound terms may sometimes be put in the quadratic form and reduced to an equation of half the degree.

**Illustrations.—**

$$1. \text{ Solve } x + 2y + 4\sqrt{x + 2y + 6} = 26 \quad (\text{A})$$

$$x^2 - 4y^2 = 20 \quad (\text{B})$$

Add 6 to (A),

$$(x + 2y + 6) + 4\sqrt{x + 2y + 6} = 32 \quad (1)$$

Complete the square,

$$(x + 2y + 6) + (\quad) + 4 = 36 \quad (2)$$

Extract the  $\sqrt{\quad}$ ,

$$\sqrt{x + 2y + 6 + 2} = \pm 6 \quad (3)$$

Subtract 2 from (3),

$$\sqrt{x + 2y + 6} = 4 \text{ or } -8 \quad (4)$$

Square (4),

$$x + 2y + 6 = 16 \text{ or } 64 \quad (5)$$

Subtract 6,

$$x + 2y = 10 \text{ or } 58 \quad (6)$$

Divide (B) by (6),

$$x - 2y = 2 \text{ or } \frac{10}{29} \quad (7)$$

Add (6) and (7),

$$2x = 12 \text{ or } 58\frac{10}{29}$$

whence,

$$x = 6 \text{ or } 29\frac{5}{29} \quad (8)$$

Subtract (7) from (6),

$$4y = 8 \text{ or } 57\frac{19}{29}$$

whence,

$$y = 2 \text{ or } 14\frac{12}{29} \quad (9)$$

$$2. \text{ Solve } \left(x^2 y^2 + \frac{1}{x^2 y^2}\right) + \left(xy + \frac{1}{xy}\right) = 42 \frac{7}{36} \quad (\text{A})$$

$$x^2 + y^2 = 13 \quad (\text{B})$$

Add 2 to (A),

$$\left(x^2 y^2 + 2 + \frac{1}{x^2 y^2}\right) + \left(xy + \frac{1}{xy}\right) = 44 \frac{7}{36} \quad (1)$$

$$\text{or, } \left(xy + \frac{1}{xy}\right)^2 + \left(xy + \frac{1}{xy}\right) = 44 \frac{7}{36} \quad (2)$$

Complete the square,

$$\left(xy + \frac{1}{xy}\right)^2 + \left(xy + \frac{1}{xy}\right) + \frac{1}{4} = 44 \frac{4}{9} \text{ or } \frac{400}{9}$$

$$\text{Extract the } \sqrt{\phantom{x}}, \quad \left(xy + \frac{1}{xy}\right) + \frac{1}{2} = \pm \frac{20}{3};$$

$$\text{whence, } xy + \frac{1}{xy} = \frac{37}{6} \text{ or } -\frac{43}{6} \quad (3)$$

Clear of fractions, and transpose terms,

$$(1) \quad 6x^2 y^2 - 37xy = -6$$

$$(2) \quad 6x^2 y^2 + 43xy = -6$$

$$\text{Solve (1), } xy = 6 \text{ or } \frac{1}{6}$$

$$\text{Solve (2), } xy = -7.024 \text{ or } -0.142.$$

$$\text{Solve } \begin{cases} x^2 + y^2 = 13 \\ xy = 6 \end{cases}, \quad \begin{cases} x^2 + y^2 = 13 \\ xy = \frac{1}{6} \end{cases},$$

$$\begin{cases} x = 3, 2, -3, -2 \\ y = -2, 3, -2, -3 \end{cases}, \quad \begin{cases} x = 3.6, .05, -3.6, -.05 \\ y = .05, 3.6, -.05, -3.6 \end{cases}$$

$$\text{Solve } \begin{cases} x^2 + y^2 = 13 \\ xy = -7.024 \end{cases}, \quad \begin{cases} x^2 + y^2 = 13 \\ xy = -0.142 \end{cases},$$

$$\begin{cases} x = \frac{1}{2}(\pm 5.2 \pm \sqrt{-1.048}) \\ y = \frac{1}{2}(\mp 5.2 \mp \sqrt{-1.048}) \end{cases}, \quad \begin{cases} x = 3.6, 0.4, -3.6, -.04 \\ y = .04, 3.6, -.04, -3.6 \end{cases}$$

#### EXERCISE 60.

Solve :

$$1. \quad x^2 + y^2 + 3\sqrt{x^2 + y^2} = 40$$

$$xy = 12$$

$$2. \quad x^3 + y^3 + 2\sqrt{x^3 + y^3 + 11} = 157$$

$$x + y + \sqrt{x + y + 2} = 10$$

$$3. \quad x - y - 2\sqrt{x-y} = -1$$

$$x^3 - y^3 + 4\sqrt{x^3 - y^3} = 60$$

$$4. \quad x^3 + y^3 + \sqrt{x^3 + y^3} = 12$$

$$x^2 - xy + y^2 = 3$$

$$5. \quad x^2 - xy + y^2 + 4\sqrt{x^2 - xy + y^2} = 32$$

$$x^4 + x^2y^2 + y^4 = 768$$

$$6. \quad \frac{1}{x} + \frac{1}{y} + \sqrt{\frac{x+y}{xy} + \frac{6}{25}} = 1\frac{1}{5}$$

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{2}{25}$$

$$7. \quad x + y = \frac{5}{6}$$

$$x^2y^2 + \frac{1}{x^2y^2} + 4\left(xy + \frac{1}{xy}\right) = 60\frac{25}{36}$$

$$8. \quad (x+y)^{\frac{1}{2}} + (x+y)^{\frac{1}{2}} = 20$$

$$(x-y)^{\frac{1}{2}} - (x-y)^{\frac{1}{2}} = -2$$

### Examples leading to Equations of the Second Degree.

**Illustrations.**—1. The difference of two numbers is 5, and their product is 14. Find the numbers.

**Solution 1:** Let

$x$  = the smaller number.

Then

$x + 5$  = the greater number,

and

$x(x + 5) = 14$ , their product.

Expand,

$$x^2 + 5x = 14$$

(1)

Reduce,

$x = 2$ , the smaller number,

and

$x + 5 = 7$ , the greater number.

**Solution 2:** Let

$x$  = the greater number,

and

$y$  = the smaller number.

Then

$x - y = 5$ , their difference, (A)

and

$xy = 14$ , their product. (B)

Reduce

$x = 7$ , the greater number,

and

$y = 2$ , the smaller number.

2. A man sold a horse for \$171, and thereby gained as many per cent as there were dollars in the cost. Required the cost.

**Solution 1:** Let  $x$  = the number of dollars the horse cost.  
 Then  $x$  = the rate per cent,  
 and  $\frac{x}{100}$  of  $x$ , or  $\frac{x^2}{100}$  = the number of dollars gained.  
 $\therefore x + \frac{x^2}{100} = 171$ , the number of dollars sold for. (A)  
 Reduce (A),  $x = 90$ .

**Solution 2:** Let  $x$  = the cost,  
 and  $y$  = the gain.  
 Then  $\frac{y}{x} \times 100$ , or  $\frac{100y}{x}$  = the rate of gain.  
 Now,  $x + y = 171$ , the selling price, (A)  
 and  $\frac{100y}{x} = x$ , the cost. (B)  
 Reduce (A) and (B),  $x = 90$ , the cost.

**Note.**—The following questions may be solved by assuming one or two unknown quantities, as may seem best to the student.

#### EXERCISE 61.

1. The sum of two numbers is 15 and their product is 54. Required the numbers.

2. The product of two numbers is 192 and their quotient is 3. Find the numbers.

3. The length of a rectangle exceeds its width by 3 rods, and its area is 180 square rods. Find its dimensions.

4. The sum of A's and B's ages is 50 years, and the difference of the squares of their ages is 500 years. What are their ages, if A is the older?

5. The sum of the lengths of two cubical hay-stacks is 27 feet, and the contents of both is 4941 cubic feet. Required the dimensions of each.

6. The hypotenuse of a right-angled triangle is 25 feet and its area is 150 square feet. Find its base and perpendicular.

7. A man bought a certain number of horses for \$1200; had he bought two more for the same sum, each would have cost \$30 less. How many did he buy?

8. At what rate will \$500, in 2 years, compound interest, amount to \$561.80?

9. The compound interest of \$400 in 2 years exceeds the simple interest at the same rate by \$1. Required the rate.

10. If the radius of a given circle were increased by 1 foot, the area would be increased by 34.5576 square feet. Required the radius.

*Note.*—The area =  $\pi r^2$ ;  $r$  = radius;  $\pi$  = 3.1416.

11. A man sold a number of sheep for \$160; had he reserved 5, and sold the remainder at \$1 apiece more, he would have received \$135. How many did he sell, and at what price?

12. A steamboat goes 140 miles down a river whose current is 2 miles an hour. On its return, it takes 4 hours longer to make the journey. What is its rate of sailing in still water?

13. The difference of the circumferences of the hind and fore-wheels of a wagon is 4 feet, and the fore-wheel makes 110 revolutions more than the hind-wheel in going a mile. What is the circumference of each wheel?

14. The area of a board is 12 square feet, and there are twice as many inches, less 2, in its width as there are feet in its length. Required its dimensions.

15. A man sold a cow for \$16, and thereby lost as many per cent as the cow cost dollars. What was the cost of the cow?

16. I bought bonds at as many per cent below par as is their rate of interest, and found that my rate of income was  $\frac{18}{47}\%$  better than the rate of the bond. Required the rate of the bond and the purchase price.

17. I bought bonds at their rate of interest above par, and found that my rate of income was  $\frac{5}{21}\%$  below that of the bond. Required the rate and purchase price of the bond.

18. I bought 6's at a certain price; had I bought them 5% lower, my rate of income would have been  $\frac{20}{57}\%$  higher. Required the rate of income and the purchase price.

19. Sold goods at 25 cents a yard. Had I paid 1 cent a yard less, my rate of gain would have been 5% greater. Required the cost and rate of gain.

20. A man had 400 shares of railroad stock, but, after receiving two stock dividends, he had 441 shares. What was the rate of dividend?

21. At what annual rate per cent must the population of a town containing 5000 inhabitants increase, that in three years it may contain 6655 inhabitants?

22. What are eggs worth per dozen, when a decrease of 5 cents in the price will make a difference of 2 dozen in \$2 worth?

23. A and B are 49 miles apart, and approach each other; when they meet, it is discovered that A could have gone B's distance in 9 hours, but B would have required 16 hours to go A's distance. Required each man's distance and rate of travel.

24. An engineer said, if his train would run 5 miles an hour faster, it would go 400 miles in 4 hours less time. What is the rate of the train?

25. Two cubical hay-stacks contain  $10\frac{14}{135}$  tons, and the sum of their lengths is 22 feet. What is the length of each, allowing 10 cubic yards to a ton?

26. The difference of two rectangular cisterns, each of which is 1 foot longer than wide, and 1 foot wider than deep, is 1410 cubic feet. What are the dimensions of each, if the larger is 3 feet longer than the smaller?

27. A and B were 700 miles apart, and approached each other. A traveled 10 miles a day faster than B. After 5 days, A diminished his speed 10 miles a day, and, after 6 days, B increased his speed  $12\frac{1}{2}$  miles a day; when they met, each had traveled half the distance. What was each man's rate of travel?

28. The sum of the squares of two numbers plus their product is 133, and the sum of the numbers plus the square root of their product is 19. What are the numbers?

29. The sum of the squares of two numbers that are reciprocals of each other exceeds the sum of the numbers by  $11\frac{13}{16}$ . What are the numbers?

30. Two men were engaged for a given time at different daily wages. The first was idle 6 days, and received \$49; the second was idle 2 days, and received \$90. Had the first been idle 2 days and the second 6 days, they together would have received \$133. For how many days were they engaged?

31. A huckster bought a certain number of eggs for \$9; after breaking 9 of them, he sold the remainder for 7 cents a dozen more than they cost, and gained \$2.28. How many did he buy, and at what price did he buy them?

32. A and B in partnership invest \$1500; A withdraws his money in 4 months, and B his in 5 months; A receives for his share of capital and profits \$980, and B for his share \$1200. How much did each invest?

33. The diagonal of a rectangle is twice its width, and its area is one acre. Required its width.

34. A and B hired a pasture, into which A put 4 horses, and B as many as cost him 18 shillings a week. Afterward, B put in 2 additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired?

35. A drover bought a certain number of horses for \$7950; had he bought 12 more at \$15 more each, they would have cost \$10725. How many did he buy?

36. A and B engage to reap a field for £4 10s.; and, as A, alone, could reap it in 9 days, they promise to complete it in 5 days. They find, however, that they are obliged to call in C to assist them for 2 days, in consequence of which B receives 3s. 9d. less than he otherwise would. In what time could B and C each reap the field?

37. A and B put out different sums at interest, amounting together to \$200. B's rate of interest was 1 per cent more than A's. At the end of 5 years, B's accumulated simple interest wanted but \$4 to be double A's. At the end of 10 years, A's principal and interest was  $\frac{5}{8}$  of B's. What were the sums paid out by each, and the rate per cent?

38. A courier proceeds from one place, P, to another, Q, in 14 hours; a second courier starts at the same time as the first from a place 10 miles behind P, and arrives at Q at the same time as the first courier. The second courier finds that he takes half an hour less than the first to accomplish 20 miles. Find the distance of Q from P.

39. A certain number of workmen can move a heap of stones in 8 hours from one place to another. If there had been 8 more workmen, and each workman had carried 5 pounds less at a time, the whole would have been completed in 7 hours. If, however, there had been 8 fewer workmen, and each had carried 11 pounds more at a time, the work would have taken 9 hours. Find the number of workmen and the weight which each carried at a time.

40. A person bought a number of £20 railway shares, when they were at a certain per cent discount, for £1500; and afterward, when they were at the same rate per cent premium, sold them, all but 60, for £1000. How many did he buy, and what price did he pay?

41. Two kinds of pears are sold in market, two more of one kind being given for a shilling than of the other ; a score of the inferior sort cost sixpence more than a dozen of the superior sort. Find the price of the pears apiece.

42. Required to divide a line of 134 yards in length into three such parts that the sum of their squares may be 6036, and that the first, twice the second, and three times the third, may together make 278.

## Generalization and Specialization.

### Definitions.

352. A problem in which some or all known quantities involved are literal, is a *general* problem.

353. A problem in which all known quantities involved are numerical, is a *special* problem, or an *example*.

354. The solution of a general problem gives rise to a *formula*, which may be translated into a *principle* or a *rule* for the solution of an entire class of examples.

**Illustration.**—A can do a piece of work in  $a$  days, and B in  $b$  days. In what time can they do it working together ?

**Solution :** Let

$x$  = the required time.

Then,

$\frac{1}{x}$  = the part they can do in one day.

$\frac{1}{a}$  = the part A can do in one day.

$\frac{1}{b}$  = the part B can do in one day.

$$\therefore \frac{1}{a} + \frac{1}{b} = \frac{1}{x};$$

whence,  $x = \frac{ab}{a+b}$ , a formula.

Therefore,

**Principle.**—The time required for two persons to do a piece of work equals the product of their respective times of doing it divided by the sum of those times.

Or,

**Rule.**—To find the time required for two persons to do a piece of work :

Divide the product of their respective times of doing it by the sum of those times.

**355.** The process of substituting literal for numerical quantities in a problem, and solving the general problem thus formed, is *Generalization*.

**356.** The process of substituting numerical for literal quantities, in a formula or a general problem, and solving the special example thus formed, is *Specialization*.

Examples in Generalization and Specialization.

**Illustrations.**—1. A man has 5 hours at his disposal. How far may he ride in a stage-coach going 6 miles an hour, in order that he may walk back at the rate of  $2\frac{1}{2}$  miles an hour? Generalize.

**Solution:** Put  $a$  for 5,  $m$  for 6, and  $n$  for  $2\frac{1}{2}$ .

Let  $x$  = the distance.

Then  $\frac{x}{m}$  = the time going,

and  $\frac{x}{n}$  = the time returning.

$$\therefore \frac{x}{m} + \frac{x}{n} = a;$$

$$\text{whence, } x = \frac{a m n}{m + n}.$$

$$\text{Substitute, } x = \frac{5 \times 6 \times 2\frac{1}{2}}{6 + 2\frac{1}{2}} = 8\frac{14}{17} \text{ miles.}$$

2. The difference of two numbers is  $b$ , and their sum multiplied by the difference of their squares is  $a$ . What are the numbers? Specialize by putting 4 for  $b$  and 1600 for  $a$ .

**Solution:** Let  $x$  = the greater number.  
and  $y$  = the less number.

Then,  $x - y = b$  (1)

and  $(x + y)(x^2 - y^2) = a$  (2)

Divide (2) by (1),  $(x + y)^2 = \frac{a}{b}$  (3)

$$x + y = \pm \sqrt{\frac{a}{b}} \quad (4)$$

Adding (4) and (1),  $2x = b \pm \sqrt{\frac{a}{b}}$  (5)

Subtracting (1) from (4),  $2y = -b \pm \sqrt{\frac{a}{b}}$  (6)

$$\therefore \quad \begin{aligned} x &= \frac{1}{2} \left( b \pm \sqrt{\frac{a}{b}} \right) \\ y &= \frac{1}{2} \left( -b \pm \sqrt{\frac{a}{b}} \right) \end{aligned}$$

Substitute  $b = 4$  and  $a = 1600$ ,

$$x = \frac{1}{2} (4 \pm \sqrt{400}) = 12 \text{ or } -8$$

$$y = \frac{1}{2} (-4 \pm \sqrt{400}) = 8 \text{ or } -12.$$

#### EXERCISE 62.

1. In an orchard,  $\frac{1}{a}$  of the trees bear apples,  $\frac{1}{b}$  bear plums,  $\frac{1}{c}$  peaches, and  $d$  bear cherries. How many trees in all? How many, if  $a = 2$ ,  $b = 4$ ,  $c = 5$ , and  $d = 20$ ?

2. The sum of two numbers is  $a$  and their difference is  $b$ . What are the numbers? What, if  $a = 25$  and  $b = 12$ ?

3. A, B, and C can perform a piece of work in 5 days. A alone can do it in 12 days, and B in 15 days. In what time can C alone do it? Generalize by putting  $5 = a$ ,  $12 = b$ , and  $15 = c$ .

4. A company at an inn paid  $a$  dollars each for a supper. Had there been  $c$  more persons, each would have paid only  $b$  dollars. How many were there? How many if  $a = 8$ ,  $b = 7$ , and  $c = 4$ ?

5. A purse holds  $a$  crowns and  $b$  guineas. But  $c$  crowns and  $d$  guineas fill  $\frac{m}{n}$  of it. How many will it hold of each? How many if  $a = 19$ ,  $b = 6$ ,  $c = 4$ ,  $d = 5$ ,  $m = 17$ , and  $n = 63$ ?

6. A and B can do a piece of work in 8 days; A and C in 6 days; and B and C in 4 days. How long would it take each to do it, and how long all? Put  $8 = a$ ,  $6 = b$ , and  $4 = c$ .

7. A person bought some sheep for  $\$m$ , and found that, if he had bought  $a$  more for the same money, they would have cost  $\$n$  less each. How many did he buy, and what was the price of each? What are the results for  $m = 72$ ,  $a = 6$ , and  $n = 1$ ?

8. A drover bought as many sheep as cost him  $\pounds m$ ; out of which he reserved  $a$ , and sold the remainder for  $\pounds n$ , gaining  $b$  shillings apiece on those he sold. How many did he buy, and at what price per head? Give results for  $m = 60$ ,  $a = 15$ ,  $n = 54$ , and  $b = 2$ .

9. Two partners, A and B, gained  $\$140$  by trade; A's money was in trade 3 months, and his gain was  $\$60$  less than his stock, and B's money, which was  $\$50$  more than A's, was in trade 5 months. What was A's stock? Put  $140 = a$ ,  $3 = r$ ,  $60 = m$ ,  $50 = n$ ,  $5 = t$ .

10. A and B hired a pasture, into which A put  $a$  horses, and B as many as cost him  $s$  shillings a week. Afterward, B put in  $c$  additional horses, and found that he must pay  $t$  shillings a week. At what rate per week was the pasture hired? Give result for  $a = 4$ ,  $s = 18$ ,  $c = 2$ , and  $t = 20$ .

11. A and B together carried 100 eggs to market, and each received the same sum. If A had carried as many as B, he would have received 18 pence for them; and if B had taken as many as A, he would have received 8 pence. How many had each? Put  $100 = a$ ,  $18 = m$ , and  $8 = n$ .

12. The difference between the hypotenuse and base of a right triangle is  $d$  feet, and the difference between the hypotenuse and perpendicular is  $b$  feet. Required the sides. Give results for  $d = 6$ , and  $b = 3$ .

**Note.**—For treatment of negative solutions, see “Numbers Symbolized,” Elementary Algebra, page 211.

## Simple Indeterminate Equations.

### Definitions and Principles.

357. If the number of simultaneous equations is less than the number of unknown quantities they contain, their solution, after all possible eliminations, generally depends upon the solution of a single equation containing two or more unknown quantities. Now, an unlimited number of values, sometimes within a particular range, may be assigned to the unknown quantities of such an equation that will satisfy it. Hence, the equation is said to be *Indeterminate*; and any problem that in its solution will finally depend upon the solution of an indeterminate equation is an *indeterminate problem*.

358. If the unknown quantities in an indeterminate equation be limited, however, to positive integers, the number of possible values of the unknown quantities will be greatly restricted, and the equation may even become impossible.

359. If  $a$  and  $b$ , in an equation of the form of  $ax + by = c$ , are both positive, and  $c$  is negative, the equation is not solvable in *positive* values of  $x$  and  $y$ .

For, if  $x$  and  $y$  were both *positive*,  $ax + by$  would be positive, and could not, therefore, equal  $c$ , a *negative* quantity.

**360.** If  $a$ ,  $b$ , and  $c$  are all positive integers, and  $c$  is less than  $a + b$ , the equation  $ax + by = c$  is not solvable in positive integers.

For, the least positive integral value that can be assigned to each of  $x$  and  $y$  is 1. Let  $x = 1$ , and  $y = 1$ , then  $ax + by = a + b$ , which is greater than  $c$ .

**361.** If  $a$  and  $b$ , in an equation of the form of  $ax + by = c$ , have a common factor not found in  $c$ , the equation is not solvable in positive integers.

For, let  $d =$  a factor common to  $a$  and  $b$ ,

and  $a = md$ , and  $b = nd$ ,

then,  $ax + by = c$  becomes  $mdx + ndy = c$  (1)

Divide (1) by  $d$ ,  $mx + ny = \frac{c}{d}$ . (2)

Now,  $mx + ny$  is integral for any integral values of  $x$  and  $y$ , since  $m$  and  $n$  are integral.

But,  $\frac{c}{d}$  is a fraction, since  $d$  is not a factor of  $c$ .

Hence, we have an integer equal to a fraction, which is impossible.

### Solution of Indeterminate Equations.

#### Illustrations.—

1. Solve  $3x + 4y = -5$  in positive integers.

**Solution:** This problem is impossible, because any positive values of  $x$  and  $y$  will make  $3x + 4y$  positive.

2. Solve  $3x + 4y = 5$  in positive integers.

**Solution:** This problem is impossible, since any positive integral values of  $x$  and  $y$  will make  $3x + 4y$  greater than 5.

3. Solve  $4x + 6y = 5$  in positive integers.

**Solution:** Divide by 2,  $2x + 3y = \frac{5}{2}$ .

Now, any integral values of  $x$  and  $y$  will make  $2x + 3y$  integral. Therefore, the problem is impossible.

4. Solve in positive integers  $3x + 4y = 10$ .

**Solution:** Given  $3x + 4y = 10$  (A)

Transpose,  $3x = 10 - 4y$  (1)

Divide by 3,  $x = \frac{10 - 4y}{3} = 3 - y + \frac{1 - y}{3}$  (2)

Now, since  $x$ ,  $y$ , and 3 are integers,

$$\frac{1 - y}{3} = \text{an integer.} \quad (3)$$

Let  $\frac{1 - y}{3} = m$ , an integer. (4)

Then  $y = 1 - 3m$ , an integer. (5)

Substitute (5) in (2) and reduce,

$$x = 2 + 4m, \text{ an integer.} \quad (6)$$

Now, any values may be assigned to  $m$  that will make  $x$  and  $y$  positive integers.

It is evident by inspection that  $m$  can not be a positive integer without making  $y$  negative. Neither can  $m$  be a negative integer without making  $x$  negative.

But  $m$  may be zero, in which case  $x = 2$  and  $y = 1$ .

5. Solve  $3x - 14y = 11$  in positive integers.

**Solution:**  $3x = 11 + 14y$  (1)

$$x = \frac{11 + 14y}{3} = 3 + 4y + \frac{2 + 2y}{3} \quad (2)$$

Since  $x$ , 3, and  $4y$  are integers,

$$\frac{2 + 2y}{3} = \frac{2(1 + y)}{3} = \text{an integer.} \quad (3)$$

Now, it is evident that if  $\frac{1 + y}{3}$  is an integer,  $\frac{2(1 + y)}{3}$  is also.

Let  $\frac{1 + y}{3} = m$ , an integer. (4)

Then,  $y = 3m - 1$  (5)

Substitute (5) in (2), and reduce,

$$x = 14m - 1. \quad (6)$$

Now, any value may be assigned to  $m$  that will render both  $x$  and  $y$  positive integers.

By inspection, we see that  $m$  can not equal zero or a negative quantity without making both  $x$  and  $y$  negative; but that it may be any positive quantity whatever. Hence, there is an indefinite number of solutions.

When  $m = 1$ ,  $x = 13$ , and  $y = 2$ ,

When  $m = 2$ ,  $x = 27$ , and  $y = 5$ ,

When  $m = 3$ ,  $x = 41$ , and  $y = 8$ ,

etc., etc., etc.

6. Solve  $3x - 14y = -11$ .

**Solution:** By a process similar to the one employed in solving Ex. 5, we find  
 $y = 3m + 1$   
 and  $x = 14m + 1$

Here  $m$  may be zero or any positive number; hence, there is again an indefinite number of solutions.

**Note.**—An equation of the form of  $ax - by = \pm c$  has an unlimited number of solutions in positive integers.

7. Solve  $3x + 5y = 37$  in positive integers.

**Solution:**  $3x = 37 - 5y$  (1)

$$x = \frac{37 - 5y}{3} = 12 - y + \frac{1 - 2y}{3} \quad (2)$$

Since  $x$ , 12, and  $y$  are positive integers,

$$\frac{1 - 2y}{3} = \text{an integer}, \quad (3)$$

and any whole number of times  $\frac{1 - 2y}{3} = \text{an integer}.$  (4)

For convenience of reduction, multiply  $\frac{1 - 2y}{3}$  by any number that will make the coefficient of  $y$  one more than a multiple of the denominator, which number in this case is 2.

$$\text{Then, } \frac{2 - 4y}{3} = -y + \frac{2 - y}{3} = \text{an integer}. \quad (5)$$

Since  $-y$  is an integer,

$$\frac{2 - y}{3} = \text{an integer}. \quad (6)$$

$$\text{Let } \frac{2 - y}{3} = m, \text{ an integer}; \quad (7)$$

$$\text{then } y = 2 - 3m \quad (8)$$

Substitute (8) in (2), and reduce,

$$x = 9 + 5m \quad (9)$$

Now, any value may be assigned to  $m$  that will make both  $x$  and  $y$  positive integers.

It readily appears by inspection that  $m$  may have any integral value from 0 to  $-1$  inclusive, and no other values.

$$\text{Let } m = 0, \quad x = 9, \quad \text{and } y = 2.$$

$$\text{Let } m = -1, \quad x = 4, \quad \text{and } y = 5.$$

8. Solve  $5x + 4y + 3z = 28$ .

$$\text{Solution: } 5x = 28 - (4y + 3z)$$

$$\therefore 5x < 28$$

$$x < 6$$

Let  $x = 1, 2, 3, 4, 5$ , successively;

then,  $4y + 3z = 23$  (1)  $4y + 3z = 18$  (2)

$4y + 3z = 13$  (3)  $4y + 3z = 8$  (4)

$4y + 3z = 3$  (5)

Solving (1),  $y = 2$  and  $z = 5$

or,  $y = 5$  and  $z = 1$

∴ Two sets of roots are  $x = 1, y = 2, z = 5$

and  $x = 1, y = 5, z = 1$ .

Solving (2),  $y = 3$  and  $z = 2$

∴ Another set of roots is  $x = 2, y = 3$ , and  $z = 2$ .

Solving (3),  $y = 1$  and  $z = 3$

∴ Another set of roots is  $x = 3, y = 1$ , and  $z = 3$ .

(4) and (5) are impossible.

∴ The equation has in all 4 sets of roots.

9. Find the least multiple of 5, which, when divided by 2, 3, or 6, will leave a remainder of 1.

**Solution :**

Let  $5x =$  the required multiple.

Then,  $\frac{5x-1}{2} =$  an integer; (1)

$\frac{5x-1}{3} =$  an integer; (2)

and  $\frac{5x-1}{6} =$  an integer. (3)

1.  $\frac{5x-1}{2} = 2x + \frac{x-1}{2} =$  an integer. (4)

∴  $\frac{x-1}{2} = m$ , an integer; (5)

whence  $x = 2m + 1$  (6)

2.  $\frac{5x-1}{3} = \frac{10m+4}{3} = 3m + 1 + \frac{m+1}{3} =$  an integer. (7)

∴  $\frac{m+1}{3} = n$ , an integer; (8)

whence  $m = 3n - 1$ , (9)

and  $x = 6n - 1$  (10)

3.  $\frac{5x-1}{6} = \frac{30n-6}{6} = 5n - 1$ , an integer.

∴  $x = 6n - 1$ , satisfies the three conditions.

∴  $5x = 30n - 5$ .

Now, the least positive value that  $5x$  can have is when  $n = 1$ , which makes  $5x = 25$ .

## EXERCISE 68.

Solve in positive integers :

1.  $2x + 4y = 10$

7.  $10x + 25y = -90$

2.  $3x + 5y = 20$

8.  $13x + 18y = 124$

3.  $4x + 7y = 26$

9.  $x + y + z = 5$

4.  $4x + 9y = 10$

10.  $2x + 3y + z = 8$

5.  $3x + 8y = 14$

11.  $x + 7y + 8z = 91$

6. 
$$\begin{cases} x + y + z = 15 \\ 2x + 3y + 5z = 28 \end{cases}$$

12. 
$$\begin{cases} 2x + 5y + 7z = 80 \\ 3x + 2y + 6z = 90 \end{cases}$$

Solve in least positive integers :

13.  $4x - 3y = 12$

17.  $15x - 30y = 21$

14.  $2x - y = 8$

18.  $25x - 32y = 100$

15.  $8x - 9y = 20$

19.  $14x - 17y = -60$

16.  $7x - 3y = 28$

20.  $28x - 32y = -45$

21. Find the least number that, being divided by 3, 5, and 6, shall leave the remainders 1, 3, and 4 respectively.

22. Divide 100 into two such parts that the smaller part divided by 4 leaves a remainder of 1, and the larger part divided by 9 leaves a remainder of 7.

23. Find the least multiple of 7 which divided by 4 leaves a remainder of 1, and divided by 6 leaves a remainder of 3.

24. A man employed two squads of men, paying \$3 a day to each of the first squad, and \$4 a day to each of the second. He paid to all \$56 a day. How many belonged to each squad?

25. In how many ways can \$100 be paid with \$5 and \$10 bills, using both kinds at each payment?

26. A man bought 100 animals for \$750 dollars, paying \$5 a head for pigs, \$8 a head for sheep, and \$15 a head for heifers. How many of each kind did he buy?

27. A in 2 days, B in 3 days, and C in 4 days, together earn \$47. A in 3 days, B in 4 days, and C in 6 days, earn \$68. What are the daily wages of each, supposing them to be an integral number of dollars?

28. A man invested \$10,000 in two kinds of town lots, paying \$190 each for the first kind, and \$130 each for the second kind. How many of each kind did he buy?

29. A boy sees that he can buy oranges at 2 cts., 3 cts., 4 cts., 5 cts., or 6 cts. apiece, and spend all his money; but, if he buys at 7 cts. apiece, he will have 5 cts. remaining. How much money has he if he has the least sum possible?

30. With 9 half-guineas and 6 half-crowns in my purse, how may I pay a debt of £4 11s. 6d.?

31. A man has less than \$200. He can buy sheep at \$4, \$6, or \$9 a head, and have \$3 remaining each time; or he can buy calves at \$7 or \$13 a head, and have \$1 over each time; or he can buy colts at \$11 a head, and have \$7 remaining. How much money has he, if he has a whole number of dollars?

32. Find four integral numbers, such that the sum of the first three shall be 18; the sum of the first, second, and fourth, 16; and the sum of the first, third, and fourth, 14.

33. What fraction is that, whose numerator being doubled, and denominator increased by 7, becomes equivalent to  $\frac{2}{3}$ ?

34. A number is expressed by three digits whose sum is 11; and, if 297 be added to the number, the result will be expressed by the same figures in reversed order.

35. A man in 36 minutes goes one mile. In  $x$  of these minutes he goes  $y$  yards,  $z$  feet, and 4 inches. Required the values of  $x$ ,  $y$ , and  $z$ , if  $x$  is the least whole number possible.

## Inequalities.

## Definitions and Principles.

**362.** An expression denoting that two quantities are unequal in value is an *Inequality*.

**363.** The symbol of inequality is an angle written between the two quantities, with the opening toward the greater one.

**Illustration.**—Thus,  $a > b$ , read  $a$  greater than  $b$ , and  $a < b$ , read  $a$  less than  $b$ , are inequalities.

**364.** The quantities separated by the symbol of inequality are the *members* of the inequality.

**365.** Two inequalities are said to subsist in the *same sense*, when the first member of each is either greater or less than the second.

**366.** In the doctrine of inequalities, a positive quantity is considered greater than a negative quantity irrespective of their numerical value; and, of two negative quantities, that is considered the greater which has the least numerical value.

**Note.**—Some authors express this fact by saying that “positive quantities are greater than zero, and negative quantities are less than zero.”

**367.** Inequalities like equations may be transformed in various ways without changing their sense; as follows:

1. *The same or equal quantities may be added to both members of an inequality.*
2. *The same or equal quantities may be subtracted from both members of an inequality.*
3. *Both members of an inequality may be multiplied by the same or equal positive quantities.*
4. *Both members of an inequality may be divided by the same or equal positive quantities.*

5. Both members of an inequality, if they are positive, may be raised to the same or equal powers.

6. The same or equal roots may be taken of both members of an inequality, provided the members are positive, and the positive values of the roots only are considered.

7. Two or more inequalities subsisting in the same sense may be added, member by member.

8. Two or more inequalities with positive members, and subsisting in the same sense, may be multiplied together, member by member.

**366.** The square of every quantity, whether positive or negative, is positive; hence,

$$(a - b)^2 > 0 \text{ for all real values of } a \text{ and } b; \quad (1)$$

$$\text{or, } a^2 - 2ab + b^2 > 0. \quad (2)$$

Adding  $2ab$  to both members [367, 1],

$$a^2 + b^2 > 2ab. \text{ Therefore,}$$

**Prin. 1.**—The sum of the squares of two quantities is greater than twice their product.

$$\mathbf{369.} \quad a^2 + b^2 > 2ab \quad (1)$$

$$a^2 + c^2 > 2ac \quad (2)$$

$$b^2 + c^2 > 2bc \quad (3)$$

$$\text{Add, } 2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc \text{ [367, 7]} \quad (4)$$

$$\text{Divide by 2, } a^2 + b^2 + c^2 > ab + ac + bc. \text{ Therefore,}$$

**Prin. 2.**—The sum of the squares of three quantities is greater than the sum of their products taken two and two.

### Problems and Examples.

**Illustration.**—Which is the greater,  $x^3 + y^3$  or

$x^2y + xy^2$ , for positive integral values of  $x$  and  $y$ ?

$$\text{Solution 1: } x^3 + y^3 > = < x^2y + xy^2 \quad (1)$$

$$\text{Factor, } (x + y)(x^2 - xy + y^2) > = < xy(x + y) \quad (2)$$

$$\text{Divide by } x + y, \quad x^2 - xy + y^2 > = < xy \quad (3)$$

$$\text{Add } xy, \quad x^2 + y^2 > = < 2xy \quad (4)$$

$$\text{But, } x^2 + y^2 > 2xy \text{ [P. 1]}$$

$$\therefore x^3 + y^3 > x^2y + xy^2.$$

**Solution 2:**  $x^2 + y^2 > 2xy$  [P. 1]  
 Subtract  $xy$ ,  $x^2 - xy + y^2 > xy$   
 Multiply by  $x + y$ ,  $x^3 + y^3 > x^2y + xy^2$ .

## EXERCISE 64.

- $a = b$  and  $c > d$ , is  $a - c > = < b - d$ ?
- $a, b, c$ , and  $d$  are positive quantities,  $a = b$  and  $c > d$ ,  
 is  $ad > = < bc$ ? Is  $-ac > = < -bd$ ?  
 Is  $\frac{a}{c} > = < \frac{b}{d}$ ?
- Changing the signs of both members of an inequality has what effect upon the sense of the inequality?
- If  $-a > -b$  and  $-c > -d$ , is  $ac > = < bd$ ?  
 $\frac{a}{c} > = < \frac{b}{d}$ ?  $-ac > = < -bd$ ?
- Which is the greater,  $\frac{a}{b}$  or  $\frac{a+x}{b+x}$ , when  $a < b$ ?  
 When  $a > b$ ?
- Which is the greater,  $\frac{a}{b}$  or  $\frac{a-x}{b-x}$ , when  $a < b$ ?  
 When  $a > b$ ?
- Show that  $\frac{a^4}{b^3} > 2a^2 - b^2$
- What integral values of  $x$  will satisfy both the inequalities:  $4x - 8 < 2x + 2$  and  $2x + 3 > 15 - 2x$ ?
- Between what two values must  $x$  lie, if  
 $\frac{ax}{5} + bx - ab > \frac{a^2}{5}$ , and  $\frac{bx}{7} + ax - ab < \frac{b^2}{7}$ ?
- If  $x^2 = m^2 + n^2$  and  $y^2 = p^2 + q^2$ , prove that  
 $xy > mp + nq$
- Which is the greater,  $\frac{x^3 + y^3}{x^2 + y^2}$  or  $\frac{x^2 + y^2}{x + y}$ , if  $x > y$ ?
- $xy > 18$  and  $x < 6$ , show that  $y > 3$ .  
 $xy = m^2$  and  $y < n^2$ , is  $x > = < \frac{m^2}{n^2}$ ?

13.  $x - y > 3$  and  $xy > 18$ , show that  $x > 6$   
and that  $y$  is indeterminate.

In the following exercises, all the letters are supposed to represent unequal positive quantities :

14. Which is the greater,  $x^3 + y^3$  or  $x^2 - xy + y^2$ ,  
if  $x + y > 1$  ?
15. Which is the greater,  
 $(m^2 + x^2)(n^2 + y^2)$  or  $(mn + xy)^2$  ?
16. Prove that  
 $ab + ac + bc < (a + b - c)^2 + (a + c - b)^2 + (b + c - a)^2$ ,  
 $a, b$ , and  $c$  being positive integers greater than 1.
17. Show that  $b^2c^2 + a^2c^2 + a^2b^2 > abc(a + b + c)$
18. Show that  $\frac{a-x}{a+x} < \frac{a^2-x^2}{a^2+x^2}$ , if  $x < a$
19. Show that  $(a + b + c)(a^2 + b^2 + c^2) > 9abc$
20. Show that  $\frac{2}{y+z} + \frac{2}{x+z} + \frac{2}{x+y} > \frac{9}{x+y+z}$
21. Show that  $(xy + xz + yz)^2 > 3xyz(x + y + z)$
22. Show that  $\frac{x}{y} + \frac{y}{x} + \frac{z}{u} + \frac{u}{z} > 4$
23. Show that  $\frac{n^3(n+2)}{(n+1)^3(n-1)} > \frac{(n+1)^3(n+3)}{(n+2)^3n}$
24. Show that  $xyz > (x + y - z)(x + z - y)(y + z - x)$ ,  
if each quantity is greater than the difference of the other two.
25. If  $x > y$ , prove that  $x - y > (\sqrt{x} - \sqrt{y})^2$
26. If  $\frac{u_1}{v_1} > \frac{u_2}{v_2} > \frac{u_3}{v_3} > \frac{u_4}{v_4} = r$ ,  
show that  $\frac{u_1 + u_2 + u_3}{v_1 + v_2 + v_3} > \frac{u_4}{v_4}$ ;  
also that  $\frac{u_4 + u_3 + u_2}{v_4 + v_3 + v_2} < \frac{u_1}{v_1}$

## CHAPTER VI.

### *RATIO, THEORY OF LIMITS, PROPORTION, VARIATION, AND LOGARITHMS.*

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#### Ratio.

##### Definitions and Principles.

**370.** The numerical measure of the relation existing between the magnitudes of two similar quantities is the *Ratio* of the quantities.

**371.** To find the ratio between two quantities, it is manifestly important to know which one is to be taken as the *base* of comparison. Thus, the ratio between  $a$  and  $b$ , if  $a = nb$ , is  $n$  if  $a$  be compared with  $b$ , but  $\frac{1}{n}$  if  $b$  be compared with  $a$ .

**372.** To obviate ambiguity in meaning, it is customary to name first the quantity to be compared and afterward the base of comparison, using the preposition *to* instead of *between* to show their relation. Thus, "the ratio of  $a$  to  $b$ " is an expression whose meaning is unmistakable.

**373.** The quantity to be compared, or the one first mentioned, is called the *antecedent*, and the base of comparison, or the one which follows is called the *consequent*.

**374.** Since the ratio of  $a$  to  $b$ , when  $a = nb$ , is  $n$ , it is evident that the ratio equals the antecedent divided by the consequent; or  $r = \frac{a}{c}$  ( $a$  denoting anteced. and  $c$  conseq.).

**375.** A ratio may be expressed by writing the antecedent before the consequent, with a colon between them, or the antecedent over the consequent in the form of a fraction. Thus, the ratio of  $a$  to  $b$  is denoted by  $a : b$ , or  $\frac{a}{b}$ .

**376.** The expression of a ratio is also called a ratio, but great care should be taken to distinguish between it and the true ratio, which is the value of the expression.

**377.** If two quantities have a common unit of measure, their ratio is said to be *commensurable*, and may be expressed by an integer or a common fraction.

Thus, if  $a$  contains  $c$   $m$  times, and  $b$  contains  $c$   $n$  times, the ratio of  $a$  to  $b$  is commensurable, and is  $\frac{m}{n}$ .

**378.** If two quantities do not contain a common unit of measure, their ratio is said to be *incommensurable*, and can not be expressed by an integer or a common fraction; but, by successive approximations, a common fraction may be obtained that will differ from it by less than any assignable quantity.

Thus: If  $b$  = the side of a square, and  $a$  = the diagonal,  $a^2 = 2b^2$ , and  $a = b\sqrt{2}$ ; and the ratio of  $a$  to  $b = \frac{b\sqrt{2}}{b} = \sqrt{2}$ , an incommensurable quantity.



Divide  $b$  into  $n$  equal parts, and let one of these parts be  $c$ ; then  $b = nc$ . Now, suppose  $c$  to be contained in  $a$  more than  $m$  times, but less than  $m + 1$  times, then  $a > mc$ , but  $a < (m + 1)c$ . Hence,  $\frac{a}{b} > \frac{mc}{nc}$ , or  $\frac{m}{n}$ , and  $\frac{a}{b} < \frac{(m + 1)c}{nc}$ , or  $\frac{m + 1}{n}$ . Now, the difference between  $\frac{m + 1}{n}$  and  $\frac{m}{n}$  is  $\frac{1}{n}$ ,  $\therefore$  the difference between  $\frac{m + 1}{n}$  and  $\frac{a}{b}$  is  $< \frac{1}{n}$ . Also, the difference between  $\frac{a}{b}$  and  $\frac{m}{n}$  is  $< \frac{1}{n}$ . Hence, if

we take either the common fraction  $\frac{m}{n}$  or  $\frac{m+1}{n}$  for the incommensurable ratio  $\frac{a}{b}$ , we will be correct to *within*  $\frac{1}{n}$ . But, as  $n$  may be continuously increased by doubling it, until it becomes greater than any assignable quantity,  $\frac{1}{n}$  may be continuously diminished by halving it until it becomes less than any assignable quantity.

**379.** Since  $r = \frac{a}{c}$  [374], it follows that

$$a = cr, \text{ and } c = \frac{a}{r}. \text{ Or,}$$

1. The antecedent equals the consequent multiplied by the ratio.

2. The consequent equals the antecedent divided by the ratio.

**380.** Since  $r = \frac{a}{c}$  [374],

$$nr = n \times \frac{a}{c} = \frac{na}{c} \text{ or } \frac{a}{c \div n}. \text{ Therefore,}$$

**Prin. 1.**—*Multiplying the antecedent or dividing the consequent multiplies the ratio.*

**381.** Since  $r = \frac{a}{c}$ ,

$$r \div n = \frac{a}{c} \div n = \frac{a \div n}{c} \text{ or } \frac{a}{cn}. \text{ Therefore,}$$

**Prin. 2.**—*Dividing the antecedent or multiplying the consequent divides the ratio.*

**382.** Since  $r = \frac{a}{c} = \frac{ma}{mc} = \frac{a \div m}{c \div m}$ ,

**Prin. 3.**—*Multiplying or dividing both terms of a ratio by the same quantity does not change the ratio.*

**383.** A compound ratio is the product of two or more simple ratios.

$$\text{Thus, } \left\{ \begin{array}{l} a : b \\ c : d \\ e : f \end{array} \right\} = \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}.$$

**384.** The *duplicate* ratio of two quantities is the ratio of their squares; the *triplicate* ratio the ratio of their cubes; the *sub-duplicate* ratio the ratio of their square roots; and the *sub-triplicate* ratio the ratio of their cube roots.

**385.** A ratio of *equality* is one whose terms are equal; a ratio of *greater inequality*, one whose antecedent is greater than the consequent; and a ratio of *lesser inequality*, one whose antecedent is less than the consequent.

## EXERCISE 65.

1. Find the ratio of 12 to 4; 18 to 90;  $\frac{3}{5}$  to 5; 8 to  $\frac{7}{8}$ ;  
 $\frac{2}{3}$  to  $\frac{5}{6}$ ;  $8\frac{1}{3}$  to  $12\frac{1}{2}$ ; 3.25 to 5.75

2. Find the value of  $a + b : a - b$ ;  $x + y : x^2 - y^2$ ;  
 $x^3 + y^3 : x^2 - xy + y^2$

3. Find the value of  $\frac{a+b}{a-b} : \frac{(a+b)^2}{a^2-b^2}$ ;

$$\frac{\sqrt{a^2-b^2}}{2ab} : \frac{4a^2b^2}{\sqrt{a^2-b^2}}; \sqrt{x(x+y)} : \sqrt{xy(x-y)}$$

4. Compound the ratios :

$$5 : 9, 6 : 8, 3\frac{1}{3} : 4\frac{1}{4}, \text{ and } 1.7 : .05$$

5. Compound the ratios :  $\frac{a+x}{a^2+x^2}$ ,  $\frac{a^4-x^4}{(a+x)^2}$ , and  $\frac{a^2-x^2}{xy}$

6. Compound the duplicate ratio of  $a+x$  to  $a-x$ , the triplicate ratio of  $a-x$  to  $a+x$ , and the simple ratio of  $a+x$  to  $a-x$ .

7. Compound the sub-duplicate ratios of  $x+y$  to  $x^2-y^2$ ;  $x-y$  to  $x+y$ ; and  $xy$  to  $x+y$ .

8. Which is the greater, a ratio of greater inequality or its duplicate?

9. Which is the greater, a ratio of lesser inequality or its duplicate?

10. What quantity must be added to both terms of the ratio  $x : y$  to produce the ratio  $m : n$ ?

11. What quantity must be subtracted from both terms of  $\frac{m+n}{m-n}$  to produce  $\frac{p+q}{p-q}$ ?

12. A in 6 days and B in 9 days earn as much as A in 9 days and B in 2 days. What are their relative wages per day?

13. The length of a rectangle increased by 12 rods bears the same relation to the width increased by 12 rods as the length increased by 6 rods does to the width increased by 6 rods. What is the ratio of the length to the width?

14. A man has two horses and a saddle. The value of the first horse and saddle bears the same relation to the value of the second horse that the value of the second horse and saddle bears to the value of the first horse. Compare the values of the two horses.

15. If the area of a rectangle equals the area of a given square, prove that the length of the rectangle bears the same relation to the side of the square that the side of the square does to the width of the rectangle.

16. A's money bears the same relation to B's that B's bears to C's; A's exceeds C's by \$600, and the three together have \$1400. How much has each?

17. Two boys are a certain distance apart. If they approach each other they will meet in 5 minutes, but if they walk in the same direction one will overtake the other in one hour. What are their relative rates of walking?

18. Arrange in the order of magnitude the ratios

$$\frac{a+b}{a-b}, \frac{a^2+b^2}{a^2-b^2}, \text{ and } \frac{(a+b)^2}{(a-b)^2},$$

when  $a$  and  $b$  are positive integers, and  $a > b$ .

## Theory of Limits.

**386.** A quantity that retains the same value throughout an operation or a discussion is a *Constant*.

**387.** A quantity whose value is supposed to change is a *Variable*.

**388.** Constants are generally expressed by the first and variables by the last letters of the alphabet.

**389.** A unit of conceivable value is a *Finite Unit*, and a quantity that can be expressed in finite units is a *finite quantity*.

**390.** A quantity too small to be expressed in finite units is said to be *infinitely small*.

**391.** An infinitely small variable is called an *Infinitesimal*, and is expressed by the character  $\circ$ , read an *infinitesimal*, or *zeroid*.

**392.** An infinitely small quantity is less than any finite unit, however small the unit be taken ; hence,

*Prin. 1.*—*A finite quantity increased or diminished by an infinitely small quantity is still a finite quantity.*

**393.** A quantity too large to be expressed in finite units is said to be *infinitely large*.

**394.** An infinitely large variable is called an *Infinite*, and is expressed by the character  $\alpha$ , read an *infinite*.

**395.** An infinite unit, however small it be taken, is greater than a finite quantity ; hence,

*Prin. 2.*—*An infinitely large quantity diminished by a finite or infinitely small quantity is still infinitely large.*

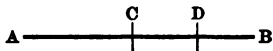
**396.** The entire absence of quantity is called *naught*, or *zero*, and is expressed by the character 0, read *zero*.

**397.** The unlimited whole of quantity, or, better, unlimited quantity, is called *Infinity*, and is expressed by the character  $\infty$ , read *infinity*.

**398.** The *limit* of a variable is a constant which the variable, from and after a fixed stage, is supposed to continually approach, but which it can never equal, although it may be made to differ from it by only an infinitesimal.

**399.** The limit of a variable may be a finite constant, zero, or infinity.

**Illustration.**—1. Let a point start at A, and move the distance A C ( $= \frac{1}{2}$  A B) the first minute, C D ( $= \frac{1}{2}$  C B) the second minute, and so on, unceasingly. It is evident that the whole distance ( $x$ ) traversed by the point continually approaches the distance ( $c$ ) from A to B, and will eventually differ from it by only an infinitesimal. Therefore,  $\lim. (x) = c$ , a finite constant.



2. If we represent the distance from the moving point to B by  $y$ ,  $y$  decreases continually toward zero, and will eventually differ from zero by only an infinitesimal. Therefore,  $\lim. (y) = 0$ .

3. If a point start at A and move unceasingly in a straight line, going each minute *twice* as far as the preceding, the entire distance ( $x$ ) traversed by the point increases without limit, which fact is expressed by  $\lim. (x) = \infty$ .

**Note.**—From these illustrations it will be seen that a variable, in approaching its limit, may pass from the finite to the infinitesimal, or the infinite state, or may remain, forever, a finite variable.

**400.** The final state of a variable as it approaches its limit is called its *ultimum*. Therefore,

**Prin. 3.**—*Lim. (x) is a finite constant if ult. (x) is a finite variable, and conversely.*

**Prin. 4.**—*Lim. (x) is 0, if ult. (x) is  $\infty$ , and conversely.*

**Prin. 5.**—*Lim. (x) is  $\infty$ , if ult. (x) is  $\alpha$ , and conversely.*

**401.** Let  $x = y$ ,  $\text{lim. } (x) = a$ , and  $\text{lim. } (y) = b$ .

Then, since  $x = y$ ,  $a - x = a - y$ .

But ult.  $(a - x) = 0$ , since  $\text{lim. } (x) = a$ .

$\therefore$  ult.  $(a - y) = 0$  ;

whence  $\text{lim. } y = a$ .

But  $\text{lim. } y = b$ .

$\therefore a = b$ . Therefore,

**Prin. 6.**—*If two variables are always equal their limits are equal.*

**402.** If, in the fraction  $\frac{x}{a}$ ,  $x$  decreases by a constant ratio until it becomes an infinitesimal and  $a$  remains a finite constant, the value of the fraction decreases in the same ratio [189, P. 3], and becomes an infinitesimal.

Therefore,

**Prin. 7.**  $\frac{0}{a} = 0$ . *An infinitesimal divided by a finite constant is an infinitesimal.*

**Corollary.**—*Lim.  $\left(\frac{0}{a}\right) = \text{lim. } 0$  [P. 6]; therefore,*  
 $\frac{0}{a} = 0$ .

**403.** Since  $\frac{0}{a} = 0$ , it follows that,

**Prin. 8.**  $0 \times a = 0$ . *An infinitesimal multiplied by a finite constant is an infinitesimal.*

**Cor.**—*Lim.  $(0 \times a) = \text{lim. } (0)$  [P. 6]; therefore,*  
 $0 \times a = 0$ .

**404.** Since  $0 \times a = 0$ , it follows that,

**Prin. 9.**  $\frac{0}{0} = a$ . *An infinitesimal divided by an infinitesimal may be any finite constant.*

**405.** If, in the fraction  $\frac{x}{a}$ ,  $x$  increases by a constant ratio until it becomes an infinite, and  $a$  remains a finite constant, the value of the fraction increases in the same ratio [187, P. 1], and becomes an infinite. Therefore,

**Prin. 10.**  $\frac{\alpha}{a} = \alpha$ . *An infinite divided by a finite constant is an infinite.*

**Cor.—Lim.**  $\left(\frac{\alpha}{a}\right) = \lim. \alpha$ ; therefore,  $\frac{\infty}{a} = \infty$ .

**406.** Since  $\frac{\alpha}{a} = \alpha$ , it follows that,

**Prin. 11.**  $\alpha \times a = \alpha$ . *An infinite multiplied by a finite constant is an infinite.*

**Cor.—Lim.**  $(\alpha \times a) = \lim. (\alpha)$ ; therefore,  $\infty \times a = \infty$ .

**407.** Since  $\alpha \times a = \alpha$ , it follows that,

**Prin. 12.**  $\frac{\alpha}{\alpha} = a$ . *An infinite divided by an infinite may be any finite constant.*

**408.** If, in the fraction  $\frac{a}{x}$ ,  $x$  decreases by a constant ratio until it becomes an infinitesimal and  $a$  remains a finite constant, the value of the fraction increases in the same ratio [188, P. 2], and becomes an infinite.

Therefore,

**Prin. 13.**  $\frac{a}{0} = \alpha$ . *A finite constant divided by an infinitesimal is an infinite.*

**Cor.—Lim.**  $\left(\frac{a}{0}\right) = \lim. \alpha$ ; therefore,  $\frac{a}{0} = \infty$ .

**409.** Since  $\frac{a}{0} = \alpha$ , it follows that,

**Prin. 14.**  $0 \times \alpha = a$ . *The product of an infinitesimal and an infinite may be any finite constant.*

410. Since  $\circ \times \alpha = a$ , it follows that,

**Prin. 15.**  $\frac{a}{\alpha} = \circ$ . A finite constant divided by an infinite is an infinitesimal.

**Cor.—Lim.**  $\left(\frac{a}{\alpha}\right) = \text{lim. } \circ$ ; therefore,  $\frac{a}{\infty} = 0$ .

411. Since  $\frac{\circ}{\circ}$ ,  $\frac{\alpha}{\alpha}$ , and  $\alpha \times \circ$  may each be satisfied by any finite constant, they are symbols of indetermination.

412. Let  $i_1$ ,  $i_2$ , and  $-i_3$  be three different infinitesimals.

1. Put  $i_1 + i_2 = I_1$ ;

Then  $i_1 = I_1 - i_2$ .

Now, since  $i_2$  is an infinitesimal,  $I_1$  can not be either finite or infinite, else would  $i_1$  be finite [392, P. 1], or infinite [395, P. 2].

$\therefore I_1$  is an infinitesimal.

2. Put  $I_1 - i_3 = I_2$ ;

Then  $I_1 = I_2 + i_3$ .

Now, since  $I_1$  and  $i_3$  are infinitesimals,  $I_2$  must also be an infinitesimal [P. 1 and P. 2]. Therefore,

**Prin. 16.**—The algebraic sum of a finite number of infinitesimals is an infinitesimal.

413. Let  $\text{lim. } x = a$ ,  $\text{lim. } y = b$ , and  $\text{lim. } z = c$ .

Put  $a - x = \circ_1$ ,  $b - y = \circ_2$ , and  $c - z = \circ_3$  [398];

then  $(a + b - c) - (x + y - z) =$

$$\circ_1 + \circ_2 - \circ_3 = \circ \text{ [P. 16];}$$

$\therefore \text{Lim. } (x + y - z) = a + b - c$  [398]. Therefore,

**Prin. 17.**—The limit of the algebraic sum of a finite number of variables equals the algebraic sum of their limits.

414. Since  $a \times x = x + x + x + \dots$  to  $a$  terms,  
 $\text{Lim. } (ax) = \text{lim. } x + \text{lim. } x + \text{lim. } x + \text{lim. } x + \dots$  to  
 $a$  terms [P. 17]  $= a \text{ lim. } (x)$ . Therefore,

**Prin. 18.**—*The limit of a constant times a variable equals the constant times the limit of the variable.*

415. Let  $\text{lim. } x = a$ , and  $\text{lim. } y = b$ .

Put  $a - x = o_1$ , and  $b - y = o_2$  [398];

then  $(a - x)(b - y) = o_1 \times o_2$ ,

or  $ab - ay - bx + xy = o_1 \times o_2 = 0$ .

$\therefore \text{Lim. } (ab - ay - bx + xy) = 0$  [P. 6],

or  $ab - a b - a b + \text{lim. } xy = 0$  [P. 17],

or  $\text{lim. } xy = ab$ . Therefore,

**Prin. 19.**—*The limit of the product of two variables equals the product of their limits.*

416. Since  $\text{lim. } x \times \text{lim. } y = \text{lim. } xy$ ,

$$\text{lim. } x = \frac{\text{lim. } xy}{\text{lim. } y},$$

$$\text{But } x = \frac{xy}{y},$$

$$\therefore \text{lim. } \left( \frac{xy}{y} \right) = \frac{\text{lim. } xy}{\text{lim. } y}. \text{ Therefore,}$$

**Prin. 20.**—*The limit of the quotient of two variables equals the quotient of their limits.*

**Illustrative Examples.**—

1. Find the limit of  $\frac{x-a}{x}$ , if  $\text{lim. } x = 0$ .

**Solution:**

$$\text{Lim. } \left( \frac{x-a}{x} \right)_{x=0} = \text{lim. } \left( 1 - \frac{a}{x} \right)_{x=0} = 1 - \infty = -\infty.$$

2. Find the limit of  $\frac{a-x}{x}$  when  $\text{lim. } x = \infty$ .

**Solution:**

$$\text{Lim. } \left( \frac{a-x}{x} \right)_{x=\infty} = \text{lim. } \left( \frac{a}{x} - 1 \right)_{x=\infty} = 0 - 1 = -1.$$

3. Find the limit of  $\frac{a-x}{x}$  when  $\lim. x = a$ .

**Solution :**

$$\lim. \left( \frac{a-x}{x} \right)_{x=a} = \lim. \left( \frac{a}{x} - 1 \right)_{x=a} = 1 - 1 = 0.$$

4. Find  $\lim. \left( \frac{x^2 - a^2}{x - a} \right)_{x=a}$

**Solution :**  $\lim. \left( \frac{x^2 - a^2}{x - a} \right)_{x=a} = \frac{a^2 - a^2}{a - a} = \frac{0}{0}$ , which is an irreducible form. But when  $\lim. x = a$ , ult.  $(x - a) = 0$ , and ult.  $\left( \frac{x^2 - a^2}{x - a} \right) = \text{ult. } (x + a)$ . Therefore,

$$\lim. \left( \frac{x^2 - a^2}{x - a} \right)_{x=a} = \lim. (x + a)_{x=a} = a + a = 2a.$$

**Note.**—It is usually best to reduce an improper fraction to the integral or mixed form before passing to the limit.

#### EXERCISE 66.

Find the value of :

1.  $\lim. \left( \frac{x+a}{x} \right)_{x=0}$

8.  $\lim. \left( x^5 + \frac{1}{x^5} \right)_{x=1}$

2.  $\lim. \left( \frac{x^2 - a^2}{x^2} \right)_{x=0}$

9.  $\lim. \left( \frac{x^3 - 1}{x^2 - 1} \right)_{x=1}$

3.  $\lim. \left( \frac{x^3 - a^3}{x - a} \right)_{x=a}$

10.  $\lim. \left\{ \frac{(x-a)^{\frac{1}{2}}}{(x^2 - a^2)^{\frac{1}{2}}} \right\}_{x=a}$

4.  $\lim. \left( \frac{x^3 + m^3}{x + m} \right)_{x=m}$

11.  $\lim. \left\{ \frac{x-a}{x^{\frac{1}{2}} - a^{\frac{1}{2}}} \right\}_{x=a}$

5.  $\lim. \left( \frac{x^4 + 1}{x^2} \right)_{x=-1}$

12.  $\lim. \left\{ \frac{x - x^{\frac{1}{2}}}{2x^{\frac{1}{2}}} \right\}_{x=0}$

6.  $\lim. \left( \frac{x^2 + x - 5}{x + 3} \right)_{x=\infty}$

13.  $\lim. \left\{ \frac{x^2 - a}{x - \sqrt{a}} \right\}_{x=\sqrt{a}}$

7.  $\lim. \left( \frac{x^2 + (a+b)x + ab}{x+a} \right)_{x=0}$

14. Lim.  $\left(\frac{x^5 - 1}{x - 1}\right)_{x=0}$

17. Lim.  $\left\{\frac{x^2 - 1}{x - 1}\right\}_{x=1}$

15. Lim.  $\left\{\frac{x + \sqrt{x}}{x - \sqrt{x}}\right\}_{x=\infty}$

18. Lim.  $\left\{\frac{x + \sqrt{a^2 - x^2}}{x - \sqrt{a^2 - x^2}}\right\}_{x=a}$

16. Lim.  $\left\{\frac{(x + a)(x + b)}{x + c}\right\}_{x=\infty}$

### Proportion.

417. A *Proportion* is an expression of equality between two equal ratios.

418. A proportion is expressed by writing a double colon, or the symbol of equality, between the equal ratios.

Thus,  $a : b :: c : d$ , read  $a$  is to  $b$  as  $c$  is to  $d$ ;

$a : b = c : d$ , read the ratio of  $a$  to  $b$  equals the ratio of  $c$  to  $d$ ; and

$\frac{a}{b} = \frac{c}{d}$ , read as either of the above, or  $a$  divided by  $b$  equals  $c$  divided by  $d$ , are proportions.

419. Since each ratio has two terms, a proportion has four terms. The first and fourth are called the *extremes*; the second and third the *means*; the first and third the *antecedents*; the second and fourth the *consequents*; the first and second the *first couplet*; and the third and fourth the *second couplet*.

420. A *mean proportional* between two quantities is a quantity to which the first bears the same relation as the mean bears to the second.

Thus,  $b$  is a mean proportional between  $a$  and  $c$  when  $a : b :: b : c$ .

**421.** A *third proportional* to two quantities is a quantity to which the second bears the same relation as the first bears to the second.

Thus,  $c$  is a third proportional to  $a$  and  $b$ , when  $a : b :: b : c$ .

**422.** A *compound proportion* is one in which one or both ratios are compound.

**423.** A *multiple proportion* is one in which three or more ratios are placed equal to one another.

Thus,  $a : b :: c : d :: e : f :: g : h :: i : j$  is a multiple proportion.

**424.** A *continued proportion* is a multiple proportion in which each consequent equals the following antecedent ; as,  $a : b :: b : c :: c : d$ , etc.

**425.** A proportion is taken by *alternation* when antecedent is compared with antecedent and consequent with consequent.

**426.** A proportion is taken by *inversion* when the consequents are made the antecedents and the antecedents the consequents.

**427.** A proportion is taken by *composition* when the sum of the terms of each ratio is compared with either term of that ratio.

**428.** A proportion is taken by *division* when the difference of the terms of each ratio is compared with either term of that ratio.

**429.** Four quantities are *inversely* or *reciprocally* proportional when the first is to the second as the fourth is to the third, or as the reciprocal of the third is to the reciprocal of the fourth.

## Theorems.

**430.** *In any proportion the product of the extremes equals the product of the means.*

Given  $a : b :: c : d$ . Prove  $a \times d = b \times c$ .

**Demonstration:**  $\frac{a}{b} = \frac{c}{d}$  [418]. Clearing of fractions,

$$a \times d = b \times c.$$

**Corollary 1.**—*Either extreme equals the product of the means divided by the other extreme.*

**Cor. 2.**—*Either mean equals the product of the extremes divided by the other mean.*

**431.** *If the product of two factors equals the product of two other factors, either pair may be taken as the means and the other pair as the extremes of a proportion.*

Given  $a \times d = b \times c$  to prove

$$(1) a : b :: c : d$$

$$(5) c : a :: d : b$$

$$(2) a : c :: b : d$$

$$(6) c : d :: a : b$$

$$(3) b : a :: d : c$$

$$(7) d : b :: c : a$$

$$(4) b : d :: a : c$$

$$(8) d : c :: b : a$$

**Demonstration:** 1. Given  $a \times d = b \times c$  (A)

Divide (A) by  $b \times d$ ,  $\frac{a}{b} = \frac{c}{d}$ ;

whence  $a : b :: c : d$ .

2. Divide (A) by  $c \times d$ ,  $\frac{a}{c} = \frac{b}{d}$ ;

whence  $a : c :: b : d$ .

In a similar manner all the rest may be derived.

**432.** *In any proportion, the means or the extremes may be interchanged, severally or jointly.*

Given  $a : b :: c : d$  to prove

$$(1) a : c :: b : d$$

$$(2) d : b :: c : a$$

$$(3) d : c :: b : a$$

**Demonstration:** Since  $a : b :: c : d$ ,  $a \times d = b \times c$  [430]; whence (1), (2), and (3) follow according to 431.

**Cor.**—*If four quantities are in proportion, they are also in proportion by alternation.*

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**433.** *In any proportion, the means may exchange places with the extremes in any order.*

Given  $a : b :: c : d$  to prove

$$(1) \ b : a :: d : c \qquad (3) \ c : a :: d : b$$

$$(2) \ b : d :: a : c \qquad (4) \ c : d :: a : b$$

**Demonstration:** Since  $a : b :: c : d$ ,  $a \times d = b \times c$  [430]; whence (1), (2), (3), and (4) follow according to 431.

**Cor.**—*If four quantities are in proportion, they are also in proportion by inversion.*

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**434.** *If four quantities are in proportion, they are also in proportion by composition.*

Given  $a : b :: c : d$  to prove

$$(1) \ a + b : b :: c + d : d$$

$$(2) \ a + b : a :: c + d : c$$

**Demonstration:** Given  $a : b :: c : d$ ,  
then  $a : c :: b : d$  by alternation.

Let  $r =$  the common ratio, then

$$\frac{a}{c} = r, \text{ and } \frac{b}{d} = r,$$

whence  $a = cr$  and  $b = dr$ ,

and  $a + b = (c + d)r$ ;

$$\text{or, } \frac{a + b}{c + d} = r = \frac{a}{c} = \frac{b}{d}.$$

$$\therefore a + b : c + d :: a : c$$

$$\text{and } a + b : c + d :: b : d;$$

whence  $a + b : a :: c + d : c$  by alternation;

also,  $a + b : b :: c + d : d$  by alternation.

---

**435.** *If four quantities are in proportion, they are also in proportion by division.*

Given  $a : b :: c : d$  to prove

$$(1) a - b : b :: c - d : d$$

$$(2) a - b : a :: c - d : c$$

**Demonstration:**  $a : c :: b : d$  by alternation.

Let  $r =$  the common ratio, then

$$\frac{a}{c} = r, \text{ and } \frac{b}{d} = r,$$

whence  $a = cr$  and  $b = dr$ ,

and  $a - b = (c - d)r$ ;

$$\text{or, } \frac{a - b}{c - d} = r = \frac{a}{c} = \frac{b}{d}.$$

$$\therefore a - b : c - d :: a : c$$

and  $a - b : c - d :: b : d$ ;

whence  $a - b : a :: c - d : c$  by alternation,

and  $a - b : b :: c - d : d$  by alternation.

**436.** *If four quantities are in proportion, they are in proportion by composition and division.*

Given  $a : b :: c : d$  to prove

$$a + b : a - b :: c + d : c - d$$

**Demonstration:** Since  $a : b :: c : d$ ,

$$a + b : b :: c + d : d \quad [434] \quad (A)$$

$$\text{and } a - b : b :: c - d : d \quad [435]. \quad (B)$$

Taking (A) and (B) by alternation,

$$\frac{a + b}{c + d} = \frac{b}{d}, \text{ and } \frac{a - b}{c - d} = \frac{b}{d}; \text{ whence}$$

$$\frac{a + b}{c + d} = \frac{a - b}{c - d}, \text{ or}$$

$$a + b : c + d :: a - b : c - d, \text{ whence,}$$

by alternation,  $a + b : a - b :: c + d : c - d$ .

**437.** *In any proportion, the antecedents or the consequents may be multiplied or divided by the same quantity, severally or jointly.*

Given  $a : b :: c : d$  to prove

$$(1) ma : b :: mc : d$$

$$(2) a : nb :: c : nd$$

$$(3) ma : nb :: mc : nd$$

$$(4) \quad \frac{a}{m} : b :: \frac{c}{m} : d$$

$$(5) \quad a : \frac{b}{n} :: c : \frac{d}{n}$$

$$(6) \quad \frac{a}{m} : \frac{b}{n} :: \frac{c}{m} : \frac{d}{n}$$

- Demonstration :** (1) is true, since both ratios are multiplied by  $m$ .  
 (2) is true, since both ratios are divided by  $n$ .  
 (3) is true, since both ratios are multiplied by  $m$  and divided by  $n$ .  
 (4) is true, since both ratios are divided by  $m$ .  
 (5) is true, since both ratios are multiplied by  $n$ .  
 (6) is true, since both ratios are divided by  $m$  and multiplied by  $n$ .
- 

**438.** *In any proportion, either extreme or mean may be multiplied and the other extreme or mean divided by the same quantity.*

Given  $a : b :: c : d$  to prove

$$(1) \quad am : b :: c : \frac{d}{m}$$

$$(5) \quad am : bn :: \frac{c}{n} : \frac{d}{m}$$

$$(2) \quad \frac{a}{m} : b :: c : dm$$

$$(6) \quad am : \frac{b}{n} :: cn : \frac{d}{m}$$

$$(3) \quad a : bn :: \frac{c}{n} : d$$

$$(7) \quad \frac{a}{m} : bn :: \frac{c}{n} : dm$$

$$(4) \quad a : \frac{b}{n} :: cn : d$$

$$(8) \quad \frac{a}{m} : \frac{b}{n} :: cn : dm$$

- Demonstration :** (1) is true, since both ratios are multiplied by  $m$ .  
 (2) is true, since both ratios are divided by  $m$ .  
 (3) is true, since both ratios are divided by  $n$ .  
 (4) is true, since both ratios are multiplied by  $n$ .  
 (5) is true, since both ratios are multiplied by  $m$  and divided by  $n$ .  
 (6) is true, since both ratios are multiplied by  $m$  and  $n$ .  
 (7) is true, since both ratios are divided by  $m$  and  $n$ .  
 (8) is true, since both ratios are divided by  $m$  and multiplied by  $n$ .
- 

**439.** *If four quantities are in proportion, any equimultiples of the first couplet are proportional to any equimultiples of the second couplet.*

Given  $a : b :: c : d$  to prove

$$ma : mb :: nc : nd$$

**Demonstration:** This proposition is evident, since the value of neither ratio is changed.

**Cor.**—*Equimultiples of two quantities are proportional to the quantities themselves.*

Thus,  $ma : mb :: a : b$ .

**Scholium.**  $m$  and  $n$  may be either integral or fractional.

**440.** *If four quantities are in proportion, the terms of the first couplet, increased or diminished by like parts of themselves, are proportional to the terms of the second couplet increased or diminished by like parts of themselves.*

Given  $a : b :: c : d$  to prove

$$a \pm \frac{m}{n}a : b \pm \frac{m}{n}b :: c \pm \frac{r}{s}c : d \pm \frac{r}{s}d$$

**Demonstration:** If  $a : b :: c : d$ ,

$$\left(1 \pm \frac{m}{n}\right)a : \left(1 \pm \frac{m}{n}\right)b :: \left(1 \pm \frac{r}{s}\right)c : \left(1 \pm \frac{r}{s}\right)d \text{ [439].}$$

Expand,  $a \pm \frac{m}{n}a : b \pm \frac{m}{n}b :: c \pm \frac{r}{s}c : d \pm \frac{r}{s}d$ .

**Cor.**—*If two quantities be increased or diminished by like parts of themselves, the results will be proportional to the quantities themselves.*

Thus,  $\left(1 \pm \frac{m}{n}\right)a : \left(1 \pm \frac{m}{n}\right)b :: a : b$ .

Whence,  $a \pm \frac{m}{n}a : b \pm \frac{m}{n}b :: a : b$ .

**441.** *If four quantities are in proportion, like powers and like roots of them are also in proportion.*

Given  $a : b :: c : d$  to prove

$$(1) \quad a^m : b^m :: c^m : d^m$$

$$(2) \quad \sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}$$

**Demonstration:** Given  $\frac{a}{b} = \frac{c}{d}$ . Raise both members to the  $m$ th power; and extract the  $n$ th root of both members.

$$1. \frac{a^m}{b^m} = \frac{c^m}{d^m}$$

$$2. \frac{\sqrt[n]{a^m}}{\sqrt[n]{b^m}} = \frac{\sqrt[n]{c^m}}{\sqrt[n]{d^m}}; \text{ whence,}$$

$$(1) a^m : b^m :: c^m : d^m$$

$$(2) \sqrt[n]{a^m} : \sqrt[n]{b^m} :: \sqrt[n]{c^m} : \sqrt[n]{d^m}.$$


---

**442.** *A mean proportional between two quantities equals the square root of their product.*

Given  $b$  a mean proportional between  $a$  and  $c$  to prove that  $b = \sqrt{ac}$ .

**Demonstration:** Since  $b$  is a mean proportional between  $a$  and  $c$ ,

$$a : b :: b : c; \text{ whence}$$

$$b^2 = ac \text{ [430]. Extracting square root of}$$

$$\text{both members, } b = \sqrt{ac}.$$


---

**443.** *If two proportions have a couplet in each the same, the remaining couplets form a proportion.*

Given  $a : b :: c : d$  and

$$a : b :: m : n \text{ to prove}$$

$$c : d :: m : n$$

**Demonstration:**  $\frac{a}{b} = \frac{c}{d}$  and  $\frac{a}{b} = \frac{m}{n}$ ; hence  $\frac{c}{d} = \frac{m}{n}$ , or

$$c : d :: m : n.$$


---

**444.** *If two proportions have their antecedents equal their consequents are in proportion, and if they have their consequents equal their antecedents are in proportion.*

1. Given  $a : b :: c : d$  and

$$a : x :: c : y \text{ to prove}$$

$$b : d :: x : y$$

**Demonstration:**  $a : c :: b : d$  and

$$a : c :: x : y, \text{ by alternation;}$$

$$\therefore b : d :: x : y \text{ [443].}$$

2. Given  $a : b :: c : d$  and  
 $m : b :: n : d$  to prove  
 $a : c :: m : n$

**Demonstration:**  $a : c :: b : d$  and  
 $m : n :: b : d$ , by alternation;  
 $\therefore a : c :: m : n$  [443].

---

**445.** *If two proportions have a couplet in each in proportion, the remaining couplets form a proportion.*

Given  $a : b :: c : d$ , (A)  
 $m : n :: p : q$ , and (B)  
 $c : d :: p : q$ , to prove (C)  
 $a : b :: m : n$

**Demonstration:** From (A) and (C) we have  
 $a : b :: p : q$  [443]. (D)  
 From (D) and (B) we have  
 $a : b :: m : n$  [443].

---

**446.** *If two proportions have three terms of the one equal to three terms of the other, each to each, the remaining terms are also equal.*

Given  $a : b :: c : d$  and  
 $a : b :: c : x$  to prove  
 $d = x$

**Demonstration:**  $d = \frac{b \times c}{a}$  [430, Cor. 1],  
 and  $x = \frac{b \times c}{a}$  [430, Cor. 1],  
 $\therefore d = x$ .

---

**447.** *The products or quotients of the corresponding terms of two proportions form a proportion.*

Given  $a : b :: m : n$  and  
 $c : d :: p : q$  to prove

$$1. a \times c : b \times d :: m \times p : n \times q$$

$$2. \frac{a}{c} : \frac{b}{d} :: \frac{m}{p} : \frac{n}{q}$$

$$\text{Demonstration: } a \times n = b \times m \quad [430] \quad (A)$$

$$c \times q = d \times p \quad [430] \quad (B)$$

1. Multiply (A) by (B),

$$(a \times c)(n \times q) = (b \times d)(m \times p); \text{ whence,}$$

$$a \times c : b \times d :: m \times p : n \times q \quad [431].$$

2. Divide (A) by (B),

$$\frac{a \times n}{c \times q} = \frac{b \times m}{d \times p}, \text{ or } \frac{a}{c} \times \frac{n}{q} = \frac{b}{d} \times \frac{m}{p}; \text{ whence,}$$

$$\frac{a}{c} : \frac{b}{d} :: \frac{m}{p} : \frac{n}{q} \quad [431].$$

**448.** *The corresponding members of two equations form a proportion.*

Given  $a = b$  and  $c = d$  to prove

$$a : c :: b : d$$

**Demonstration:** Dividing  $a = b$  by  $c = d$ , member by member,

$$\frac{a}{c} = \frac{b}{d}; \text{ whence } a : c :: b : d.$$

**449.** *In any multiple proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Given  $a : b :: c : d :: e : f :: g : h$  to prove

$$a + c + e + g : b + d + f + h :: a : b$$

**Demonstration:** Let  $\frac{a}{b} = r$ ;  $\frac{c}{d} = r$ ;  $\frac{e}{f} = r$ ;  $\frac{g}{h} = r$ ; then

$$a = br; c = dr; e = fr; g = hr; \text{ adding}$$

$$a + c + e + g = (b + d + f + h)r; \text{ dividing}$$

$$\frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b}; \text{ therefore,}$$

$$a + c + e + g : b + d + f + h :: a : b.$$

**450.** *If any number of quantities are in continued proportion, the first is to the third as the square of the first*

is to the square of the second; the first is to the fourth as the cube of the first is to the cube of the second, etc.

Given  $a : b :: b : c :: c : d :: d : e :: e : f$  to prove

$$(1) a : c :: a^2 : b^2$$

$$(3) a : e :: a^4 : b^4$$

$$(2) a : d :: a^3 : b^3$$

$$(4) a : f :: a^5 : b^5$$

**Demonstration:** 1.  $\frac{a}{b} = \frac{a}{b}$

3.  $\frac{c}{d} = \frac{a}{b}$

5.  $\frac{e}{f} = \frac{a}{b}$

2.  $\frac{b}{c} = \frac{a}{b}$

4.  $\frac{d}{e} = \frac{a}{b}$

Take the product of 1 and 2,

$$\frac{a}{c} = \frac{a^2}{b^2}; \text{ whence } a : c :: a^2 : b^2.$$

Take the product of 1, 2, and 3,

$$\frac{a}{d} = \frac{a^3}{b^3}; \text{ whence } a : d :: a^3 : b^3.$$

Take the product of 1, 2, 3, and 4,

$$\frac{a}{e} = \frac{a^4}{b^4}; \text{ whence } a : e :: a^4 : b^4.$$

Take the product of 1, 2, 3, 4, and 5,

$$\frac{a}{f} = \frac{a^5}{b^5}; \text{ whence } a : f :: a^5 : b^5.$$

**451.** *If the successive approximations of two incommensurable ratios are equal, each to each, the ratios themselves are equal.*

**Demonstration:** The successive approximations of each ratio may be regarded as successive values of a variable whose limit is the incommensurable ratio. But, if the successive approximations are always equal, each to each, the variables are always equal, and, therefore, their limits, or the incommensurable ratios, are equal [401, P. 6].

**Illustrative Examples.**—1. If  $a : b :: c : d$ , prove that

$$ma + nb : ma - nb :: mc + nd : mc - nd.$$

**Solution:** Given  $a : b :: c : d$ .

Multiply antecedents by  $m$  and consequents by  $n$ ,

$$ma : nb :: mc : nd \text{ [437].}$$

Take this proportion by composition and division,

$$ma + nb : ma - nb :: mc + nd : mc - nd \text{ [436].}$$

2. Given  $\frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} = b$ , to find the values of  $x$ .

**Solution :** Write the equation in the form of a proportion,

$$a - \sqrt{a^2 - x^2} : a + \sqrt{a^2 - x^2} :: b : 1.$$

Take this proportion by composition and division,

$$2a : 2\sqrt{a^2 - x^2} :: b + 1 : 1 - b;$$

Divide the terms of the first couplet by 2,

$$a : \sqrt{a^2 - x^2} :: b + 1 : 1 - b;$$

Square both couplets,

$$a^2 : a^2 - x^2 :: b^2 + 2b + 1 : 1 - 2b + b^2;$$

Take this proportion by division,

$$x^2 : a^2 :: 4b : b^2 + 2b + 1;$$

Extract the square root of both couplets,

$$x : a :: \pm 2\sqrt{b} : b + 1; \text{ whence}$$

$$x = \pm \frac{2a\sqrt{b}}{b + 1}.$$

#### EXERCISE 67.

If  $a : b :: c : d$ , prove that

1.  $md : nc :: mb : na$

2.  $a - b : c - d :: a + b : c + d$

3.  $2a + 3b : 2a - 3b :: 2c + 3d : 2c - 3d$

4.  $a^2 - c^2 : b^2 - d^2 :: a^2 : b^2$

5.  $3a + 2b : 3c + 2d :: 2a + 3b : 2c + 3d$

6.  $a + \frac{p}{q}a : b - \frac{r}{s}b :: c + \frac{p}{q}c : d - \frac{r}{s}d$

7.  $pa + qc : pb + qd :: a : b$

8.  $a^2x + aby + b^2y : c^2x + cdy + d^2y :: b^2 : d^2$

If  $a : b :: b : c$ , prove that

9.  $a + b : b + c :: b : c$

10.  $na + mb : nb + mc :: a : b$

11.  $(a + b)^2 : (b + c)^2 :: a : c$

12. If  $m \times y = x \times n$ , prove that

$$am + bn : ax + by :: m : x$$

13. If  $(m+n+p+q)(m-n-p+q) =$   
 $(m-n+p-q)(m+n-p-q),$   
 prove that  $m:n::p:q$

14. If  $p = \sqrt{\frac{1}{3}q}$ , prove that  $p:1::\sqrt{q}:\sqrt{3}$

15. If  $a:b::c:d::e:f$ , prove that  
 $a+b:a-b::c+d:c-d::e+f:e-f$

16. If  $a-b:x::b-c:y::c-a:z,$   
 prove that  $x+y+z=0$

17. If  $a:b::c:d$ , prove that  $a+x:b+x::c+x:d+x$   
 is false for any finite value of  $x$ .

18. Given  $x+1:x+4::2x-1:x+6$   
 to find the value of  $x$ .

19. Given  $x:x^2-1::15-7x:8-8x$   
 to find the value of  $x$ .

20. Given  $a+\sqrt{x}:a-\sqrt{x}::b+\sqrt{c}:b-\sqrt{c}$   
 to find the value of  $x$ .

21. Solve  $\left\{ \begin{array}{l} x+y:x-y::2:1 \\ x^2+y^2=80 \end{array} \right\}$

22. Solve  $\left\{ \begin{array}{l} x^3+y^3:x+y::7:1 \\ x^2-y^2:x-y::5:1 \end{array} \right\}$

23. Solve  $\left\{ \begin{array}{l} x^3+y^3:(x+y)^3::xy(x+y):27 \\ x+y=3 \end{array} \right\}$

24. Solve  $\frac{x+\sqrt{3-x}}{x-\sqrt{3-x}}=3$       25. Solve  $\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}=\frac{a}{b}$

26. Solve  $\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}=\frac{a+x}{a-x}$

27. Solve  $\frac{\sqrt{4x+1}+2\sqrt{x}}{\sqrt{4x+1}-2\sqrt{x}}=9$

28. Solve  $\frac{\sqrt[3]{x+1}+\sqrt[3]{x-1}}{\sqrt[3]{x+1}-\sqrt[3]{x-1}}=2$

$$29. \text{ Solve } \left\{ \begin{array}{l} x^3 + y^3 : x^3 - xy + y^3 :: \frac{1}{2} : \frac{1}{12} \\ x^3 - y^3 : x^3 + xy + y^3 :: \frac{1}{3} : \frac{1}{6} \end{array} \right\}$$

### Concrete Examples.

**Illustration.**—1. A's age is to B's as 3 is to 4; but, 20 years hence, their ages will be to each other as 5 is to 6. Required the age of each.

**Solution:** Let  $3x = \text{A's age, then will}$   
 $4x = \text{B's age; and}$   
 $3x + 20 = \text{A's age 20 years hence, and}$   
 $4x + 20 = \text{B's age 20 years hence; whence}$   
 $3x + 20 : 4x + 20 :: 5 : 6; \text{ or}$   
 $x : 3x + 20 :: 1 : 5, \text{ by division.}$   
 Therefore,  $5x = 3x + 20, \text{ or}$   
 $2x = 20, \text{ and } x = 10;$   
 $3x = 30, \text{ A's age,}$   
 $4x = 40, \text{ B's age.}$

2. If $a$ men, in $b$ days, working $c$ hours a day, can dig a ditch $d$ rods long, $e$ feet wide, and $f$ feet deep,	How many $(x)$ men, in $m$ days, working $n$ hours a day, can dig a ditch $r$ rods long, $s$ feet wide, and $t$ feet deep?
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**Solution:** Since the number of men needed *diminishes* as the number of days and the number of hours per day increase, and *increases* as the number of rods in length, feet in width, and feet in depth increase, it is in *indirect* ratio with the first two mentioned, and in *direct* ratio with the last three; therefore,

$$x : a :: \left\{ \begin{array}{l} b : m \\ c : n \\ r : d \\ s : e \\ t : f \end{array} \right.$$

whence  $x = \frac{a \times b \times c \times r \times s \times t}{m \times n \times d \times e \times f}.$

## EXERCISE 68.

1. The length of a certain rectangle is to its width as 5 to 4; and, if 8 feet be added to each dimension, the length will be to the width as 7 to 6. Required the length and the breadth.

2. The sum of two numbers is to their difference as 7 to 3, and the sum of their squares equals 1856. What are the numbers?

3. Eight silver and 3 gold coins are worth as much as 4 silver and 4 gold ones. What is the relative value of a silver to a gold coin?

4. In what time will  $a$  dollars at  $r$  per cent gain as much as  $b$  dollars in  $t$  years at  $n$  per cent?

5. A farmer, having mixed a certain number of bushels of corn and oats, found that, if he had mixed 10 more bushels of each, there would have been 4 bushels of oats to 3 bushels of corn; but, if he had mixed 5 bushels less of each, there would have been 5 bushels of oats to 3 bushels of corn. How many bushels of each did he mix?

6. A hare is 50 of her leaps before a hound, and takes 4 leaps to the hound's 3; but two of the hound's leaps are equal to three of the hare's. How many leaps must the hound take to catch the hare?

7. When two trains run in opposite directions they pass each other in 2 seconds; but, if they run in the same direction, the faster train passes the slower one in 30 seconds. Compare the rates of the two trains.

8. A passenger on a fast train observes that he passes a slow train in 20 seconds when the trains run in the same direction, and in 2 seconds when they run in opposite directions. Compare the rates of the trains.

9. What number must be added to each of the numbers 3, 7, and 13, to make the second one a mean proportional between the other two?

10. There is a number consisting of three digits, the first (or hundreds') of which is to the second as the second is to the third; the number itself is to the sum of the digits as 124 to 7; and, if 594 be added to the number, the digits will be inverted. What is the number?

11. Show that no number added to three consecutive numbers will make the second a mean proportional between the other two.

12. Two cog-wheels, one having  $a$  cogs and the other  $b$ , run together. In how many revolutions of the former will the latter gain  $c$  revolutions?

13. If an  $a$ -cent loaf of bread weighs  $b$  ounces when flour is  $c$  dollars a barrel, how many ounces should an  $r$ -cent loaf weigh when flour is  $t$  dollars a barrel?

14. One cask contains 10 gallons of alcohol and 6 gallons of water, while another contains 12 gallons of alcohol and 5 gallons of water. If one gallon be taken from each cask and mixed, what will be the relative quantities of alcohol and water in the mixture?

15. What quantity must be added to each of the quantities  $p$ ,  $q$ ,  $r$ , and  $s$ , to make them proportional?

16. If  $a$  men in  $b$  days can build  $c$  rods of wall, how many men can build  $d$  rods of wall  $\frac{m}{n}$  as high and  $\frac{p}{q}$  as thick in  $e$  days?

17. The quantities of water that flow through cylindrical pipes are to each other as the squares of the diameters of the pipes. If a pipe whose diameter is  $a$  inches fills a cistern in  $c$  hours, in what time will a pipe whose diameter is  $n$  inches fill it?

18. Similar volumes are to each other as the cubes of their like dimensions. What is the diameter of a ball that weighs as much as the sum of two balls whose diameters are  $a$  inches and  $b$  inches respectively?

## Variation.

**452.** Two quantities may be so related that when one of them varies the other varies also.

**Illustration.**—The value of a pile of lumber varies as the amount in the pile varies.

**453.** One quantity varies *directly* as another when it retains a constant ratio to the other.

**Illustration.**—The distance a train goes varies directly as the time it runs, since it retains a constant ratio to the time.

**454.** One quantity varies *inversely* as another when it varies directly as the reciprocal of the other.

**Illustration.**—The number of days required to do a given piece of work varies inversely as the number of men employed, since it varies directly as the reciprocal of the number of men employed.

**455.** One quantity varies *jointly* as two others when it varies directly as their product.

**Illustration.**—The area of a rectangle varies jointly as the base and altitude, since it varies directly as their product.

**456.** One quantity varies *directly* as a second and *inversely* as a third when it varies directly as their quotient.

**Illustration.**—The altitude of a rectangle varies directly as the area and inversely as the base, since it varies directly as their quotient.

**457.** A *Variation* is an expression denoting that one algebraic quantity varies directly as another. The symbol of variation is  $\propto^*$ , read *varies as*, meaning *varies directly as*.

**Illustration.**—That  $x$  varies directly as  $y$  is expressed by  $x \propto y$ , read  $x$  varies as  $y$ .

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\* As a symbol of quantity,  $\propto$  denotes *an infinite*; as a symbol of relation, *varies as*. No ambiguity can arise from this double use of the symbol.

That  $x$  varies inversely as  $y$  is expressed by  $x \propto \frac{1}{y}$ , read  $x$  varies as one divided by  $y$ .

That  $x$  varies jointly as  $y$  and  $z$  is expressed by  $x \propto yz$ , read  $x$  varies as  $y$  times  $z$ , or as  $yz$ .

That  $x$  varies directly as  $y$  and inversely as  $z$  is expressed by  $x \propto \frac{y}{z}$ , read  $x$  varies as  $y$  divided by  $z$ .

### Principles.

**458.** *If one quantity varies as another it is a constant number of times the other.*

Given  $x \propto y$  to prove that  $x = ry$ , in which  $r$  is a constant.

**Demonstration.**—Since  $x \propto y$ ,  $\frac{x}{y} = r$ , a constant ratio [453],

Clear of fractions,  $x = ry$ .

**459.** *If one variable quantity is a constant number of times another it varies as the other.*

Given  $x = ry$ , in which  $r$  is constant, to prove that  $x \propto y$ .

**Demonstration.**—Since  $x = ry$ ,  $\frac{x}{y} = r$ , a constant;

whence  $x \propto y$  [453].

**460.** *If one quantity varies as a second and the second varies as a third, the first varies as the third.*

Given  $x \propto y$  and  $y \propto z$ , to prove that  $x \propto z$ .

**Demonstration.**—  $x = ry$ , in which  $r$  is a constant, (A)

$y = r'z$ , in which  $r'$  is a constant, (B)

Substitute (B) into (A),  $x = rr'z$ , in which  $rr'$  is a constant.

Therefore,  $x \propto z$  [453].

**461.** *If two quantities vary as the same quantity, their sum, their difference, and the square root of their product vary as that quantity.*

Given  $x \propto z$  and  $y \propto z$  to prove that

$$x + y \propto z, \quad x - y \propto z, \quad \text{and} \quad \sqrt{x y} \propto z.$$

**Demonstration.**—

$$x = rz, \text{ in which } r \text{ is constant.} \quad (\text{A})$$

$$y = r'z, \text{ in which } r' \text{ is constant.} \quad (\text{B})$$

Add (A) and (B),  $x + y = (r + r')z$ , in which  $r + r'$  is constant;

hence,  $x + y \propto z$ .

Subtract (B) from (A),  $x - y = (r - r')z$ , in which  $r - r'$  is constant;

hence,  $x - y \propto z$ .

Multiply (A) by (B),  $xy = rr' \times z^2$ .

Extract  $\sqrt{\quad}$ ,  $\sqrt{xy} = \sqrt{rr'} \times z$ , in which  $\sqrt{rr'}$  is constant;

hence,  $\sqrt{xy} \propto z$ .

**462.** *The product or quotient of the first members of two variations vary as the product or quotient of the second members.*

Given  $x \propto y$  and  $z \propto w$ .

**Demonstration.**—1.  $x = ry$ , in which  $r$  is constant. (\text{A})

$z = r'w$ , in which  $r'$  is constant. (\text{B})

Multiply (A) by (B),  $xz = rr' \times yw$ , in which  $rr'$  is constant;

hence,  $xz \propto yw$ .

2. Divide (A) by (B),  $\frac{x}{z} = \frac{r}{r'} \times \frac{y}{w}$ , in which  $\frac{r}{r'}$  is constant;

hence,  $\frac{x}{z} \propto \frac{y}{w}$ .

**463.** *If one quantity varies as another, any power or root of the first varies as the same power or root of the second.*

Given  $x \propto y$  to prove that  $x^n \propto y^n$  and  $\sqrt[n]{x} \propto \sqrt[n]{y}$ .

**Demonstration.**—  $x = ry$ , in which  $r$  is constant. (\text{A})

Raise both members to the  $n$ th power,

$$x^n = r^n y^n, \text{ in which } r^n \text{ is constant;}$$

hence,  $x^n \propto y^n$ .

Extract the  $n$ th root of both members of (A),

$$\sqrt[n]{x} = \sqrt[n]{r} \times \sqrt[n]{y}, \text{ in which } \sqrt[n]{r} \text{ is constant;}$$

hence,  $\sqrt[n]{x} \propto \sqrt[n]{y}$ .

**464.** *If  $x \propto y$  when  $z$  is constant, and  $x \propto z$  when  $y$  is constant, then  $x \propto yz$  when  $y$  and  $z$  both vary.*

**Demonstration.**  $x = ry$  when  $y$  varies and  $z$  is constant. Since  $r$  is constant it can not contain the variable  $y$ ; hence,  $x$  can contain  $y$  only once as a factor.

Again,  $x = r'z$  when  $z$  varies and  $y$  is constant. Since  $r'$  is constant it can not contain the variable  $z$ ; hence,  $x$  can contain  $z$  only once as a factor.

Again, since  $y$  may be constant when  $z$  varies, and  $z$  constant when  $y$  varies, neither of these quantities can contain the other as a factor; nor can they contain a common variable factor.

Therefore,  $y$  and  $z$  are both factors of  $x$ , and are the only factors that may vary; or  $x$  is some constant number of times  $yz$ , as  $r''yz$ ; whence  $x \propto yz$ .

**Illustrative Examples.** — 1. If  $x \propto y$ ; and  $x = 6$  when  $y = 2\frac{1}{2}$ , what will  $x$  equal when  $y = 4\frac{3}{4}$ ?

**Solution:** If  $x \propto y$ ,  $x = ry$ , in which  $r$  is constant. Since  $x = 6$  when  $y = 2\frac{1}{2}$ ,  $6 = r \times 2\frac{1}{2}$ ; whence  $r = \frac{12}{5}$ . Therefore,  $x = \frac{12}{5}y$ .

Substituting  $y = 4\frac{3}{4}$  in this equation,  $x = \frac{12}{5} \times \frac{19}{4} = 11\frac{2}{5}$ .

2. If  $x = u + v$  and  $u \propto y$  and  $v \propto y^2$ ; also when  $y = 2$ ,  $v = 4$ , and  $u = 6$ ; what is the equation between  $x$  and  $y$ ?

**Solution:** If  $u \propto y$  and  $v \propto y^2$ ,  $u = ry$  and  $v = r'y^2$ , in which  $r$  and  $r'$  are constants.

Since  $v = 4$  and  $u = 6$  when  $y = 2$ ,  
 $6 = r \times 2$  and  $4 = r' \times 4$ ; whence  
 $r = 3$  and  $r' = 1$ ; therefore,  
 $u = 3y$  and  $v = y^2$ ; and  
 $u + v = x = 3y + y^2$ .

3. If 12 sheep cost \$96, what will 25 sheep cost at the same price per head?

**Solution:** Let  $x$  = the cost of any number of sheep,  
 and  $y$  = any number of sheep.

Since the cost varies directly as the number,

$x \propto y$ , and  $x = ry$ , in which  $r$  is constant.

Since  $x$  is 96 when  $y$  is 12,  $96 = r \times 12$ , or  $r = 8$ .

Therefore,  $x = 8y$ . Substituting 25 for  $y$ ,  
 $x = 8 \times 25 = 200$ , the cost in dollars.

4. The attraction of bodies varies inversely as the square of the distance they are apart. If the body A attracts the body B with a force of 10 pounds when they are 5 feet apart, with what force does it attract it when they are 25 feet apart?

**Solution:** Let  $x$  = the force of attraction at any distance,  
and  $y$  = any distance measured in feet.

Now,  $x \propto \frac{1}{y^2}$  or  $x = \frac{r}{y^2}$ , in which  $r$  is constant.

Clearing of fractions,  $r = xy^2$ ; but  $y = 5$  when  $x = 10$ ; hence,  
 $r = 10 \times 5^2 = 250$ ; and

$$x = \frac{250}{y^2}. \text{ Now substitute 25 for } y,$$

$$x = \frac{250}{25^2} = \frac{2}{5} \text{ of a pound.}$$

#### EXERCISE 89.

1. If  $x \propto y$  prove that  $x \propto my$  in which  $m$  is constant.
2. If  $x \propto y$  prove that  $vx \propto vy$  in which  $v$  may be either a constant or a variable.
3. If  $x \propto vy$ , prove that  $v \propto \frac{x}{y}$ .
4. If  $x \propto vy$  and  $v \propto z$  and  $y \propto z^2$ , prove that  $x \propto z^3$ .
5. If  $x \propto y$  when  $v$  and  $z$  are constant,  $x \propto v$  when  $y$  and  $z$  are constant, and  $x \propto z$  when  $v$  and  $y$  are constant, prove that  $x \propto yz$  when  $v$  alone is constant,  $x \propto vy$  when  $z$  alone is constant,  $x \propto vz$  when  $y$  alone is constant, and  $x \propto vyz$  when all vary.

6.  $x \propto y$  and when  $x = 8$ ,  $y = 20$ . What is the value of  $y$  when  $x = 20$ ?

7.  $x \propto y$  and  $y \propto \frac{1}{z}$ ; also, when  $z = 5$ ,  $x = 10$ . What is the value of  $x$  when  $z = \frac{1}{2}\sqrt{2}$ ?

8.  $x + y \propto x - y$ ,  $x \propto v$  and  $y \propto z$ ; when  $v = 4$ ,  $x = 10$ ; when  $z = 6$ ,  $y = 15$ ; and when  $y = 10$ ,  $x = 25$ . What is the ratio of  $v$  to  $z$ ?

9.  $x = v + w$ , in which  $v \propto y$  and  $w \propto \frac{1}{y}$ ; when  $y = 1$ ,  $x = 6$ ; and when  $y = 2$ ,  $x = 5$ . Find the equation between  $x$  and  $y$ .

10. The space ( $s$ ) through which a falling body passes varies as the square of the time ( $t^2$ ) when the force of gravity ( $f$ ) is constant; and  $s \propto f$ , when  $t$  is constant; also,  $2s = f$ , when  $t = 1$  second. Find the equation between  $f$ ,  $s$ , and  $t$ .

11. What will  $a$  pounds of sugar cost if  $b$  pounds cost  $c$  cents?

12. If  $a$  men can do a piece of work in  $c$  days, in how many days can  $d$  men do it?

13. The area of a circle varies jointly as the circumference and diameter, and the circumference varies as the diameter. Prove that the area varies as the square of the diameter.

14. The volume of a sphere varies jointly as the surface and diameter, and the surface varies as the square of the diameter. Prove that the volume varies as the cube of the diameter.

15. The circumference of a circle varies as the diameter, and when the diameter is *one* the circumference is 3.1416. What must the diameter be that the circumference may be *one*?

16. The velocity acquired by a falling body varies as the square root of the height of its fall. If the height is 16.08 feet when the velocity is 32.16 feet, what is the velocity when the height is 257.28 feet?

17. The amount of water flowing through a pipe varies as the square of the diameter of the pipe when the time is constant, and as the time when the diameter of the pipe is constant. Prove that the time varies inversely as the square of the diameter when the amount is constant.

18. The height through which a body falls varies as the square of the time. If in 3 seconds it falls 144·72 feet, in how many seconds will it fall 402 feet?

19. The velocity with which a liquid escapes from an orifice varies as the square root of the depth of the liquid above the orifice; when the depth is 402 inches the velocity is 160·8 inches. What is the velocity when the depth is 16·08 inches?

20. The square of the time required for a planet to revolve around the sun varies as the cube of its distance from the sun. The earth is 93,000,000 miles from the sun, and revolves around it in one year; in how many years will Saturn, whose distance from the sun is 890,000,000 miles, make a revolution?

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### Logarithms.

#### General Definitions and Principles.

465. The logarithm of a quantity is the exponent by which an assumed quantity, called the base, must be affected to produce the quantity.

**Illustration.**—If we assume  $a$  as the base, and suppose  $a^x = m$ , then will  $x$  be the logarithm of  $m$ , which is expressed by  $\log. m = x$ .

466. Since  $a^1 = a$  and  $a^0 = 1$ ,  $\log. a = 1$  and  $\log. 1 = 0$ . Therefore,

**Prin. 1.**—*The logarithm of the base is unity and the logarithm of unity is zero.*

467. Let  $a^x = m$  and  $a^y = n$ ,

then,  $\log. m = x$  and  $\log. n = y$ .

Now,  $a^{x+y} = m \times n$  and  $a^{x-y} = m \div n$ ;

$\therefore \log. (m \times n) = x + y$ , and  $\log. (m \div n) = x - y$ .

Therefore,

**Prin. 2.**—*The logarithm of the product of two quantities equals the sum of their logarithms.*

**Prin. 3.**—*The logarithm of the quotient of two quantities equals the logarithm of the dividend minus the logarithm of the divisor.*

**468.** Suppose  $a^x = m$ , then  $\log. m = x$ .

Now,  $(a^x)^y = m^y$ , or  $a^{xy} = m^y$ .

$\therefore \log. (m^y) = xy = y \log. m$ . Therefore,

**Prin. 4.**—*The logarithm of a quantity affected by any exponent equals the exponent times the logarithm of the quantity.*

### The Common System of Logarithms.

**469.** In the common system of logarithms, the assumed base is 10, the radix of the decimal system of notation.

Since  $10^1 = 10$ ,  $\log. 10 = 1$ .

Since  $10^2 = 100$ ,  $\log. 100 = 2$ .

Therefore, the logarithm of any number lying between 10 and 100 lies between 1 and 2, or is  $1 +$  a decimal. From this we see that a logarithm may consist of two parts, an integral and a decimal part. The integral part is called the *characteristic* and the decimal part the *mantissa*.

**470.**  $10^0 = 1 \quad \therefore \log. 1 = 0$

$10^1 = 10 \quad \therefore \log. 10 = 1$

$10^2 = 100 \quad \therefore \log. 100 = 2$

$10^3 = 1000 \quad \therefore \log. 1000 = 3$

etc., etc.,                      etc., etc.

Since  $\log. 1 = 0$  and  $\log. 10 = 1$ , the logarithm of any number between 1 and 10, as 3.25, lies between 0 and 1, or is a pure decimal. That is, if a number contains one and only one integral place, the characteristic of its logarithm is zero.

Since  $\log. 10 = 1$  and  $\log. 100 = 2$ , the logarithm of any number between 10 and 100, as 42.3, lies between 1 and 2, or is  $1 +$  a decimal.

That is, if a number contains two and only two integral places, the characteristic of its logarithm is 1.

In a similar manner it may be shown that if a number contains 3, 4, 5, etc., integral places, the characteristic of its logarithm will be 2, 3, 4, etc. Therefore,

**Prin. 5.**—*The characteristic of the logarithm of any whole or mixed number is positive, and numerically one less than the number of integral places in the number.*

$$\begin{array}{llll}
 471. \quad 10^0 = 1 & \therefore \log. 1 = 0 \\
 10^{-1} = .1 & \therefore \log. .1 = -1 \\
 10^{-2} = .01 & \therefore \log. .01 = -2 \\
 10^{-3} = .001 & \therefore \log. .001 = -3 \\
 \text{etc.,} & \text{etc.,} & \text{etc.,} & \text{etc.}
 \end{array}$$

For convenience, the characteristic of a logarithm will always be so taken as to make the mantissa positive.

Since  $\log. 1 = 0$  and  $\log. .1 = -1$ , the logarithm of any number between 1 and .1, as .35, lies between 0 and  $-1$ , or is  $-1 +$  a decimal. That is, if a pure decimal contains no cipher between the decimal point and the first significant figure, the characteristic of its logarithm is  $-1$ .

Since  $\log. .1 = -1$  and  $\log. .01 = -2$ , the logarithm of any decimal between .1 and .01, as .025 lies between  $-1$  and  $-2$ , or is  $-2 +$  a decimal. That is, if a pure decimal contains one cipher between the decimal point and the first significant figure, the characteristic of its logarithm is  $-2$ .

In a similar manner it may be shown that if a pure decimal contains two, three, four, five, etc., ciphers before the first significant figure, the characteristic of its logarithm will be  $-3$ ,  $-4$ ,  $-5$ ,  $-6$ , etc. Therefore,

**Prin. 6.**—*The characteristic of the logarithm of a pure decimal is negative, and is numerically one greater than the number of ciphers between the decimal point and the first significant figure.*

**472.** To designate that the characteristic is negative and the mantissa positive, it is customary to write the negative sign above the characteristic.

Thus,  $\bar{3}.1472 = -3 + .1472$ .

**473.** By adding 10 to a logarithm with a negative characteristic, then indicating the subtraction of 10 from the result, the characteristic may be made positive.

Thus,  $\bar{3}.4675 = 7.4675 - 10$ .

**474.** Since  $\log. (n \times 10) = \log. n + \log. 10$  [P. 2] =  $\log. n + 1$ , it follows that annexing a cipher to a number, or moving the decimal point one place rightward, increases the characteristic of its logarithm by unity, but does not alter the mantissa.

Again, since  $\log. (n \div 10) = \log. n - \log. 10$  [P. 3] =  $\log. n - 1$ , it follows that moving the decimal point one place leftward in a number, decreases the characteristic of its logarithm by unity, but does not alter the mantissa.

Therefore,

*Prin. 7.*—*The logarithms of all numbers composed of the same significant figures arranged in the same order, irrespective of the position of the decimal point, have the same mantissa.*

Thus, Log.	355 = 2.55023	log.	35.5 = 1.55023
	Log. 3550 = 3.55023	log.	3.55 = 0.55023
	Log. 35500 = 4.55023	log.	.355 = $\bar{1}$ .55023
Also, Log.	7 = 0.84510	log.	.7 = $\bar{1}$ .84510
	Log. 70 = 1.84510	log.	.07 = $\bar{2}$ .84510
	Log. 700 = 2.84510	log.	.007 = $\bar{3}$ .84510
	Log. 7000 = 3.84510	log.	.0007 = $\bar{4}$ .84510

**475.** The following five-place table of mantissas, arranged from Fünfstellige logarithmisch-trigonometrische Tafeln von Theodor Wittstein, Hannover, will answer all purposes here intended, namely, to familiarize the pupil with the use of logarithms in multiplication, division, involution, and evolution.

Table of Logarithms.

N.	0	1	2	3	4	5	6	7	8	9
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743
11	04189	04532	04922	05308	05690	06070	06446	06819	07188	07555
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059
13	11394	11727	12057	12385	12710	13033	13354	13672	13988	14301
14	14618	14922	15229	15534	15836	16137	16435	16732	17026	17319
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140
16	20412	20688	20952	21219	21484	21748	22011	22272	22531	22789
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25289
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646
19	27875	28108	28330	28556	28780	29003	29226	29447	29667	29885
20	30108	30320	30535	30750	30963	31175	31387	31597	31806	32015
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984
23	36178	36361	36549	36736	36922	37107	37291	37475	37658	37840
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330
26	41497	41664	41830	41996	42160	42325	42488	42651	42818	42975
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379
32	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097
40	60208	60314	60423	60531	60638	60746	60853	60959	61066	61172
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225
45	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159
54	73239	73320	73400	73480	73560	73640	73719	73799	73879	73957

Table of Logarithms.

N.	0	1	2	3	4	5	6	7	8	9
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511
57	75597	75664	75740	75815	75891	75967	76042	76118	76193	76268
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550
64	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448
75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154
78	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795
81	90849	90902	90956	91009	91062	91116	91169	91222	91275	91328
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855
83	91908	91960	92012	92065	92117	92169	92221	92273	92324	92376
84	92428	92480	92531	92583	92634	92686	92737	92788	92840	92891
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399
88	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890
89	94939	94988	95036	95085	95134	95182	95231	95279	95328	95376
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520
99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957

## Problems.

## 1. To find the logarithm of a number by the Table.

**Illustrations.**—1. Find the logarithm of 247.

**Solution:** In column headed "N" look for 24. Opposite this, in column headed 7, see 39270, the required mantissa. The characteristic is 2 [P. 5].

Therefore,  $\log. 247 = 2.39270$ .

## 2. Find the logarithm of .035.

**Solution:** Man. .035 = man. 350 [P. 7].

In column headed "N" look for 35. Opposite this, in column headed 0, see 54407, the required mantissa. The characteristic is 2 [P. 6].

$\therefore \log. .035 = \bar{2}.54407$ , or  $8.54407 - 10$ .

3. Find  $\log. 6$ .

**Solution:** Man. 6 = man. 600 [P. 7].

In column headed "N" look for 60, and opposite it, in column headed 0, see 77815, the required mantissa. The characteristic is 0 [P. 5].

$\therefore \log. 6 = 0.77815$ .

4. Find  $\log. 43.752$ .

**Solution:** Man. 43.752 = man. 437.52 [P. 7].

Man. 437 = 64048; man. 438 = 64147; difference = 99.

If a difference of 1 in two numbers makes a difference of 99 in the mantissas, a difference of .52 in the numbers makes a difference of  $.52 \times 99 = 51.48$  in the mantissas. Discard .48, being less than .5.

$\therefore$  Man. 43.752 = 64048 + 51 = 64099, and  $\log. 43.752 = 1.64099$ .

## EXERCISE 70.

Find the logarithms of :

1. 375	7. .002	13. 4862
2. 856	8. .0043	14. 3476
3. 904	9. .0001	15. 34.24
4. 700	10. .0463	16. 900.9
5. 401	11. 1.47	17. .0046325
6. 1.00	12. 34.8	18. .187643

**2. To find the number corresponding to a given logarithm.**

**476.** The number corresponding to a given logarithm is called the *antilogarithm*.

**Illustrations.**—1. To find the antilog. of 1·48144.

**Solution:** Look for the mantissa 48144. This is found in column headed 3, opposite 30, in column headed "N." Therefore, the number, irrespective of the decimal point, is 303; but the characteristic is 1, hence the number of integral places in the number is 2 [P. 5].

∴ Antilog. 1·48144 = 30·3.

2. Find the antilog. of  $\bar{2}$ ·63749.

**Solution:** Mantissa 63749 is found in column 4, opposite 43, in column "N." Therefore, the number, irrespective of decimal point, is 434. But the characteristic is  $\bar{2}$ .

∴ Antilog. of  $\bar{2}$ ·6375 = ·0434 [P. 6].

3. Find antilog. of 4·51250.

**Solution:** The mantissa 51250 is not found in the table. It, however, lies between 51188 (the mantissa corresponding to 325) and 51322 (the mantissa corresponding to 326).

The difference between 51250 and 51188 is 62.

The difference between 51322 and 51188 is 134.

If a difference of *one* in two numbers makes a difference of 134 in their mantissas, to make a difference of 62 in their mantissas it will require a difference of  $\frac{62}{134}$  of 1, or ·46 in the numbers.

∴ ·51250 is the mantissa of 325·46, irrespective of the position of the decimal point; but, since the characteristic is 4,

Antilog. of 4·5125 = 32546 [P. 5].

**EXERCISE 71.**

Find the antilogarithms of :

1. 2·54777

3. 3·71096

5.  $\bar{2}$ ·51851

2. 1·62014

4. 0·50243

6.  $\bar{2}$ ·00000

7.  $\bar{1}$ ·41162

10. 6·43775 — 10

8. 6·66181 — 10

11. 3·42564 — 10

9. 0·47712

12. 7·30400 — 10

13. 3·00012	15. 0·11111	17. 1·00000
14. 6·32015	16. 0·00000	18. 7·34260
19. 2·00000 — 10	22. 5·04044	
20. 3·47635 — 10	23. 5·55555 — 10	
21. 8·00064 — 10	24. 8·40603 — 10	

### 3. To find the cologarithm of a number.

477. The cologarithm of a number is the logarithm of the reciprocal of the number.

Thus,  $\text{colog. } N = \log. \frac{1}{N}$ .

478.  $\log. \frac{1}{N} = \log. 1 - \log. N = -\log. N = 10 - \log. N - 10 = (10 - \log. N) - 10$ . Therefore,

$\text{Colog. } N = (10 - \log. N) - 10$ .

To find the value of  $10 - \log. N$ , begin at the left hand and subtract each figure of  $\log. N$  from 9, except the last significant figure, which subtract from 10.

### EXERCISE 72.

Find the colog. of :

1. 42	7. ·0135	13. 4·732
2. 350	8. 43·21	14. ·0086
3. 425	9. ·6375	15. ·00043
4. 6·75	10. 4000	16. ·00001
5. ·0432	11. 3·06	17. 7·002
6. 4270	12. ·0004	18. ·000018

### 4. To multiply by logarithms.

**Illustrations.**—1. Find the product of 24·36 and 3·45.

**Solution:**  $\log. (24·36 \times 3·45) = \log. 24·36 + \log. 3·45$  [P. 2].

$$\log. 24·36 = 1·38668$$

$$\log. 3·45 = 0·53782$$

$$\log. \text{prod.} = 1·92450$$

$$\text{antilog.} = 84·042 \text{ Ans.}$$

2. Find the product of 2.45, .0016, and .0346.

**Solution:** Log.  $(2.45 \times .0016 \times .0346) = \text{log. } (2.45 \times .0016) + \text{log. } .0346 = \text{log. } 2.45 + \text{log. } .0016 + \text{log. } .0346.$

$$\text{log. } 2.45 = 0.88917$$

$$\text{log. } .0016 = 7.20412 - 10$$

$$\text{log. } .0346 = 8.53908 - 10$$

$$\text{log. prod.} = 16.13237 - 20 = 4.13237$$

$$\text{antilog.} = .0001356.$$

### EXERCISE 73.

Find the value of :

1.  $325 \times 478$

7.  $.0436 \times 2.45 \times .072$

2.  $4.36 \times 3.48$

8.  $21.42 \times 21.42 \times 3.1416$

3.  $.0645 \times 3.047$

9.  $.00036 \times 40.04 \times 3.24$

4.  $560.2 \times .00082$

10.  $2.02 \times 20.2 \times .0202$

5.  $3.746 \times 2.415$

11.  $4763 \times 3451 \times .00625$

6.  $3456 \times 3.1416$

12.  $.7854 \times .6666 \times 3.303$

### 5. To divide by logarithms.

479. Log.  $\left(\frac{a}{b}\right) = \text{log. } a - \text{log. } b$  [P. 3]  $= \text{log. } a + 10 - \text{log. } b - 10 = \text{log. } a + \text{colog. } b$ . Therefore,

**Prin. 8.**—*The logarithm of the quotient of two quantities equals the logarithm of the dividend, plus the cologarithm of the divisor.*

**Illustrations.**—1. Divide 432 by .0143.

**Solution 1:** Log.  $(432 \div .0143) = \text{log. } 432 - \text{log. } .0143$

$$\text{log. } 432 = 2.63548$$

$$\text{log. } .0143 = 2.15534$$

$$\text{log. of quot.} = 4.48014$$

$$\text{antilog.} = 30208.$$

**Solution 2:** Log.  $(432 \div .0143) = \text{log. } 432 + \text{colog. } .0143$  [P. 8].

$$\text{log. } 432 = 2.63548$$

$$\text{colog. } .0143 = 11.84466 - 10$$

$$\text{log. of quot.} = 14.48014 - 10$$

$$\text{antilog.} = 30208.$$

2. Find the value of  $\frac{.0436 \times 3.072}{4072 \times .00016}$

**Solution:** Log.  $\frac{.0436 \times 3.072}{4072 \times .00016} = \log. .0436 + \log. 3.072 + \text{colog. } 4072 + \text{colog. } .00016.$

log. .0436 = 8.63949 - 10  
 log. 3.072 = 0.48742  
 colog. 4072 = 6.39020 - 10  
 colog. .00016 = 13.79588 - 10  
 log. of quot. = 29.31299 - 30 = ̄9.31299  
 antilog. = .20558.

#### EXERCISE 74.

Find the value of :

- |   |   |
|---|---|
| 1. $1342 \div 67$   | 7. $3.25 \times 2.14 \div 5.16$   |
| 2. $4063 \div 24.5$                                       | 8. $1.42 \times 3.42 \div .0014$  |
| 3. $.0075 \div 2.45$                                      | 9. $3.004 \div 245 \times 5.05$   |
| 4. $3000 \div 1.24$                                       | 10. $(18.6 \times 300) \div (42.5 \times 3.05)$   |
| 5. $\frac{42.75}{.0016} \times \frac{3.85}{4.27}$         | 11. $\frac{8}{9} \times \frac{.001}{.7} \times \frac{5.6}{.008} \times \frac{3.25}{.006}$ |
| 6. $\frac{56.3}{3.75} \times \frac{.0016}{98.7} \div 6.3$ | 12. $\frac{40.09 \times 3.702 \times 25}{.075 \times 100 \times 2050}$                    |

Find the value of  $x$  in :

13.  $40.5 : 3x :: 7.45 : 48.6$     14.  $\frac{125}{327} : \frac{5042}{7189} :: \frac{347}{529} : x$

#### 6. To affect a number by any exponent.

**Illustrations.**—1. Find the value of  $(2.45)^5$ .

**Solution:** Log.  $(2.45)^5 = 5 \log. 2.45$  [P. 4]  
 $5 \log. 2.45 = 5 \times 0.38917 = 1.94585$   
 antilog. = 88.277 *Ans.*

2. Find  $\sqrt[7]{.0345}$ .

**Solution:** Log.  $(.0345)^{\frac{1}{7}} = \frac{1}{7} \log. .0345 = \frac{1}{7} \text{ of } \bar{2}.53782 =$

$\frac{1}{7} \text{ of } (7 + \bar{2}.53782 - 7) = \frac{1}{7} \text{ of } (5.53782 - 7) = .79112 - 1 = \bar{1}.79112$   
 antilog. = .61819.

**Note.**—Before dividing a logarithm with a negative characteristic by a number not a factor of the characteristic, add to the logarithm the divisor, or such a multiple of the divisor as will make the characteristic positive, and indicate the subtraction of the same number from the result.

3. Find the value of  $(2.45^2 \times .046^3)^{\frac{2}{7}}$ .

$$\begin{aligned} \text{Solution: Log. } (2.45^2 \times .046^3)^{\frac{2}{7}} &= \frac{2}{7} \log. (2.45^2 \times .046^3) = \\ & \frac{2}{7} [2 \log. 2.45 + 3 \log. .046] \\ 2 \log. 2.45 &= 2 \times 0.38917 = 0.77834 \\ 3 \log. .046 &= 3 \times \overline{2.66276} = \overline{5.98828} \\ & \quad \underline{4.76662} \\ \frac{2}{7} \text{ of } 4.76662 &= \frac{1}{7} \text{ of } 7.53324 = 1.07618 \\ \text{antilog.} &= .11917. \end{aligned}$$

EXERCISE 73.

Find the value of :

1.  $3.24^3$
2.  $.875^8$
3.  $42.6^4$
4.  $\sqrt[5]{46.7}$
5.  $280^5 \div 136^3$
5.  $3684^2 \times 2.45^{\frac{1}{2}}$
9.  $8^{\frac{2}{3}} \times 9^{\frac{1}{2}} \times 10^{\frac{3}{4}}$
6.  $4.72^{\frac{1}{2}} \times .0472^{\frac{1}{2}}$
10.  $(400^2 \times 6000^3)^{\frac{1}{2}}$
7.  $\left( \frac{4.36^2 \times 98.6^3}{7.52^4 \div 6.36^2} \right)^{\frac{2}{3}}$
11.  $\sqrt[10]{.5 \times \frac{7}{8} \times .25}$
12.  $\left\{ \frac{\left( \frac{7}{8} \right)^2 \times \left( \frac{5}{6} \right)^3 \times \left( 2\frac{1}{2} \right)^4}{5\frac{1}{6}} \right\}^{\frac{1}{2}}$

Exponential Equations.

**480.** An exponential equation is an equation in which the unknown quantities appear as exponents; as,  $a^x = b$ , or  $a^{x+y} = b$  and  $a^{x-y} = c$ .

**Illustrative Examples.**—1. Solve  $3^x = 7$ .

**Solution :** Take the logarithm of both members,

$$x \log. 3 = \log. 7 \quad (1)$$

$$\therefore x = \frac{\log. 7}{\log. 3} = \frac{.84510}{.47712} = \frac{84510}{47712} = 1.77$$

2. Solve  $\begin{cases} 5^{x+y} = 100 \\ 3^{x-y} = 10 \end{cases}$

**Solution :**  $(x + y) \log. 5 = \log. 100 = 2 \quad (1)$

$$(x - y) \log. 3 = \log. 10 = 1 \quad (2)$$

$$x + y = \frac{2}{\log. 5} = \frac{2}{.6990} = 2.86 \quad (3)$$

$$x - y = \frac{1}{\log. 3} = \frac{1}{.4771} = 2.09 \quad (4)$$

$$2x = 4.95 \quad (5), \text{ whence } x = 2.48 \quad (6)$$

$$2y = .77 \quad (7), \text{ whence } y = .39 \quad (8)$$

3. Solve  $(a - b)^x = a^3 - b^3$ .

**Solution :**  $x \log. (a - b) = \log. (a^3 - b^3)$

$$x = \frac{\log. (a^3 - b^3)}{\log. (a - b)} = \frac{\log. (a - b) + \log. (a^2 + ab + b^2)}{\log. (a - b)}.$$

#### EXERCISE 76.

Solve :

1.  $10^x = 1000$

3.  $1.25^x = 8.42$

5.  $m^{x^2} = c^3$

2.  $7^x = 300$

4.  $a^x = b$

6.  $a r^{x-1} = l$

7.  $(m - n)^{3x} = p^4$

15.  $a^{x+2} = a^2 - b^3$

8.  $p^x q^{x-1} = r^m$

16.  $p^{-x} q^x = s^{-r}$

9.  $\begin{cases} a^{x+y} = b \\ c^{x-y} = d \end{cases}$

17.  $\begin{cases} 10^{x+y} = 1000 \\ 10^{x-y} = 100 \end{cases}$

10.  $\begin{cases} a^{x-y} = m^3 \\ m^{x-y} = a^2 \end{cases}$

18.  $\begin{cases} a^{xy} = m \\ a^{x+y} = n \end{cases}$

11.  $\begin{cases} a^{3x+2y} = p^{-3} \\ a^{3x-2y} = q^{-5} \end{cases}$

19.  $\begin{cases} (3a)^{x-y} = p^{x+y} \\ xy = \log. p + \log. 3a \end{cases}$

12.  $\frac{a r^x - a}{r - 1} = s$

20.  $\left(\frac{a}{b}\right)^{2xx} - 2\left(\frac{a}{b}\right)^{xx} = b$

13.  $(a - b)^{2x} + (a - b)^x = a^2 - b^2$

14.  $10^{2x} + 4 \times 10^x = 12$

## CHAPTER VII.

### PROGRESSIONS, INTEREST, AND ANNUITIES.

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#### Arithmetical Progression.

**481.** A *Progression* is a series of quantities that vary according to some definite law.

The quantities which form the series are called its *terms*.

**482.** An *arithmetical* progression is one in which each term after the first is derived from the preceding one by the addition of a *common difference*.

**483.** When the common difference is a positive quantity the series is *ascending*; when negative, it is *descending*.

**Illustration.**—Thus,  $a, 3a, 5a, 7a, 9a$  is an ascending series whose common difference is  $+2a$ ; and

$9a, 7a, 5a, 3a, a$  is a descending series whose common difference is  $-2a$ .

**484.** In the general discussion of arithmetical series,  $a$  represents the first term,  $d$  the common difference,  $l$  the last term,  $n$  the number of terms, and  $S$  the sum of all the terms.

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**485.** If we represent the first term by  $a$  and the common difference by  $d$ , the

$$2\text{d term} = a + d$$

$$3\text{d term} = a + 2d$$

$$4\text{th term} = a + 3d$$

$$5\text{th term} = a + 4d$$

Here, we observe that each term equals the first term plus the common difference multiplied by one less than the number of the term; hence, the

$$n\text{th term} = a + (n - 1)d;$$

but the  $n$ th term is the last term  $l$ ; therefore,

$$l = a + (n - 1)d. \quad (\text{A})$$

**486.** If we represent the sum by  $S$ , we have

$$S = a + (a + d) + (a + 2d) + \dots + (l - d) + l; \text{ also,}$$

$$S = l + (l - d) + (l - 2d) + \dots + (a + d) + a; \text{ adding}$$

$$2S = (l + a) + (l + a) + (l + a) + \dots + (l + a) + (l + a); \text{ or}$$

$$2S = (l + a)n; \text{ and}$$

$$S = (l + a)\frac{n}{2} \text{ or } \left(\frac{l + a}{2}\right)n. \quad (\text{B})$$

**Illustrative Examples.**—1. Find the sum of 12 terms of the series  $5 + 9 + 13 + 17 + \text{etc.}$

**Solution:** Here  $a = 5$ ,  $d = 4$ ,  $n = 12$ , to find  $l$  and  $S$ .

$$\begin{aligned} 1. \quad l &= a + (n - 1)d \\ l &= 5 + 11 \times 4 = 49. \end{aligned}$$

$$\begin{aligned} 2. \quad S &= (l + a)\frac{n}{2} \\ S &= (49 + 5) \times 6 = 324. \end{aligned}$$

2. Given  $a = 20$ ,  $l = 4$ , and  $d = -4$ , find  $n$ .

**Solution:** Substitute the given values in formula (A).

$$\begin{aligned} l &= a + (n - 1)d, \\ 4 &= 20 + (n - 1) \times (-4); \text{ whence} \\ 4(n - 1) &= 16; \text{ and} \\ n &= 5. \end{aligned}$$

3. Given  $d = 3$ ,  $l = 26$ , and  $S = 124$ , find  $a$ .

**Solution:** 1.  $l = a + (n - 1)d$ ; substitute given values,

$$\begin{aligned} 26 &= a + (n - 1)3; \text{ whence} \\ a + 3n &= 29 \quad (\text{A}) \end{aligned}$$

$$2. \quad S = (l + a)\frac{n}{2}; \text{ substitute given values,}$$

$$\begin{aligned} 124 &= (26 + a)\frac{n}{2}; \text{ whence} \\ 26n + an &= 248 \quad (\text{B}) \end{aligned}$$

From equation (A),  $n = \frac{29 - a}{3}$  (1)

From equation (B),  $n = \frac{248}{26 + a}$  (2)

Compare (1) and (2),

$$\frac{29 - a}{3} = \frac{248}{26 + a} \quad (3)$$

Clear of fractions,

$$754 + 3a - a^2 = 744 \quad (4)$$

Simplify,  $a^2 - 3a = 10$  (5)

whence  $a = 5$  or  $-2$ .

4. Insert 4 arithmetical means between 6 and 31.

**Solution:** Since the number of means is 4, the number of terms is 6; also,  $a = 6$  and  $l = 31$ . Use formula (A).

$$l = a + (n - 1)d; \text{ or}$$

$$31 = 6 + 5d; \text{ whence}$$

$$d = 5; \text{ and}$$

11, 16, 21, 26 are the means.

5. The sum of  $n$  terms of an arithmetical progression is  $\frac{1}{2}(3n^2 + n)$ , find the series.

**Solution:**  $S = \frac{1}{2}(3n^2 + n)$ , in which  $n$  is a variable.

Let  $n$  be successively 1, 2, 3, 4,

$$S_1 = 2 = a, \quad S_2 = 15$$

$$S_3 = 7, \quad S_4 = 26$$

Since the first term is 2 and the sum of two terms is 7, the second term is 5; the third term is  $15 - 7 = 8$ ; the fourth term is  $26 - 15 = 11$ ; hence, the series is 2, 5, 8, 11, etc.

#### EXERCISE 77.

1. Find the 10th term of the series : 3, 5, 7, 9, etc.
2. Find the 50th term of the series :  $\frac{1}{2}, \frac{5}{6}, \frac{7}{6}$ , etc.
3. Find the  $n$ th term of the series : 15, 12, 9, 6, etc.
4. Given  $l = 35$ ,  $a = 5$ ,  $n = 11$ , find  $d$
5. Given  $l = 36$ ,  $n = 9$ , and  $d = -3$ , find  $a$
6. Given  $l = 51$ ,  $a = 2\frac{1}{2}$ , and  $d = 3\frac{1}{4}$ , find  $n$

7. Find the sum of :

(1) 1, 3, 5, 7, etc., to 20 terms.

(2) 3, 6, 9, 12, etc., to 45 terms.

(3)  $2\frac{1}{2}$ , 4,  $5\frac{1}{2}$ , etc., to  $n$  terms.

(4) 50, 47, 44, etc., to  $n$  terms.

8. Given  $S = 143\frac{1}{3}$ ,  $a = \frac{2}{3}$ , and  $n = 20$ , find  $l$  and  $d$

9. Given  $S = 200$ ,  $n = 12$ , and  $d = 5$ , find  $l$  and  $a$

10. Given  $n = 20$ ,  $a = 5$ , and  $d = 2\frac{2}{3}$ , find  $l$  and  $S$

11. Given  $l = 100$ ,  $a = 40$ , and  $d = 10$ , to find  $n$  and  $S$

12. Given  $a = 100\frac{2}{3}$ ,  $l = 20\frac{1}{6}$ , and  $n = 16$ , to find  $S$  and  $d$

13. Given  $d = 12\frac{1}{2}$ ,  $n = 40$ , and  $l = 200$ , to find  $S$  and  $a$

14. Given  $d = 6$ ,  $l = 154\frac{1}{2}$ , and  $S = 2062\frac{1}{2}$ ,

to find  $a$  and  $n$

15. Given  $n = 20$ ,  $l = 166\frac{2}{3}$ , and  $S = 3000$ ,

to find  $a$  and  $d$

16. Given  $a = 200$ ,  $l = 88$ , and  $S = 2160$ , to find  $d$  and  $n$

17. Given  $a = 8$ ,  $d = 2\frac{2}{3}$ , and  $S = 200$ , to find  $l$  and  $n$

18. Prove that the arithmetical mean between  $a$  and  $b$  is  $\frac{a+b}{2}$ . What principle is deducible from this example?

19. Insert an arithmetical mean between 18 and 25; between  $16\frac{2}{3}$  and  $18\frac{3}{4}$ ; between  $m^2 + n^2$  and  $m^2 - n^2$

20. Insert 4 arithmetical means between 32 and 45; between  $x$  and  $y$

21. Insert 3 arithmetical means between  $x - 2y$  and  $x + 2y$ ; between  $a$  and  $a + 4b$

22. Prove the truth of the following formulæ :

No.	Given.	Required.	Formulæ.
1	$a d n$	$l$	$l = a + (n - 1) d$
2	$a d S$		$l = -\frac{1}{2} d \pm \sqrt{2 d S + (a - \frac{1}{2} d)^2}$
3	$a n S$		$l = \frac{2 S}{n} - a$
4	$d n S$		$l = \frac{S}{n} + \frac{n-1}{2} d$
5	$a d n$	$S$	$S = \frac{1}{2} n [2 a + (n - 1) d]$
6	$a d l$		$S = \frac{l+a}{2} + \frac{l^2 - a^2}{2 d}$
7	$a n l$		$S = (l + a) \frac{n}{2}$
8	$d n l$		$S = \frac{1}{2} n [2 l - (n - 1) d]$
9	$d n l$	$a$	$a = l - (n - 1) d$
10	$d n S$		$a = \frac{S}{n} - \frac{n-1}{2} d$
11	$d l S$		$a = \frac{1}{2} d \pm \sqrt{(l + \frac{1}{2} d)^2 - 2 d S}$
12	$n l S$		$a = \frac{2 S}{n} - l$
13	$a n l$	$d$	$d = \frac{l-a}{n-1}$
14	$a n S$		$d = \frac{2(S - a n)}{n(n-1)}$
15	$a l S$		$d = \frac{l^2 - a^2}{2 S - l - a}$
16	$n l S$		$d = \frac{2(n l - S)}{n(n-1)}$
17	$a d l$	$n$	$n = \frac{l-a}{d} + 1$
18	$a d S$		$n = \frac{d - 2 a \pm \sqrt{(2 a - d)^2 + 8 d S}}{2 d}$
19	$a l S$		$n = \frac{2 S}{l + a}$
20	$d l S$		$n = \frac{2 l + d \pm \sqrt{(2 l + d)^2 - 8 d S}}{2 d}$

23. Find the series whose  $n$ th term is  $2n + 1$   
 24. Find the series whose  $2n$ th term is  $6n + 2$   
 25. Find the series of  $n$  terms whose sum is  $(n + 1)\frac{n}{2}$   
 26. The sum of 20 terms of a series is  $400x$  and the common difference is  $2x$ . Find the series.  
 27. Insert  $m$  arithmetical means between  $a$  and  $b$   
 28. Insert 4 arithmetical means between  $a^3$  and  $a^{-3}$

487. To represent an *odd* number of quantities in arithmetical progression, it is often convenient to put  $x$  for the middle term and  $y$  for the common difference.

Thus,  $x - 2y$ ,  $x - y$ ,  $x$ ,  $x + y$ , and  $x + 2y$  represent five quantities in arithmetical progression.

488. An *even* number of quantities in arithmetical progression may be expressed by placing  $x + y$  and  $x - y$  for the two middle terms, making  $2y$  the common difference.

Thus,  $x - 5y$ ,  $x - 3y$ ,  $x - y$ ,  $x + y$ ,  $x + 3y$ , and  $x + 5y$  represent six quantities in arithmetical progression.

**Illustrative Examples.**—1. The sum of five numbers in arithmetical progression is 40, and the product of the two extremes is 28. Find the numbers.

**Solution:** Let  $x - 2y$ ,  $x - y$ ,  $x$ ,  $x + y$ , and  $x + 2y$  represent the numbers; then

$$(x - 2y) + (x - y) + x + (x + y) + (x + 2y) = 40;$$

Collecting terms,  $5x = 40$ ; whence  $x = 8$  (A)

$$\text{Again, } (x - 2y)(x + 2y) = 28; \text{ or}$$

$$x^2 - 4y^2 = 28 \quad \text{(B)}$$

Substitute (A) in (B),  $64 - 4y^2 = 28$ ; whence

$$4y^2 = 36, \text{ and } y = \pm 3.$$

$$x - 2y = 8 - (\pm 6) = 2 \text{ or } 14$$

$$x - y = 8 - (\pm 3) = 5 \text{ or } 11$$

$$x = 8 \text{ or } 8$$

$$x + y = 8 + (\pm 3) = 11 \text{ or } 5$$

$$x + 2y = 8 + (\pm 6) = 14 \text{ or } 2$$

2. The product of the means of four numbers in arithmetical progression is 88, and the product of the extremes is 70. Find the numbers.

**Solution:** Let  $x - 3y$ ,  $x - y$ ,  $x + y$ , and  $x + 3y$  be the numbers,

$$(x + y)(x - y) = 88, \text{ or} \\ x^2 - y^2 = 88 \quad (\text{A})$$

$$(x + 3y)(x - 3y) = 70, \text{ or} \\ x^2 - 9y^2 = 70 \quad (\text{B})$$

Reduce (A) and (B),  $y = \pm \frac{3}{2}$  and  $x = \pm \frac{19}{2}$

$$x - 3y = \left( \pm \frac{19}{2} \right) - \left( \pm \frac{9}{2} \right) = 5, 14, -5, -14$$

$$x - y = \left( \pm \frac{19}{2} \right) - \left( \pm \frac{3}{2} \right) = 8, 11, -8, -11$$

$$x + y = \left( \pm \frac{19}{2} \right) + \left( \pm \frac{3}{2} \right) = 11, 8, -11, -8$$

$$x + 3y = \left( \pm \frac{19}{2} \right) + \left( \pm \frac{9}{2} \right) = 14, 5, -14, -5$$

3. The sum of  $2r$  terms of the series 1, 3, 5, etc., is to the sum of the last  $r$  terms as  $x$  is to 1. Find the value of  $x$ .

1. Find the sum of  $2r$  terms.

$$l = a + (n - 1)d = 1 + (2r - 1)2 = 4r - 1$$

$$S = (l + a) \frac{n}{2} = 4r \times r = 4r^2. \quad (\text{A})$$

2. Find the sum of the last  $r$  terms.

The first term of the last  $r$  terms is the  $(r + 1)$ th term of the series,

$$(r + 1)\text{th term} = 1 + (r + 1 - 1)2 = 1 + 2r$$

$$S = (l + a) \frac{n}{2} = (4r - 1 + 1 + 2r) \frac{r}{2} = 3r^2 \quad (\text{B})$$

$\therefore 4r^2 : 3r^2 :: x : 1$ ; whence

$$x = \frac{4}{3} = 1\frac{1}{3}.$$

#### EXERCISE 78.

1. The product of the third and fourth terms of an arithmetical progression exceeds the product of the first and sixth by 24; and the product of the second and fifth divided by the product of the third and fourth is  $\frac{27}{35}$ . Find the series.

2. There are three numbers in arithmetical progression whose sum is 24 and the sum of whose squares is 224. Find the numbers.

3. The sum of four numbers in arithmetical progression is 42, and the difference of the squares of the extremes exceeds the difference of the squares of the means by 210. Find the numbers.

4. If a body falls  $16\frac{1}{12}$  feet in 1 second, three times that distance in the 2d second, five times that distance in the 3d second, how far will it fall in the 30th second?

5. A sum of money at simple interest will amount to \$189 in 1 year, \$198 in 2 years, \$207 in 3 years, etc. What will it amount to in 25 years?

6. The sum of five numbers in arithmetical progression is 60, and the sum of their squares is 880. Find the numbers.

7. How many arithmetical means must be inserted between 4 and 34 in order that the sum of the first two may be to the sum of the last two as 23 to 53?

8. A travels uniformly 30 miles a day; B starts two days later and travels 6 miles the first day, 10 miles the second, 14 miles the third, and so on, in arithmetical progression. In how many days will B overtake A?

9. Two men are 147 miles apart and approach each other; the first day they travel 2 and 5 miles respectively, the second day 6 and 8 miles, the third day 10 and 11 miles, and so on. In what time will they meet?

10. The three sides of a right-angled triangle, whose area is 54 square rods, are in arithmetical progression. Find the sides.

11. The first of 16 numbers in arithmetical progression is 2, and the sum of the first half is to the sum of the last half as 25 to 73. Find the common difference and the last term.

12. The three sides of a triangle are in arithmetical progression, and the rectangle on the two longer sides exceeds the square on the shorter side by 68 square feet; and the sum of the three sides is 36 feet. Required the length of the sides.

13. The three dimensions of a room are in arithmetical progression; the sum of its edges is 180 feet, and its entire surface is 1300 square feet. Required its dimensions.

14. A man started at a certain place and traveled 4 miles the first day,  $5\frac{1}{3}$  miles the second,  $6\frac{2}{3}$  miles the third, and so on. Two days later another man started from the same place and traveled 3 miles the first day,  $5\frac{5}{7}$  miles the second,  $8\frac{3}{7}$  miles the third, and so on. In how many days will they be together?

15. A and B are 539 miles apart and approach each other. A travels 4 miles the first day, 8 the second, 12 the third, and so on. B travels 50 miles the first day, 45 the second, 40 the third, and so on. In how many days will they meet?

16. The sum of  $n$  arithmetical means between 10 and 50 : the sum of the first  $n - 3$  of them ::  $7 : 3$ . Find  $n$  and  $d$ .

17. Find the difference between the sum of the even and the sum of the odd numbers from 1 to 100 inclusive.

18. Show that, if the same quantity be added to every term of an arithmetical progression, the sums will be in arithmetical progression; also, if every term of an arithmetical progression be multiplied by the same quantity, the products will be in arithmetical progression.

19. Show that, if a series of terms in arithmetical progression be separated into groups of  $n$  terms each, and the sum of each group be taken, the results will form an arithmetical progression whose common difference : the original common difference ::  $n^2 : 1$ .

20. The sum of  $n$  terms of the series 3, 5, 7, etc., exceeds the  $n$ th term by 63. Find the value of  $n$ .

21. If  $S$  be the sum of any number of terms of the series  $1 + 2 + 3 + \dots$ , prove that  $8S + 1$  is always a square. How many terms of the series must be taken to make 21?

22. If four quantities are in A. P., show that the sum of the squares of the extremes is greater than the sum of the squares of the means, and that the product of the extremes is less than the product of the means.

23. Find the sum of  $n$  terms of the series obtained by taking the 1st,  $r$ th, 2 $r$ th, 3 $r$ th, etc., terms of the A. P. whose first term is  $a$  and common difference  $b$ .

### Geometrical Progressions.

489. A *Geometrical Progression* is a series of quantities in which each term after the first is derived from the preceding one by multiplying it by a common *ratio*.

490. When the ratio is greater than unity the series is *ascending*; when less than unity it is *descending*.

Thus, 2, 4, 8, 16, 32, etc., is an ascending series.

32, 16, 8, 4, 2, etc., is a descending series.

491. In the general discussion of a geometrical series,  $a$  represents the first term,  $r$  the ratio,  $l$  the last term,  $n$  the number of terms, and  $S$  the sum of all the terms.

492. If we represent the first term by  $a$  and the ratio by  $r$ , the

$$2\text{d term} = ar$$

$$4\text{th term} = ar^3$$

$$3\text{d term} = ar^2$$

$$5\text{th term} = ar^4$$

Here we observe that each term equals the first term multiplied by the ratio raised to a power denoted by the number of the term less *one*; hence, the

$$n\text{th term} = ar^{n-1};$$

but the  $n$ th is the last term, since  $n$  is the number of terms.

$$\therefore l = ar^{n-1}. \quad (\text{A})$$


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**493.** If we represent the sum of the series by  $S$ , we have

$$S = a + ar + ar^2 + \dots + \frac{l}{r} + l;$$

Multiplying by  $r$ ,  $rS = ar + ar^2 + \dots + \frac{l}{r} + l + lr;$

Subtracting,  $(r-1)S = lr - a;$

$$\text{Dividing by } r-1, \quad S = \frac{lr-a}{r-1}. \quad (\text{B})$$


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**Illustrative Examples.**—1. Find the sum of 10 terms of the series 3, 9, 27, etc.

**Solution:** Here  $a = 3$ ,  $r = 3$ , and  $n = 10$ , to find  $l$  and  $S$ .

$$1. \quad l = ar^{n-1} = 3 \times 3^9 = 59049.$$

$$2. \quad S = \frac{lr-a}{r-1} = \frac{59049 \times 3 - 3}{2} = 88572.$$

2. Given  $S = 93$ ,  $l = 48$ , and  $a = 3$ , to find  $n$ .

**Solution:**

$$\begin{aligned} l &= ar^{n-1}; \text{ substituting} \\ 48 &= 3r^{n-1}; \text{ dividing by } 3, \\ r^{n-1} &= 16 \end{aligned} \quad (\text{A})$$

$$\begin{aligned} S &= \frac{lr-a}{r-1}; \text{ substituting} \\ 93 &= \frac{48r-3}{r-1}; \text{ clearing of fractions,} \end{aligned}$$

$$\begin{aligned} 93r - 93 &= 48r - 3; \text{ whence} \\ r &= 2. \end{aligned} \quad (\text{B})$$

Substitute (B) in (A),

$$2^{n-1} = 16 = 2^4;$$

$$\therefore (n-1) \log. 2 = 4 \log. 2;$$

$$n-1 = 4$$

$$n = 5.$$

3. Find the series whose sum is  $3^n - 1$ .

**Solution:**  $S = 3^n - 1$ , in which  $n$  is a variable.

Let  $n$  be successively 1, 2, 3, 4,

$$S_1 = 2, S_2 = 8, S_3 = 26, S_4 = 80.$$

Since the first term is 2, and the sum of two terms is 8, the second term is 6; the third term is  $26 - 8 = 18$ ; the fourth term is  $80 - 26 = 54$ ; hence, the series is 2, 6, 18, 54, etc.

4. Find 3 geometrical means between 4 and 324.

**Solution:** Since the number of means is 3, the number of terms is 5.  $a = 4$  and  $l = 324$ .

$$l = ar^{n-1}; \text{ substituting,}$$

$$324 = 4r^4; \text{ whence,}$$

$$r^4 = 81, \text{ and}$$

$$r = 3; \text{ therefore,}$$

$$12, 36, 108 \text{ are the means.}$$

#### EXERCISE 79.

1. Find the 8th term of 3, 9, 27, etc.

7th term of 8, 20, 50, etc.

9th term of  $6\frac{1}{4}$ ,  $2\frac{1}{12}$ ,  $\frac{25}{36}$ , etc.

6th term of  $2b$ ,  $-6ab$ ,  $18a^2b$ , etc.

2. Find the sum of :

$$8 + 24 + 72 + \dots \text{ to 6 terms.}$$

$$20 + 5 + 1\frac{1}{4} + \dots \text{ to 8 terms.}$$

$$\frac{3}{5} + \frac{2}{5} + \frac{4}{15} + \dots \text{ to 4 terms.}$$

$$a + b + \frac{b^2}{a} + \dots \text{ to 5 terms.}$$

3. Given  $a = 6$ ,  $r = 4$ , and  $n = 8$ , to find  $l$  and  $S$

4. Given  $a = 4$ ,  $r = 3$ , and  $S = 1456$ , to find  $l$  and  $n$

5. Given  $a = 5$ ,  $n = 4$ , and  $S = 200$ , to find  $r$  and  $l$

6. Given  $r = -\frac{2}{3}$ ,  $n = 6$ , and  $S = 11\frac{29}{81}$ , to find  $a$  and  $l$

7. Given  $a = 5$ ,  $r = 2$ , and  $l = 80$ , find  $S$  and  $n$

8. Given  $a = 64$ ,  $n = 6$ , and  $l = 2$ , find  $r$  and  $S$
9. Given  $r = \frac{5}{6}$ ,  $n = 5$ , and  $l = 104\frac{1}{6}$ , find  $a$  and  $S$
10. Given  $r = 3$ ,  $l = 90$ , and  $S = 130$ , find  $a$  and  $n$
11. Given  $n = 3$ ,  $l = 100$ , and  $S = 124$ , find  $a$  and  $r$
12. Given  $a = 5$ ,  $l = 125$ , and  $S = 155$ , find  $r$  and  $n$
13. Prove that a geometrical mean between two quantities equals the square root of their product.
14. Insert a geometrical mean between 27 and 48 ;  
 $4\frac{25}{36}$  and  $21\frac{7}{9}$  ;  $a$  and  $b$  ;  $3a^2$  and  $12b^2$
15. Insert two geometrical means between 8 and 216 ;  
 $a$  and  $a(a+b)^3$  ;  $\frac{a^3}{b}$  and  $\frac{b^3}{a}$
16. The  $n$ th term of a geometrical series is  $3^{n-1}$ . Find the series.
17. The sum of  $n$  terms of a geometrical progression is  $5^{n-1} - 1$ . Find the series.
18. Find the  $n$ th term of the series  $1, -\frac{1}{2}, +\frac{1}{4}, \text{etc.}$
19. Find the sum of  $n$  terms of the series  $1, -\frac{2}{3}, \frac{4}{9}, \text{etc.}$
20. In an odd number of terms, show that the product of the two extremes equals the square of the middle term.
21. How many terms of the series  $\cdot 5, \cdot 25, \cdot 125, \text{etc.}$ , will amount to  $\cdot 99609375$  ?
22. The first two terms of a series are 2 and 6. Extend the series so that the first, second, and third terms shall be in geometrical progression ; the second, third, and fourth terms in arithmetical progression ; the third, fourth, and fifth terms in geometrical progression, and so on alternately.

23. Prove the truth of the following formulæ :

No.	Given.	Required.	Formulæ.
1	$a r n$	$l$	$l = ar^{n-1}$
2	$a r S$		$l = \frac{a + (r-1)S}{r}$
3	$a n S$		$l(S-l)^{n-1} - a(S-a)^{n-1} = 0$
4	$r n S$		$l = \frac{(r-1)S r^{n-1}}{r^n - 1}$
5	$a r n$	$S$	$S = \frac{a(r^n - 1)}{r - 1}$
6	$a r l$		$S = \frac{rl - a}{r - 1}$
7	$a n l$		$S = \frac{^{n-1}\sqrt{l^n} - ^{n-1}\sqrt{a^n}}{^{n-1}\sqrt{l} - ^{n-1}\sqrt{a}}$
8	$r n l$		$S = \frac{l r^n - l}{r^n - r^{n-1}}$
9	$r n l$	$a$	$a = \frac{l}{r^{n-1}}$
10	$r n S$		$a = \frac{(r-1)S}{r^n - 1}$
11	$r l S$		$a = rl - (r-1)S$
12	$n l S$		$a(S-a)^{n-1} - l(S-l)^{n-1} = 0$
13	$a n l$	$r$	$r = \sqrt[n-1]{\frac{l}{a}}$
14	$a n S$		$r^n - \frac{S}{a}r + \frac{S-a}{a} = 0$
15	$a l S$		$r = \frac{S-a}{S-l}$
16	$n l S$		$r^n - \frac{S}{S-l}r^{n-1} + \frac{l}{S-l} = 0$
17	$a r l$	$n$	$n = \frac{\log. l - \log. a}{\log. r} + 1$
18	$a r S$		$n = \frac{\log. [a + (r-1)S] - \log. a}{\log. r}$
19	$a l S$		$n = \frac{\log. l - \log. a}{\log. (S-a) - \log. (S-l)} + 1$
20	$r l S$		$n = \frac{\log. l - \log. [lr - (r-1)S]}{\log. r} + 1$

**494.** Since the geometric mean between two quantities equals the square root of their product [Ex. 13], three quantities in geometrical progression may be represented by

$$x, \sqrt{xy}, \text{ and } y, \text{ or} \\ x^2, xy, \text{ and } y^2.$$

**495.** Four numbers in geometrical progression may be represented by  $\frac{x^2}{y}, x, y, \frac{y^2}{x}$ , in which the ratio is  $\frac{y}{x}$ .

**496.** Any number of quantities in geometrical progression may be represented by  $x, xy, xy^2, xy^3$ , etc.

**Illustrative Examples.**—1. Find three numbers in geometrical progression whose sum is 13, and the sum of whose squares is 91.

**Solution:** Let  $x, \sqrt{xy}$ , and  $y$  be the three numbers;

$$\text{then} \quad x + \sqrt{xy} + y = 13 \quad (\text{A})$$

$$\text{and} \quad x^2 + xy + y^2 = 91 \quad (\text{B})$$

$$\text{Divide (B) by (A),} \quad x - \sqrt{xy} + y = 7 \quad (\text{1})$$

$$\text{Subtract (1) from (A),} \quad 2\sqrt{xy} = 6 \quad (\text{2})$$

$$\text{Divide by 2,} \quad \sqrt{xy} = 3 \quad (\text{3})$$

$$\text{Square (3),} \quad xy = 9 \quad (\text{4})$$

$$\text{Subtract (3) from (A),} \quad x + y = 10 \quad (\text{5})$$

$$\text{Reduce (4) and (5),} \quad x = 9 \text{ or } 1$$

$$y = 1 \text{ or } 9.$$

The numbers are 9, 3, and 1, or 1, 3, and 9.

**2.** In a geometrical series of an odd number of terms, show that the square of the middle term is equal to the product of the first and last terms.

**Solution:** Since  $2n + 1$  is an odd number for all integral values of  $n$ , let  $2n + 1$  be the number of terms,  $a$  the first term, and  $r$  the ratio.

1. The  $(2n + 1)$ th term  $= ar^{2n}$ , the last term.

2. The  $(n + 1)$ th term  $= ar^n$ , the middle term.

$$\text{Now,} \quad a \times ar^{2n} = a^2 r^{2n} = (ar^n)^2.$$

$\therefore$  The square of the middle term equals the product of the first and last terms.

## EXERCISE 80.

1. Find three numbers in geometrical progression whose sum is 39, and whose product is 729.

2. Find three numbers in geometrical progression whose sum is 21, and the third is four times the product of the first two.

3. There are four numbers in geometrical progression. The sum of the means is 60 and the sum of the extremes is 140. Find the numbers.

4. If four numbers are in geometrical progression, prove that the first is to the third as the second is to the fourth.

5. If any number of quantities are in geometrical progression, prove that the first plus the second is to the second plus the third as the third plus the fourth is to the fourth plus the fifth, and so on.

6. Show that the compound amounts of \$1 for 1, 2, 3, 4, 5, 6 years, form a geometrical progression. What is the compound amount of \$1 for 6 years at 5 per cent?

7. The sum of the first and third of four numbers in geometrical progression is 40, and the sum of the second and fourth is 120. What are the numbers?

8. There are three numbers in geometrical progression; the sum of the first and second is 28, and the difference of the third and second is 42. Required the numbers.

9. The perimeter of a pentagon whose sides are in geometrical progression is 31, and the sum of the squares of the sides is 341. Find the sides.

10. Show that, if a series of numbers are in geometrical progression, like powers and like roots of them are also in geometrical progression.

11. The sum of six numbers in geometrical progression is 189, and the sum of the third and fourth is 36. Find the numbers.

12. The three sides of a triangle are in geometrical progression. The sum of the squares of the sides is 2275 square feet, and the perimeter is 65 feet. Required the length of each side.

13. One third of a stick was broken off; then  $\frac{1}{3}$  of the remainder; then  $\frac{1}{3}$  of what then remained, and so on, until 10 parts were broken off. What part of the stick remained?

14. One fifth of a keg of wine was drawn off, and the keg filled with water; then one fifth of that was drawn off, and the keg filled with water. This process was performed five times. What part of the contents was then wine?

15. A has \$1000 and B \$15·625; if A now doubles his capital and B quadruples his every decade, in how many years will B be worth as much as A, and how much will each be worth?

16. The sides of a triangle form a geometrical progression whose ratio is 2, and the sum of the squares on the sides is 2100. Required the length of the sides.

17. A's, B's, and C's fortunes form a geometrical progression whose ratio is 3, and the sum of their interests at 6% amount to \$2340. What is the fortune of each?

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### Infinite Series.

497. A series of quantities containing an unlimited number of terms is called an *Infinite Series*.

**Illustration.**  $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \text{ad infinitum.}$

498. The limit of the sum of an infinite series is a quantity which the sum of  $n$  terms continually approaches as  $n$  is increased indefinitely, and from which it will event-

ually differ by less than any assignable quantity, but which it can never equal.

**Note.**—The limit of the sum of an infinite series is often called the limit of the series.

**499.** The sum ( $S$ ) of any descending geometrical series of  $n$  terms is  $\frac{a - lr}{1 - r}$ . If  $n$  is made to increase indefinitely,  $l$  will decrease by a constant ratio, and its limit will be zero; whence  $\lim. lr$  will be zero, and  $\lim. S = \frac{a}{1 - r}$ .

Therefore,

**Prin.**—The limit of an infinite descending geometrical series is  $\frac{a}{1 - r}$ , in which  $a$  is the first term and  $r$  is the ratio.

**Illustrative Examples.**—

1. Find the limit of  $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16},$  *ad infinitum*.

$$\text{Solution: } \lim. S = \frac{a}{1 - r} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

2. Find the limit of  $\cdot 12121212$  etc., *ad infinitum*.

**Solution:**  $\cdot 1212$  etc.  $= \frac{12}{100} + \frac{12}{10000} +$  etc. Therefore,

$$\lim. \cdot 1212 \text{ etc.} = \frac{\frac{12}{100}}{1 - \frac{1}{100}} = \frac{12}{99} = \frac{4}{33}.$$

3. Find the limit of  $\cdot 3222$  etc., *ad infinitum*.

**Solution:**  $\cdot 322$  etc.  $= \frac{3}{10} + \left( \frac{2}{100} + \frac{2}{1000} + \text{etc.} \right)$

$$\lim. \left( \frac{2}{100} + \frac{2}{1000} + \text{etc.} \right) = \frac{\frac{2}{100}}{1 - \frac{1}{10}} = \frac{2}{90}; \text{ and}$$

$$\lim. \cdot 322 \text{ etc.} = \frac{3}{10} + \frac{2}{90} = \frac{29}{90}.$$

4. At what time after 3 o'clock are the hour and minute hands of a watch together?

**Solution :** While the minute hand goes over an arc the hour hand goes over  $\frac{1}{12}$  of an equal arc. At 3 o'clock, the minute hand is 15 minute spaces behind the hour hand; while the minute hand goes these 15 spaces the hour hand goes  $\frac{15}{12}$  spaces; while the minute hand goes  $\frac{15}{12}$  spaces the hour hand goes  $\frac{15}{144}$  spaces; while the minute hand goes  $\frac{15}{144}$  spaces the hour hand goes  $\frac{15}{1728}$  spaces, and so on *ad infinitum*; hence, the whole distance the minute hand goes is  $15 + \frac{15}{12} + \frac{15}{144} + \frac{15}{1728} + \text{etc.}$ , or  $\frac{15}{1 - \frac{1}{12}}$  spaces =  $16\frac{4}{11}$  spaces. Therefore they are together at 3 o'clock  $16\frac{4}{11}$  minutes.

## EXERCISE 81.

Find the limit of :

1.  $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \text{etc.}$

2.  $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \text{etc.}$

3.  $3 - \frac{3}{4} + \frac{3}{16} - \frac{3}{64} + \text{etc.}$

4.  $3\frac{1}{3} + 2\frac{6}{7} + 2\frac{22}{49} + 2\frac{34}{343} + \text{etc.}$

5.  $a + 1 + a^{-1} + a^{-2} + \text{etc.}$ , if  $a > 1$

6.  $b + \frac{b}{a} + \frac{b}{a^2} + \frac{b}{a^3} + \text{etc.}$ , if  $a > 1$

7.  $a - a^{\frac{1}{2}} + 1 - a^{-\frac{1}{2}} + \text{etc.}$ , if  $a > 1$

8.  $a - b + \frac{b^2}{a} - \frac{b^3}{a^2} + \text{etc.}$ , if  $a > b$

9.  $\cdot\dot{3}\dot{4}$ ,  $\cdot\dot{0}\dot{3}$ ,  $\cdot\dot{1}\dot{2}\dot{4}$ ,  $\cdot\dot{0}\dot{0}\dot{6}$       10.  $\cdot\dot{3}\dot{4}$ ,  $\cdot\dot{0}\dot{0}\dot{3}$ ,  $\cdot\dot{1}\dot{2}\dot{4}$ ,  $\dot{1}\dot{2}$

11. At what time after 8 o'clock are the minute and hour hands of a clock together?

12. A dog is 90 rods behind a fox, and runs 5 rods while the fox runs 3. How many rods must the dog run to catch the fox?

13. A man has \$60, and earns \$2 as often as he spends \$5. How much must he spend to become bankrupt?

14. A cistern contains 400 barrels of water, and receives 10 barrels as often as 12 barrels are used. How many barrels may be used before the cistern is empty?

15. If the ratio of an infinite series is less than  $\frac{1}{2}$ , show that each term is greater than the sum of the following terms.

16. The first term of a geometrical series is 9, and the sum of the first four terms is to the sum of an infinite number of terms as 80 to 81. Find the series.

### Arithmetico-Geometric Series.

500. An arithmetico-geometric series is a series of the form of  $a, (a + d)x, (a + 2d)x^2, (a + 3d)x^3, \dots \{a + (n - 1)d\}x^{n-1}, \dots$

**Problem.** To find the sum of  $n$  terms of an arithmetico-geometric series.

**Solution:**

Put  $S = a + (a + d)x + (a + 2d)x^2 + \dots$

then,  $xS = ax + (a + d)x^2 + \dots + \{a + (n - 2)d\}x^{n-1} + \{a + (n - 1)d\}x^n;$

whence,  $(1 - x)S = a + dx + dx^2 + \dots + dx^{n-1} - \{a + (n - 1)d\}x^n$   
 $= a + \frac{(1 - x^n - 1)dx}{1 - x} - \{a + (n - 1)d\}x^n,$

and  $S = \frac{a}{1 - x} + \frac{(1 - x^n - 1)dx}{(1 - x)^2} - \frac{\{a + (n - 1)d\}x^n}{1 - x}$  [D]

**Cor.**—Let  $x < 1$  and  $\lim. n = \infty$ , and write the series in the form

$$S = \frac{a}{1-x} + \frac{dx}{(1-x)^2} - \frac{x^a d}{(1-x)^2} - \frac{\{a + (n-1)d\} x^n}{1-x};$$

$$\text{then will } \lim. S = \frac{a}{1-x} + \frac{dx}{(1-x)^2} \quad (\text{E})$$

**Illustrative Examples.**—1. Find the sum of 10 terms of the series :  $1 + 3 \cdot 2 + 5 \cdot 2^2 + 7 \cdot 2^3 + \dots$

**Solution :** Here  $a = 1$ ,  $d = 2$ ,  $x = 2$ , and  $n = 10$ . Substitute these values in formula [D],

$$S = \frac{1}{1-2} + \frac{(1-2^9)4}{(1-2)^2} - \frac{\{(1+9)2\}2^{10}}{1-2} = -1 - 2044 + 20480 = 18435.$$

2. Find the sum of the infinite series :

$$1 + 3 \cdot \frac{1}{2} + 5 \cdot \frac{1}{4} + \dots$$

**Solution :** Here  $a = 1$ ,  $d = 2$ ,  $x = \frac{1}{2}$ , and  $n = \infty$ ,

$$\therefore S = \frac{a}{1-x} + \frac{dx}{(1-x)^2} = \frac{1}{1-\frac{1}{2}} + \frac{1}{\left(1-\frac{1}{2}\right)^2} = 6.$$

#### EXERCISE 82.

1. Sum  $1 + 2r + 3r^2 + 4r^3 + \dots$  to  $n$  terms.
2. Sum  $2 + 4 \cdot 3 + 6 \cdot 9 + 8 \cdot 27 + \dots$  to 7 terms.
3. Sum  $3 + 5 \cdot \frac{1}{2} + 7 \cdot \frac{1}{4} + 9 \cdot \frac{1}{16} + \dots$  to 8 terms.
4. Sum  $1 + \frac{3}{4} + \frac{5}{16} + \frac{7}{64} + \dots$  to infinity.
5. Sum  $1 + 3r + 5r^2 + 7r^3 + \dots$  to infinity, if  $r < 1$
6. Sum  $1 + \frac{4}{5} + \frac{7}{25} + \frac{10}{125} + \dots$  to infinity.
7. Sum  $2 + \frac{5}{2} + \frac{8}{4} + \frac{11}{8} + \dots$  to infinity.

### Harmonical Progressions.

**501.** A *Harmonical Progression* is a series the reciprocals of whose terms are in arithmetical progression.

General Form.

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d} \cdots \frac{1}{a+(n-1)d}$$

#### EXERCISE 88.

1. Find the 9th term of the series  $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ , etc.

**Suggestion.**—The 9th term of the series  $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ , etc., is the reciprocal of the 9th term of the series 3, 5, 7, etc.

2. Insert four harmonical means between  $\frac{1}{2}$  and  $\frac{1}{17}$

**Suggestion.**—The four harmonical means between  $\frac{1}{2}$  and  $\frac{1}{17}$  are the reciprocals of the four arithmetical means between 2 and 17.

3. Find the 20th term of the series  $\frac{1}{3}, \frac{1}{8}, \frac{1}{13}$ , etc.

4. Find the 15th term of the series  $1, \frac{3}{5}, \frac{3}{7}, \frac{3}{9}$ , etc.

5. Find the  $n$ th term of the series  $\frac{1}{a}, \frac{1}{3a}, \frac{1}{5a}$ , etc.

6. Insert three harmonical means between 5 and 15.

7. Insert a harmonical mean between  $a$  and  $b$ , and derive therefrom a principle.

8. Arrange in the order of magnitude the arithmetical, the geometrical, and the harmonical means between 2 and 8.

9. Prove that the geometrical mean between two quantities is a mean proportional between the arithmetical and harmonical means.

10. When  $a$ ,  $b$ , and  $c$  are in harmonic progression, show that  $a : c :: a - b : b - c$ .

11. If  $a$ ,  $b$ , and  $c$  are in arithmetical progression, and  $b$ ,  $c$ , and  $d$  are in harmonic progression, show that  $a : b :: c : d$ .

12. Show that, if  $a$ ,  $b$ ,  $c$  be in harmonic progression, then will

$$1. \frac{2}{b} = \frac{1}{b-a} + \frac{1}{b-c} \qquad 2. \frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$$

$$3. a - \frac{b}{2} : \frac{b}{2} :: \frac{b}{2} : c - \frac{b}{2}$$

13. Three numbers are in geometrical progression ; if each be increased by 15, they are in harmonic progression. Required the middle number.

14. Between two quantities a harmonic mean is inserted ; and between each adjacent pair of the three quantities thus obtained is inserted a geometric mean. It is now found that the three inserted means are in arithmetical progression. Show that the ratio of the two quantities is  $7 - 4\sqrt{3} : 1$ .

### Miscellaneous Series.

502. Any series whose  $n$ th term is given may be derived by substituting successively 1, 2, 3, 4, etc., for  $n$  in the  $n$ th term.

**Illustration.**—Find the series whose  $n$ th term is  $n + n^2$ . Let  $n = 1, 2, 3, 4, 5 \dots$  ; then will result the series :

$1 + 1, 2 + 4, 3 + 9, 4 + 16, 5 + 25 \dots$ ,  
or the series :    2            6            12            20,    30    .....

503. If the corresponding terms of two series be added or subtracted, a new series will be formed whose  $n$ th term is the sum or difference of the  $n$ th terms of the two series.

**Illustration.**—Assume the two series :

$$1, 4, 9, 16, 25 \dots n^2, \text{ and}$$

$$1, 2, 3, 4, 5 \dots n,$$

then will the series formed by the sum and the difference of the corresponding terms be

$$2, 6, 12, 20, 30 \dots n^2 + n, \text{ and}$$

$$0, 2, 6, 12, 20 \dots n^2 - n.$$

**504.** The sum of a series whose terms are the sums or differences of the corresponding terms of two other series, is the sum or the difference of the sums of those series.

Thus, if we let  $\Sigma n$  represent the sum of a series whose  $n$ th term is  $n$ , and  $\Sigma n^2$  the sum of a series whose  $n$ th term is  $n^2$ , then will  $\Sigma(n^2 + n) = \Sigma n^2 + \Sigma n$ , and  $\Sigma(n^2 - n) = \Sigma n^2 - \Sigma n$ . This is evident from the commutative law of addition. Similarly,

$$\Sigma(n^3 + n^2 - n) = \Sigma n^3 + \Sigma n^2 - \Sigma n,$$

$$\Sigma(an^r - bn^r - cn^r) = a(\Sigma n^r) - b(\Sigma n^r) - c(\Sigma n^r), \text{ etc.}$$

### Problems.

**505. Problem 1.** To find the value of  $\Sigma n$ .

**Solution:** The series whose  $n$ th term is  $n$  is evidently the series of natural numbers 1, 2, 3, 4...  $n$ , whose sum is  $(n+1)\frac{n}{2}$  [486, B].

$$\text{Therefore, } \Sigma n = (n+1)\frac{n}{2}. \quad (\text{A})$$

**506. Problem 2.** To find the value of  $\Sigma n^2$ .

**Solution:**  $\Sigma n^2 = 1^2 + 2^2 + 3^2 + 4^2 + \dots n^2$ .

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1 \text{ for any value of } n. \quad (1)$$

Put  $n$  successively equal to 1, 2, 3, 4...; then,

$$1. \quad 1^3 - 0^3 = 3 \times 1^2 - 3 \times 1 + 1$$

$$2. \quad 2^3 - 1^3 = 3 \times 2^2 - 3 \times 2 + 1$$

$$3. \quad 3^3 - 2^3 = 3 \times 3^2 - 3 \times 3 + 1$$

$$4. \quad 4^3 - 3^3 = 3 \times 4^2 - 3 \times 4 + 1$$

$$\dots$$

$$n. \quad n^3 - (n-1)^3 = 3 \times n^2 - 3 \times n + 1$$

Taking the sum of these equations,

$$\begin{aligned} n^3 &= 3(\Sigma n^2) - 3(\Sigma n) + n. \\ \therefore \Sigma n^3 &= \frac{3(\Sigma n) + n^3 - n}{3} = \frac{n(n+1)(2n+1)}{6} \quad (B) \end{aligned}$$

507. Problem 3. To find the value of  $\Sigma n^3$ .

$$\begin{aligned} \text{Solution:} \quad \Sigma n^3 &= 1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3 \\ n^4 - (n-1)^4 &= 4n^3 - 6n^2 + 4n - 1 \quad (1) \end{aligned}$$

Put  $n = 1, 2, 3, 4, \dots$ ;

$$\begin{aligned} 1. \quad 1^4 - 0^4 &= 4 \times 1^3 - 6 \times 1^2 + 4 \times 1 - 1 \\ 2. \quad 2^4 - 1^4 &= 4 \times 2^3 - 6 \times 2^2 + 4 \times 2 - 1 \\ 3. \quad 3^4 - 2^4 &= 4 \times 3^3 - 6 \times 3^2 + 4 \times 3 - 1 \\ &\vdots \\ n. \quad n^4 - (n-1)^4 &= 4 \times n^3 - 6 \times n^2 + 4 \times n - 1 \end{aligned}$$

Taking the sum of these equations,

$$\begin{aligned} n^4 &= 4(\Sigma n^3) - 6(\Sigma n^2) + 4(\Sigma n) - n. \\ \therefore \Sigma n^3 &= \frac{6(\Sigma n^2) - 4(\Sigma n) + n^4 + n}{4} = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2 \\ \therefore \Sigma n^3 &= (\Sigma n)^2. \quad (C) \end{aligned}$$

508. Problem 4. To find the value of  $\Sigma \frac{n(n+1)}{2}$ .

$$\begin{aligned} \text{Solution:} \quad \Sigma \frac{n(n+1)}{2} &= \frac{1}{2} \Sigma (n^2 + n) = \frac{1}{2} \{ \Sigma n^2 + \Sigma n \} \\ &= \frac{1}{2} \left\{ \frac{n(n+1)(2n+1)}{6} + \frac{(n+1)n}{2} \right\} = \frac{n(n+1)(n+2)}{6} \\ \therefore \Sigma \frac{n(n+1)}{2} &= \frac{n(n+1)(n+2)}{6} \quad (D) \end{aligned}$$

509. Problem 5. To find the value of  $\Sigma n(a-b+n)$ .

$$\begin{aligned} \text{Solution:} \quad \Sigma n(a-b+n) &= \Sigma (an - bn + n^2) = (a-b) \Sigma n + \Sigma n^2 \\ &= (a-b) \left( \frac{n(n+1)}{2} \right) + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(3a-3b+2n+1)}{6} \\ \therefore \Sigma n(a-b+n) &= \frac{n(n+1)(3a-3b+2n+1)}{6} \quad (E) \end{aligned}$$

Cor.—If  $b = n$ , we have

$$\Sigma n(a-n+n) = \frac{n(n+1)(3a-n+1)}{6} \quad (F)$$

## Applications to Piles of Shot and Shell.

**510.** *To find the number of shot in a complete square pyramid.*

**Solution :** Let  $n$  be the number in one row of the lower course; then will  $n$  also be the number of courses.

The top course contains 1 shot, the next course 4, the next 9, and so on. Therefore, the whole number of shot equals

$$\Sigma n^2 = 1 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ [506, B].}$$

**Example.**—What is the number of shot in a square pyramid that contains 25 shot in one row of the base?

$$\text{Solution : } \Sigma n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{25 \times 26 \times 51}{6} = 5525.$$

**511.** *To find the number of shot in a complete pyramid whose base is an equilateral triangle.*

**Solution :** The top course contains 1 shot, the second course 1 + 2, the third 1 + 2 + 3, the fourth 1 + 2 + 3 + 4, and the  $n$ th course 1 + 2 + 3 + ...  $n$ , or  $\frac{n(n+1)}{2}$ . Therefore, a pyramid of  $n$  courses contains  $\Sigma \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$  [508, D].

**Example.**—Find the number of cannon balls in a pyramid whose base is an equilateral triangle containing ten balls in the longest row.

$$\text{Solution : } \Sigma \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6} = \frac{10 \times 11 \times 12}{6} = 220.$$

**512.** *To find the number of shot in a complete rectangular pile whose base is  $m$  shots long and  $n$  shots wide.*

**Solution :** Since the base is  $n$  shots wide, and  $m$  shots long, there will be  $n$  courses of shot, each containing one row less, and each row one shot less than the one immediately below it. Therefore, the top course will contain  $(m-n)$  shot; the second course,  $2(m-n+1)$ ; the third course,  $3(m-n+2)$ ; the  $n$ th course,  $n(m-n+n)$ . Therefore, the pile will contain

$$\Sigma n(m-n+n) = \frac{n(n+1)(3m-n+1)}{6} \text{ [509, F].}$$

**Example.**—Find the number of cannon balls in a rectangular pile whose base is 20 balls long and 16 balls wide.

**Solution :** Here  $m = 20$  and  $n = 16$ ; therefore,

$$\Sigma n(m - n + 1) = \frac{n(n + 1)(3m - n + 1)}{6} = \frac{16 \times 17 \times 45}{6} = 2040.$$

**513.** To find the number of shot in an incomplete pile, it is generally best to compute the number for a complete pile and subtract therefrom the number required to complete the pile.

**Example.**—Find the number of shot in an incomplete rectangular pile of 10 layers, whose top layer is 8 shots long and 4 shots wide.

**Solution :** The lower layer is  $8 + 9$  or 17 shots long, and  $4 + 9$  or 13 shots wide; hence, the pile, if complete, would contain

$$\Sigma n(m - n + 1) = \frac{n(n + 1)(3m - n + 1)}{6} = \frac{13 \times 14 \times 39}{6} = 1183 \text{ shots.}$$

The lower base of the top removed is  $8 - 1$  or 7 shots long, and  $4 - 1$  or 3 shots wide; hence, it would contain  $\frac{3 \times 4 \times 19}{6} = 38$  shots.

$\therefore$  The incomplete pile contains  $1183 - 38 = 1145$  shots.

#### EXERCISE 84.

1. Find the number of balls in a square pile of 15 courses.

2. Find the number of balls in a triangular pile of 16 courses.

3. Find the number of balls in a rectangular pile whose base is 19 balls long and 12 balls wide.

4. Find the number of balls in an incomplete triangular pile of 8 courses whose base is 14 balls long on each side.

5. Find the number of balls in an incomplete square pile, whose base course is 44 balls long, and whose top course is 22 balls long.

6. A triangular pile contains 1140 balls. How many are there on each side of the base?

7. Find the number of shot in an incomplete rectangular pile, the number of shot in the sides of its upper course being 11 and 18, and the number in the shorter side of the lowest course 30.

8. Show that the number of shot in a square pile is one fourth the number in a triangular pile of twice as many courses.

9. The number of balls in a triangular pile is to the number in a square pile having the same number in one side of the base, as 6 to 11. Required the number in each pile.

10. How many shot will be required to complete a rectangular pile having 15 and 6 shot in the longer and shorter sides respectively, of its upper course?

$$11. \text{ Show that } \Sigma (2^{n-1} + 8n^3 - 6n^2) = 2^n - 1 + n(n+1)(2n^2 - 1)$$

$$12. \text{ Show that } \Sigma \{3(4^n + 2n^2) - 4n^3\} = 4^{n+1} - 4 - n(n+1)(n^2 - n - 1)$$

## INTEREST AND ANNUITIES.

### Interest.

#### Definitions.

514. Interest is money charged for the use of money. The amount of interest charged ( $I$ ) is called *the Interest*.

515. The sum on which interest is reckoned is called *the Principal* ( $P$ ).

516. The number ( $n$ ) of fixed periods (usually years) the principal is on interest is called *the time*.

**Note.**  $n$  may be integral, fractional, or mixed.

**517.** The interest on a unit of money for a unit of time is called *the rate* ( $r$ ).

**Note.**—The rate is usually stated at a certain per cent per annum. Thus, 5% per annum = \$0.05 on \$1 for 1 year.

**518.** The sum of the principal and interest is called *the amount* ( $A$ ). The amount of \$1 for 1 year, or  $1 + r$ , is represented by  $R$ .

**519.** If interest is reckoned on the principal only, it is called *Simple Interest*.

**Illustration.**—The simple interest of \$500 for 4 years at 6% =  $\$500 \times .06 \times 4 = \$120$ .

**520.** In the settlement of interest accounts, when interests are due at stated periods, it is customary, in some States, to add to the interest on the principal the interest of each unpaid interest from the time it was due to the day of settlement.

Interest thus reckoned is called *annual* or *periodic interest*.

**Illustration.**—The interest of \$500 for 4 years at 6%, payable annually =

$$\begin{aligned} &\$500 \times .06 \times 4 + \$30 \times .06 \times 3 + \\ &\$30 \times .06 \times 2 + \$30 \times .06 \times 1 = \\ &\$500 \times .06 \times 4 + \$30 \times .06 \times 6 = \$130.80. \end{aligned}$$

**521.** If the interest is added to the principal at the end of each period to form a new principal for the next period, the interest charged is called *the Compound Interest*.

**Illustration.**—The compound amount of \$500 for 4 yr. at 6% =  $\$500 \times 1.06 \times 1.06 \times 1.06 \times 1.06 = \$500 \times 1.06^4$ .

**Note.**—Though not generally recognized by law in the settlement of interest accounts, any person or corporation that annually collects all interests due, and places them on interest for the following year, is realizing compound interest on investments.

## Simple Interest.

**522.** If  $r$  represents the interest of \$1 for 1 year, the interest of \$1 for  $n$  years will be  $nr$ , and for \$ $P$  it will be  $Pnr$ ; and the amount of \$ $P$  will be  $P + Pnr$ , or  $P(1 + nr)$ . Therefore,

$$I = Pnr \quad [0] \qquad A = P(1 + nr) \quad [1]$$

These formulas enable us to solve all ordinary problems in simple interest. If any three of the terms  $P$ ,  $I$ ,  $n$ ,  $r$ ,  $A$  be given, the remaining two may be found.

**523.** The *present worth* ( $P$ ) of a debt ( $A$ ), due some time hence ( $n$  years), when the current rate is  $r$ , is such a sum as, paid at once, is an equitable equivalent for the debt. This is evidently the principal ( $P$ ), which will in  $n$  years, at the rate  $r$ , amount to  $A$ .

Since, at simple interest,

$$A = P(1 + nr) \quad [2] \qquad P = \frac{A}{1 + nr} \quad [3]$$

**524.** *Discount* is a deduction allowed for the immediate payment of a debt due at some future time. The discount ( $D$ ) is evidently the difference between the debt ( $A$ ) and the present worth ( $P$ ).

$$\therefore D = A - P = A - \frac{A}{1 + nr} = \frac{Anr}{1 + nr} \quad [4]$$

**525.** In practice, for short periods of time, it has become customary to deduct the *interest of the debt* for immediate payment. Such deduction is called *Bank Discount* ( $B$ ).

The difference between bank and true discount is evidently

$$B - D = Anr - \frac{Anr}{1 + nr} = \frac{An^2r^2}{1 + nr} \quad [5]$$

$$\text{Cor. } \frac{B}{D} = A n r \div \frac{A n r}{1 + n r} = 1 + n r$$

$$\therefore \lim. \frac{B}{D_{n=0}} = \lim. (1 + n r)_{n=0} = 1.$$

That is, the shorter the time a debt has to run, the nearer will the substitution of bank for true discount approach equitable exactness.

### Annual Interest.

**526.** Let  $P$  represent the principal,  $r$  the stipulated rate,  $r_1$  the current rate,  $n$  the time in years,  $I$  the entire interest, and  $A$  the amount. Suppose all interests unpaid until the debt becomes due.

1. The interest on the principal alone =  $P n r$ .

2. The interest due at the end of each year is  $P r$ . The interests for the 1st, 2d, 3d, etc., years, will be on interest at the rate  $r_1$  for  $n - 1$ ,  $n - 2$ ,  $n - 3$ , etc., years, respectively. That is,  $P r$  will be on simple interest for a number of years equivalent to the sum of the arithmetical series:  $1 + 2 + 3 + \dots + n - 1 = \frac{1}{2} n (n - 1)$ .

Now, the interest on  $P r$  for  $\frac{1}{2} n (n - 1)$  years at the rate  $r_1 = \frac{1}{2} P r r_1 n (n - 1)$ . Therefore,

$$I = P n r + \frac{1}{2} P r r_1 n (n - 1) = \frac{1}{2} P n r \{ 2 + r_1 (n - 1) \} \quad [6]$$

$$A = \frac{1}{2} P \{ 2 + 2 n r + n (n - 1) r r_1 \} \quad [7]$$

$$\text{Cor.}—\text{If } r_1 = r, I = \frac{1}{2} P n r \{ 2 + r (n - 1) \} \quad [8]$$

$$A = \frac{1}{2} P \{ 2 + 2 n r + n (n - 1) r^2 \} \quad [9]$$

## Compound Interest.

**527.** Let  $P$  represent the principal,  $r$  the annual rate,  $n$  the number of years,  $I$  the compound interest,  $A$  the compound amount, and  $R$  the amount of \$1 for 1 year. Then,

$$1. \text{ Amt. for 1 yr.} = P(1+r) = PR$$

$$2. \text{ Amt. for 2 yr.} = PR \times R = PR^2$$

$$3. \text{ Amt. for 3 yr.} = PR^2 \times R = PR^3$$

$$4. \text{ Amt. for 4 yr.} = PR^3 \times R = PR^4$$

$$\therefore \text{ Amt. for } n \text{ yr.} = PR^n = P(1+r)^n, \text{ or}$$

$$A = PR^n = P(1+r)^n \quad [10]$$

$$I = A - P = P(R^n - 1) = P\{(1+r)^n - 1\} \quad [11]$$

**Cor. 1.**—If the interest is convertible into principal semi-annually, the periodic rate is  $\frac{r}{2}$  and the number of periods  $2n$ . Therefore,

$$A = P \left(1 + \frac{r}{2}\right)^{2n} \quad [12]$$

**Cor. 2.**—If the interest is convertible quarterly, the periodic rate is  $\frac{r}{4}$  and the number of periods  $4n$ .

Therefore,

$$A = P \left(1 + \frac{r}{4}\right)^{4n} \quad [13]$$

**Cor. 3.**—If the interest is convertible  $q$  times a year, the periodic rate is  $\frac{r}{q}$  and the number of periods  $qn$ .

Therefore,

$$A = P \left(1 + \frac{r}{q}\right)^{qn} \quad [14]$$

**Cor. 4.**—The present worth ( $P$ ) of a debt ( $A$ ) due in  $n$  years,

1. When interest is convertible annually, is  $\frac{A}{R^n}$
2. When interest is convertible semi-annually, is  $\frac{A}{\left(1 + \frac{1}{2}r\right)^{2n}}$
3. When interest is convertible quarterly, is  $\frac{A}{\left(1 + \frac{1}{4}r\right)^{4n}}$
4. When interest is convertible  $q$  times a year, is  $\frac{A}{\left(1 + \frac{r}{q}\right)^{qn}}$

$$\therefore P_1 = \frac{A}{R^n} \quad [15] \qquad P_2 = \frac{A}{\left(1 + \frac{1}{2}r\right)^{2n}} \quad [16]$$

$$P_4 = \frac{A}{\left(1 + \frac{1}{4}r\right)^{4n}} \quad [17] \qquad P_q = \frac{A}{\left(1 + \frac{r}{q}\right)^{qn}} \quad [18]$$

### Partial Payments by Equal Annual Installments.

**528. Problem.** If a debt of  $\$P$ , bearing interest at the rate  $r$ , is to be paid in  $n$  equal annual installments, what must each payment be?

1. Solution by the U. S. Rule :

Let  $x$  = the annual payment, then the debt at the end of the

$$\text{1st year} = P(1+r) - x$$

$$\text{2d year} = P(1+r)^2 - x(1+r) - x$$

$$\text{3d year} = P(1+r)^3 - x(1+r)^2 - x(1+r) - x$$

$$\text{nth year} = P(1+r)^n - x(1+r)^{n-1} - x(1+r)^{n-2} - \dots - x$$

But in  $n$  years the debt is canceled,

$$\therefore P(1+r)^n - x(1+r)^{n-1} - x(1+r)^{n-2} - \dots - x = 0$$

$$\text{whence, } x \{(1+r)^{n-1} + (1+r)^{n-2} + \dots + 1\} = P(1+r)^n$$

$$\text{or } x \left\{ \frac{(1+r)^n - 1}{r} \right\} = P(1+r)^n \quad [493, B]$$

$$\text{whence, } x = \frac{Pr(1+r)^n}{(1+r)^n - 1} \quad [19]$$

## 2. Solution by the Merchants' Rule.

According to this rule, the sum of the amounts of the payments is regarded equivalent to the amount of the debt.

$$\text{The amount of the debt} = P(1 + nr)$$

$$\text{The amount of the 1st payment} = x + (n-1)rx$$

$$\text{The amount of the 2d payment} = x + (n-2)rx$$

$$\text{The amount of the 3d payment} = x + (n-3)rx$$

$$\text{The amount of the } (n-2)\text{th payment} = x + 2rx$$

$$\text{The amount of the } (n-1)\text{th payment} = x + rx$$

$$\text{The amount of the } n\text{th payment} = x$$

$$\text{The sum of the amounts of the payments} = nx + \frac{n(n-1)}{2}rx$$

$$\therefore nx + n \frac{(n-1)}{2}rx = P(1 + nr)$$

$$\{2n + n(n-1)r\}x = 2P(1 + nr)$$

$$x = \frac{2P(1 + nr)}{2n + n(n-1)r} \quad [20]$$

*Cor.—If compound interest is allowed,*

$$\text{The amount of the debt} = P(1 + r)^n$$

$$\text{The amount of the 1st payment} = x(1 + r)^{n-1}$$

$$\text{The amount of the 2d payment} = x(1 + r)^{n-2}$$

$$\text{The amount of the } n\text{th payment} = x$$

The sum of the amounts of the payments equals the sum of the geometrical progression

$$x + (1 + r)x + \dots + (1 + r)^{n-2}x + (1 + r)^{n-1}x,$$

which is  $\frac{x\{(1 + r)^n - 1\}}{r}$

$$\therefore x \left\{ \frac{(1 + r)^n - 1}{r} \right\} = P(1 + r)^n$$

$$x = \frac{Pr(1 + r)^n}{(1 + r)^n - 1} \quad [21]$$

**Illustration.**—If a debt of \$10,000, bearing 6% interest, is to be paid in 5 equal annual installments, what must the annual installment be?

**Solution:** Here  $P = \$10,000$ ,  $r = .06$ , and  $n = 5$ .

Substituting these values in formulas 19, 20, and 21, we have,

1st. By the U. S. Rule:

$$x = \frac{Pr(1+r)^n}{(1+r)^n - 1} = \frac{\$10,000 \times .06 \times (1.06)^5}{(1.06)^5 - 1} = \$2373.96.$$

2d. By Merchants' Rule, at Simple Interest:

$$x = \frac{2P(1+nr)}{2n + n(n-1)r} = \frac{\$20,000(1+5 \times .06)}{10 + 5 \times 4 \times .06} = \$2321.43.$$

3d. By Merchants' Rule, at Compound Interest:

$$x = \frac{Pr(1+r)^n}{(1+r)^n - 1} = \$2373.96, \text{ same as U. S. Rule.}$$

*Query.* Why do the U. S. Rule and the Merchants' Rule, at compound interest, produce the same result?

### Annuities.

**529.** An *Annuity* is a stated sum of money paid annually, or at other regular intervals of time.

**530.** An annuity is generally secured by a mortgage or some other lien, of which it is the interest.

**531.** The *principal sum* of an annuity is the sum of which the annuity is the interest.

**532.** An annuity that begins and ends at fixed times is an *annuity certain*.

**533.** An annuity whose beginning or ending depends upon some uncertain event, as the death of an individual, is a *contingent annuity*.

**534.** A *deferred annuity*, or an *annuity in reversion*, is an annuity that begins some time in the future.

**535.** A *Perpetuity* is an annuity that continues forever.

**536.** The *final*, or *forborne*, value of an annuity is its value at the conclusion, and equals the sum of all the annuities, together with interest on them from the time they were due.

**537.** The *present worth* of an annuity is the present worth of its final value.

**538. Problem.** What is the final value ( $A$ ) of an annuity ( $a$ ) that runs  $n$  years, when the rate of interest is  $r$ ?

1. *At annual interest.* The last annuity is worth  $a$ , the preceding  $a + ar$ , the next preceding  $a + 2ar \dots$ , the first  $a + (n - 1)ar$ ; hence, the final value is the sum of the arithmetical series of  $n$  terms:

$$a + (a + ar) + (a + 2ar) + \dots + a + (n - 1)ar; \text{ or,}$$

$$A = \{2a + (n - 1)ar\} \frac{n}{2} \quad [22]$$

2. *At compound interest.* The last annuity is worth  $a$ , the preceding  $a(1 + r)$ , the next preceding  $a(1 + r)^2 \dots$ , the first  $a(1 + r)^{n-1}$ ; therefore, the final value is the sum of the geometrical series of  $n$  terms:

$$a + a(1 + r) + a(1 + r)^2 + \dots + a(1 + r)^{n-1}, \text{ or}$$

$$A = \frac{a\{(1 + r)^n - 1\}}{r} = \frac{a(R^n - 1)}{R - 1} \quad [23]$$

**539. Problem.** Find the present worth ( $P$ ) of an annuity ( $a$ ) deferred  $m$  years and running  $n$  years, when the rate of interest is  $r$ .

1. *At annual interest.*

The final value is  $\{2a + (n - 1)ar\} \frac{n}{2}$ . The amount of \$1 for  $m + n$  years is  $1 + (m + n)r$ ; hence,

$$P = \frac{\{2a + (n - 1)ar\} \frac{n}{2}}{2 + (m + n)2r} \quad [24]$$

$$\text{Cor.}—\text{When } m = 0, P = \frac{\{2a + (n - 1)ar\} \frac{n}{2}}{2 + 2nr} \quad [25]$$

2. *At compound interest.* The final value is  $\frac{a(R^n - 1)}{R - 1}$ . The amount of \$1 for  $m + n$  years is  $R^{m+n}$ ; hence,

$$P = \frac{a(R^n - 1)}{R^{m+n}(R - 1)} = \frac{a}{R^n(R - 1)} \times \frac{R^n - 1}{R^n} = \frac{a}{R^n(R - 1)} \left(1 - \frac{1}{R^n}\right) \quad [26]$$

$$\text{Cor. 1.}—\text{When } \lim. n = \infty, \lim. P = \frac{a}{R^n(R - 1)} \quad [27]$$

$$\text{Cor. 2.}—\text{When } m = 0, P = \frac{a}{R - 1} \left(1 - \frac{1}{R^n}\right) \quad [28]$$

$$\text{Cor. 3.}—\text{When } m = 0, \text{ and } \lim. n = \infty, \lim. P = \frac{a}{R - 1} \quad [29]$$

**Illustration.**—Find the final value and present worth of an annuity of \$60, deferred 12 years and to run 9 years, at  $4\frac{1}{2}\%$ , compound interest.

I. **Solution:** Here  $a = 60$ ,  $r = .045$ ,  $n = 9$ .

$$A = \frac{a \{(1 + r)^n - 1\}}{r}$$

$$\therefore \log. A = \log. a + \log. \{(1 + r)^n - 1\} + \text{colog. } r$$

$$\log. a = \log. 60 = 1.77815$$

$$\log. \{(1 + r)^n - 1\} = \log. .486095 = \overline{1}.68672$$

$$\text{colog. } r = \text{colog. } .045 = 1.34679$$

$$\log. A = 2.81166$$

$$\text{Antilog. } A = \$648.12$$

$$\text{II. } P = \frac{a(R^n - 1)}{R^{m+n}(R - 1)} = \frac{a \{(1 + r)^n - 1\}}{(1 + r)^{m+n} \times r}$$

$$\therefore \log. P = \log. a + \log. \{(1 + r)^n - 1\} - (m + n) \log. (1 + r) - \log. r$$

$$\log. a = \log. 60 = 1.77815$$

$$\log. \{(1 + r)^n - 1\} = \log. .486095 = \overline{1}.68672$$

$$- (m + n) \log. (1 + r) = -21 \log. 1.045 = -0.40131$$

$$- \log. r = - \log. .045 = -\overline{2}.65321$$

$$\therefore \log. P = 2.41035$$

$$\text{Antilog. } P = \$257.24$$

**Note.**—The value of  $(1 + r)^n$  and  $R^{m+n}$  may be most readily obtained from a compound interest table, such as is found in nearly every work on higher arithmetic.

## Mortgages and Bonds.

**540.** A *Mortgage* is a conveyance of an estate by way of pledge for the security of a debt, to become void on payment of the debt.

**541.** A *Bond* is a writing of obligation, under seal, to pay a sum of money, usually with interest.

**542.** Bonds may be secured by mortgages on real estate, personal or corporate property ; or they may rest solely on the promise of the maker, as do Government bonds.

**543. Problem.** How much must be invested in a bond, price  $p$ , rate of bond  $r$ , current rate of interest  $r'$ , to yield an annual income of  $a$ , if the interest is payable  $q$  times a year?

**Solution:** Let  $x$  = the amount invested.

Then  $\frac{x}{p}$  = the face value of the bond.

$\frac{rx}{qp}$  = the periodic interest on the bond.

$\frac{rx}{p}$  = the annual interest on the bond.

The first periodic interest will be on interest for  $q-1$  periods, the second for  $q-2$  periods, etc. The interest on all the periodic interests will be equivalent to the interest on one of them for a number of periods equal to the sum of the series  $0+1+2+\dots+q-1$ , or for  $(q-1)\frac{q}{2}$  periods.

$$\therefore \frac{rx}{qp} \times \frac{r'}{q} \times (q-1)\frac{q}{2}, \text{ or } \frac{rr'(q-1)x}{2pq} = \text{the additional interest.}$$

$$\therefore \frac{rx}{p} + \frac{rr'(q-1)x}{2pq} = a$$

$$\text{whence, } x = \frac{2apq}{2qr + rr'(q-1)} \quad [30]$$

$$\text{Cor. 1.}—\text{When } r' = r, x = \frac{2apq}{2qr + r^2(q-1)} \quad [31]$$

$$\text{Cor. 2.}—\text{When } q = 1, x = \frac{ap}{r} \quad [32]$$

**544. Problem.** What rate of interest will a purchaser realize who buys a bond, face  $a$ , rate of bond  $r$ , time to run  $n$ , price  $p$ , when the current rate of interest is  $r'$ ?

**Solution:** Let  $x$  = the annual rate on the investment.  
 and  $ap = P$  = the cost of the bond.  
 then  $P(1+x)^n$  = the amount of the purchase money.  
 $ar$  = the annual income.

The amount of the first income  $= ar(1+r')^{n-1}$

The amount of the second income  $= ar(1+r')^{n-2}$

The amount of the  $n$ th income  $= ar$ .

$\therefore$  The sum of the amounts of all the incomes is the sum of the geometrical progression :

$$ar + ar(1+r') + ar(1+r')^2 + \dots + ar(1+r')^{n-1} = \frac{ar\{(1+r')^n - 1\}}{r'}$$

$$\therefore P(1+x)^n = ar \frac{\{(1+r')^n - 1\}}{r'} + a$$

$$\text{whence } 1+x = \left[ \frac{ar\{(1+r')^n - 1\} + ar}{Pr'} \right]^{\frac{1}{n}} \quad [33]$$

**Illustration.**—How much must be invested in bonds at 90, bearing 5% interest, payable quarterly, when the current rate of interest for short periods is 4%, to yield an annual income of \$406?

**Solution:** Here  $a = \$406$ ,  $p = .90$ ,  $q = 4$ ,  $r = .05$ , and  $r' = .04$ .

$$x = \frac{2apq}{2qr + rr'(q-1)}$$

$$\therefore x = \frac{2 \times \$406 \times 4 \times .90}{2 \times 4 \times .05 + .05 \times .04 \times 3} = .07200$$

### Life Insurance.

**545.** Life insurance is a contract whereby an insurer agrees for an annual consideration to pay to the insured at a given time, or to his heirs at his death, a sum of money.

**546.** The annual consideration paid by the insured is called the *premium*, and is due at the beginning of each year.

**547. Problem.** If the annual premium is  $p$ , the amount insured  $a$ , and death occurs at the end of  $n$  years, will the insurer gain or lose, and how much?

**Solution:** The amount of the first premium  $= p(1+r)^0$   
 The amount of the second premium  $= p(1+r)^1$   
 The amount of the third premium  $= p(1+r)^2$   
 $\vdots$   
 The amount of the  $n$ th premium  $= p(1+r)^{n-1}$

$\therefore$  The final value of all the premiums is the sum of the series:

$$p(1+r) + p(1+r)^2 + \dots + p(1+r)^n = \frac{p}{r}(1+r)\{(1+r)^n - 1\}$$

The difference between the insurance and the amount of the premiums is  $d = a - \frac{p}{r}(1+r)\{(1+r)^n - 1\}$  [34]

Now,  $a > < \frac{p}{r}(1+r)\{(1+r)^n - 1\}$ , accordingly as

$$\frac{ar + p(1+r)}{p(1+r)} > = < (1+r)^n, \text{ or as}$$

$$\log. \{ar + p(1+r)\} - \log. p - \log. (1+r) > = < n \log. (1+r), \text{ or as}$$

$$\frac{\log. \{ar + p(1+r)\} - \log. p - \log. (1+r)}{\log. (1+r)} > = < n.$$

**Illustration.**—A paid \$150 a year to have his life insured for \$5000; he died in 20 years. Did the insurance company gain or lose, and how much, money worth 6%?

**Solution:** Here  $a = \$5000$ ,  $p = 150$ ,  $n = 20$ , and  $r = .06$ .

$$\therefore d = a - \frac{p}{r}(1+r)\{(1+r)^n - 1\} =$$

$$5000 - \frac{150}{.06}(1.06)\{1.06^{20} - 1\} =$$

$$5000 - 2500(1.06^{21} - 1.06) =$$

$$5000 - 2500(2.3395636) = -848.91$$

$\therefore$  The company gains \$848.91.

#### EXERCISE 88.

1. What principal will in  $3\frac{1}{4}$  years, at 6%, simple interest, amount to \$669.20?

2. A certain principal will in 7 years, at a certain rate, amount to \$1136, and in 10 years, at the same rate, to \$1280. Required the principal and rate (simple interest).

3. At what rate will \$ $a$  in  $q$  years  $n$ -tuple itself at simple interest?

4. In what time will \$ $a$ , at the rate  $r$ ,  $n$ -tuple itself at simple interest?

5. What principal will in 4 years amount to \$243 at 5%, payable annually, if all interests remain unpaid and draw simple interest at the same rate until the end of the time?

6. In what time will \$860 at 6%, payable annually, amount to \$1054.53, if the annual interest bear simple interest at 6%?

7. In what time will \$800 at 6%, compound interest, amount to \$1351.56?

8. At what rate will a given principal in 5 yr. 6 mo., compounded semi-annually, treble itself?

9. Find the difference between the annual interest and the compound interest of \$1200 for 6 yr. 6 mo. at  $6\frac{1}{2}\%$ .

10. Find the present worth of \$8000, due in 5 years, money worth  $5\frac{1}{2}\%$ , compound interest, payable quarterly.

11. Show that the difference between the simple interest and the true discount of a sum of money equals the interest on the true discount.

12. Show that, if the simple interest of a certain sum equals  $\frac{a}{b}$  of the principal, the true discount equals  $\frac{a}{a+b}$  of the principal.

13. Show that the true discount is half the harmonic mean between the sum due and the simple interest on it.

14. If the true discount for 1 year is  $\frac{m}{n}$  of the simple interest, what is the rate per cent?

15. A man borrows \$1000, and renews his note every six months at an increase of 10%; in what time will it reach \$4594.97?

16. In how many years will a sum at 10%, interest convertible annually, amount to as much as twice the sum at 5%, convertible semi-annually?

17. If a bank charges  $r\%$  discount, what per cent interest does it receive?

18. Find the present worth of a perpetuity of \$325, deferred 5 years, if money is worth 8%, convertible semi-annually.

19. What annual appropriation to its sinking-fund must a school make to pay a debt of \$15,000, due in 17 years, money worth 7%, compound interest?

20. A person borrows a sum of money, and pays off at the end of each year as much of the principal as he pays interest for that year. Find how much he owes at the end of  $n$  years.

21. What annuity, beginning  $n$  years hence, and lasting for  $n$  years, is equivalent to an annuity of \$ $A$ , beginning now and lasting for  $n$  years?

22. If a loan of \$1000 is to be paid off in 10 equal monthly installments, what must the monthly payments be, reckoning compound interest at 6%?

23. At what price must bonds bearing 5% interest, payable quarterly, be purchased, in order that an investment of \$3600 will bring in an annual income of \$203, when the current rate of interest for short periods is 4%, simple interest?

24. What rate of interest will a purchaser realize who buys a \$5000 bond, payable in 20 years, bearing 4% interest, payable annually, at 90, if the current rate of interest is 5%?

25. If a man pays annually \$20 to have his life insured for \$1000, payable at his death, how long must he live that the company may neither gain nor lose, money worth 5%, compound interest?

## CHAPTER VIII.

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### Permutations and Combinations.

**548.** A *combination*, or *selection*, of any number of things is a group of two or more of them without regard to order.

Thus, if we have the four letters,  $a, b, c, d$ , we may make six combinations or selections of *two* each, as follows:  $ab, ac, ad, bc, bd, cd$ ; or four of *three* each:  $abc, abd, acd, bcd$ .

**549.** A *permutation*, or *arrangement*, of any number of things is a group of two or more of them, when regard is had to the order in which they are taken.

Thus, each selection of two, as  $ab$ , will form two permutations,  $ab$  and  $ba$ ; and each selection of three, as  $abc$ , will form six permutations:  $abc, acb, bac, bca, cab, cba$ .

**550. Proposition 1.**—If one thing can be done in  $a$  ways, another in  $b$  ways, another in  $c$  ways...., and the ways are all independent of each other, all may be done in  $a \times b \times c \times \dots$  ways.

For, each of the  $a$  ways in which the first may be done may be taken with each of the  $b$  ways in which the second may be done; hence, the first two may be done in  $ab$  ways. Again, each of the  $ab$  ways in which the first two may be done may be taken with each of the  $c$  ways in which the third may be done; hence, the first three may be done in  $abc$  ways, etc.

**Illustration.**—From 3 Latin books and 5 Greek books I can choose one of each in  $3 \times 5$ , or 15 ways.

**Cor.**—If  $n$  things can each be done in  $r$  ways, they can be done together in  $r^n$  ways.

**551.** The expression  $|n$  denotes the product of all the natural numbers from 1 to  $n$ , and is read *factorial  $n$* . Thus,  $|n = 1 \times 2 \times 3 \times 4 \times \dots n$ .

**552. Prop. 2.**—The number of permutations or arrangements of  $n$  things taken all together is  $|n$ .

**Demonstration.**—There are  $n$  places to be filled by the  $n$  things. The first place may be filled by any one of the  $n$  things, or in  $n$  ways; the second by any one of the remaining  $n - 1$  things, or in  $n - 1$  ways; the third in  $n - 2$  ways, and so on until the last place is reached, which must be filled by the only one remaining. Therefore, the number of ways in which all the places can be filled, which is evidently the whole number of arrangements that can be made of the  $n$  things taken all together is  $n(n-1)(n-2) \dots 1$  [P. 1] =  $|n$ .

**Illustration.**—The number of arrangements that can be made of the letters in the word *bird* is

$$|4 = 1 \times 2 \times 3 \times 4 = 24.$$

**553. Prop. 3.**—The number of arrangements of  $n$  things taken  $r$  together is  $n(n-1)(n-2) \dots (n-r+1)$ .

**Demonstration.**—There are  $r$  places to be filled from  $n$  things. The first place may be filled in  $n$  ways, after which the second may be filled in  $n - 1$  ways, then the third in  $n - 2$  ways. . . . , and finally the  $r$ th place in  $n - (r - 1)$ , or  $n - r + 1$  ways. Therefore, the whole number of arrangements is  $n(n-1)(n-2) \dots (n-r+1)$  [P. 1].

**Illustration.**—The number of four-lettered words that can be made from the letters  $a, b, c, d, e, f$  is

$$6 \times 5 \times 4 \times 3 = 360.$$

**554. Prop. 4.**—The number of arrangements of  $n$  things of which  $p$  are alike is  $\frac{|n|}{|p|}$ .

**Demonstration.**—Suppose the  $p$  like things all unlike and different from the remaining  $n - p$  things, then the whole number of arrangements will be  $|n|$  [P. 2].

Conceive any one of these arrangements. Now, permute the  $p$  things among themselves without disturbing the positions of the  $n - p$  things. You will have  $|p|$  arrangements with the  $n - p$  things fixed in position and order. Now, it becomes evident that, if the  $p$  things are alike, these  $|p|$  arrangements will be all alike, or, in other words, will reduce to one arrangement; therefore, there will be only  $\frac{1}{|p|}$  as many arrangements of  $n$  things when  $p$  are alike as when they are all different, or  $\frac{|n|}{|p|}$ .

**Cor.**—The number of arrangements of  $n$  things when  $p$  are alike,  $q$  others are alike,  $r$  others are alike.... is  $\frac{|n|}{|p| |q| |r| \dots}$ .

**Illustration.**—The number of arrangements of the letters in the word “ecclesiastical,” which contains 2  $e$ ’s, 3  $c$ ’s, 2  $l$ ’s, 2  $s$ ’s, 2  $i$ ’s, 2  $a$ ’s, and 1  $t$ , is

$$\frac{|14|}{|2| |3| |2| |2| |2| |2|} = 454,053,600.$$

**555. Prop. 5.**—The number of arrangements of  $n$  different things taken  $r$  together when repetitions are allowed is  $n^r$ .

**Demonstration.**—There are  $r$  places to fill, and each place may be filled in  $n$  ways. Therefore, the whole number of arrangements is  $n \times n \times n \times \dots$  to  $r$  factors  $= n^r$ .

**Cor.**—If  $r = n$ , the number of arrangements is  $n^n$ .

**Illustration.**—The number of three-lettered words that can be made of  $a, b, c, d$ , when repetitions are allowed, is  $4^3 = 64$ .

**556.** A circular arrangement is an arrangement of a number of things in the circumference of a circle or in the periphery of any plane figure.

**557. Prop. 6.**—*The number of circular arrangements of  $n$  things taken all together is  $\underline{n-1}$ .*

**Demonstration.**—Every arrangement of the things may be moved forward in a body through each of the  $n$  positions in the periphery without changing the arrangement; therefore, there will be only  $\frac{1}{n}$  as many circular arrangements as ordinary arrangements, or the number of arrangements will be  $\frac{n}{n} = \underline{n-1}$ .

**Illustration.**—The number of arrangements that can be made of 8 persons around a round table is

$$\underline{7} = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040.$$

**558. Prop. 7.**—*The whole number of combinations, or selections, of  $r$  things from  $n$  things is*

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{r}}.$$

**Demonstration.**—Each selection of  $r$  things may be permuted in  $\underline{r}$  ways [P. 2]; therefore, there are only  $\frac{1}{\underline{r}}$  as many selections as there are arrangements, or  $\frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{r}}$  [P. 3].

**Illustration.**—The number of selections of 4 guards from 20 men is  $\frac{20 \times 19 \times 18 \times 17}{1 \times 2 \times 3 \times 4} = 4845$ .

**Cor. 1.**—*The number of selections of  $r$  things from  $n$  things equals the number of selections of  $n-r$  things from  $n$  things.*

For every time a selection of  $r$  things is made, a selection of  $n-r$  things remains.

**Illustration.**—The number of ways 15 guards may be selected from 20 men equals the number of ways 5 guards may be selected from 20 men, or

$$\frac{20 \times 19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4 \times 5} = 15504.$$

**Cor. 2.**—The number of selections of  $r$  things from  $n$  things equals  $\frac{n}{r \mid n-r}$ .

$$\text{For, } \frac{n(n-1)(n-2)\dots(n-r+1)}{r} = \frac{\{n(n-1)(n-2)\dots(n-r+1)\} \mid n-r}{r \mid n-r} = \frac{n}{r \mid n-r}$$

**Illustration.**—The number of ways of selecting 4 kinds of ribbon from 7 kinds is

$$\frac{7}{4 \mid 3} = \frac{2 \times 3 \times 4 \times 5 \times 6 \times 7}{2 \times 3 \times 4 \times 2 \times 3} = 35.$$

**559. Prop. 8.**—The number of ways  $x+y$  things can be divided into two classes, putting  $x$  into one class and  $y$  into another, is  $\frac{x+y}{x \mid y}$ .

**Demonstration.**—The number of ways is evidently the same as the number of ways of selecting either  $x$  or  $y$  things from  $x+y$  things, which is  $\frac{x+y}{x \mid y}$  [P. 7, Cor. 2].

**Cor.**—The number of ways  $x+y+z$  things can be divided into three classes, putting  $x$  into one class,  $y$  into another, and  $z$  into another.... is  $\frac{x+y+z}{x \mid y \mid z}$ .

For  $x+y+z$  things can be divided into two classes, containing  $x$  and  $y+z$  things in  $\frac{x+y+z}{x \mid y+z}$  ways. Then the class of  $y+z$  things can be subdivided into two classes of  $y$  and  $z$  things in  $\frac{y+z}{y \mid z}$  ways. Therefore, the number of ways  $x+y+z$  can be divided into three classes of  $x$ ,  $y$ , and  $z$  things is

$$\frac{x+y+z}{x \mid y+z} \times \frac{y+z}{y \mid z} = \frac{x+y+z}{x \mid y \mid z}.$$

**Scholium.**—By continued subdivisions, this principle may be extended to any number of classes.

**Illustration.**—Ten boys may be divided into 3 classes of 2, 3, and 5 in  $\frac{10}{\begin{array}{|c|c|c|} \hline 2 & 3 & 5 \\ \hline \end{array}}$  ways = 2520 ways.

**560. Prop. 9.**—*The number of ways in which a selection can be made from  $n$  different things is  $2^n - 1$ .*

**Demonstration.**—Each thing can be either selected or rejected, or can be disposed of in two ways; hence, the number of ways all can be disposed of is  $2 \times 2 \times 2 \times \dots$  to  $n$  factors, which is  $2^n$ ; but this includes the way in which all are rejected; hence, the number of ways of selecting is  $2^n - 1$ .

**561. Prop. 10.**—*The number of ways in which a selection can be made from  $p + q + r + \dots$  things, of which  $p$  are alike,  $q$  others are alike,  $r$  others are alike, etc., is  $\{(p + 1)(q + 1)(r + 1)\dots\} - 1$ .*

**Demonstration.**—The  $p$  things may be disposed of in  $p + 1$  ways by taking 0, 1, 2, 3... or  $p$  of them. Similarly, the  $q, r, \dots$  things may each be disposed of in  $q + 1, r + 1, \dots$  ways. Hence, all may be disposed of in  $(p + 1)(q + 1)(r + 1)\dots$  ways. But the way in which none are taken must be rejected. Therefore, etc.

**Illustrative Examples.**—1. Two persons get into a railway carriage where there are six vacant seats. In how many different ways can they seat themselves?

**Solution:** One of them can take any one of the six seats, and the other any one of the remaining five; hence, they may be seated in  $6 \times 5$  or 30 different ways.

2. In how many ways can six different things be divided among two boys?

**Solution:** Each thing may be disposed of in two ways by giving it to either boy; hence, all may be disposed of in  $2 \times 2 \times 2 \times 2 \times 2 \times 2$  ways = 64 ways. But this includes the two ways in which one boy receives all or none, which cases are not admissible. Therefore,  $64 - 2$ , or 62, is the number of ways of making the division.

3. How many six-lettered words can be made of five letters if repetitions are allowed?

**Solution:** Each place in the word may be filled in five ways; hence, all the places may be filled in  $5^6$  ways, or 15625 ways.

4. How many signals can be made of six lights of different colors displayed singly, or any number at a time, side by side?

**Solution:** Let  $P_1, P_2, P_3, P_4, P_5, P_6$  represent respectively the number of signals made when the lights are taken singly, two, three, four, five, and six together; then

$$\begin{aligned} P_1 &= 6 & P_4 &= 6 \times 5 \times 4 \times 3 = 360 \\ P_2 &= 6 \times 5 = 30 & P_5 &= 6 \times 5 \times 4 \times 3 \times 2 = 720 \\ P_3 &= 6 \times 5 \times 4 = 120 & P_6 &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \\ \therefore P_{1-6} &= 6 + 30 + 120 + 360 + 720 + 720 = 1956. \end{aligned}$$

5. A man has six friends, and he invites three of them to dinner every day for twenty days. In how many ways can he do this without having the same party twice?

**Solution:** Three can be selected from six in  $\frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20$  ways.

A different party of three can, therefore, be selected for each day, and when selected they may be arranged in 20 ways.

### EXERCISE 86.

1. If two dice be thrown together, in how many ways can they fall?

**Suggestion.**—Each can fall in 6 ways, and together in  $6 \times 6$  ways.

2. Out of 8 pairs of gloves, in how many ways may a right-hand and a left-hand glove be chosen, which are not a pair?

**Suggestion.**—A right-hand glove may be chosen in 8 ways, and then a left-hand glove that will not make a pair, in 7 ways, and together in  $8 \times 7$  ways.

3. In how many ways can five men vote for sheriff, if there are ten candidates for the office?

**Suggestion.**—Each man can vote in 10 ways, and together they can vote in  $10 \times 10 \times 10 \times 10 \times 10$  ways.

4. How many even numbers of four digits can be made from the ten digits?

**Suggestion.**—The thousands' place can be filled in 9 ways, the units' place in 5 ways, and each of the other two places in 10 ways, and all in  $9 \times 10 \times 10 \times 5$  ways.

5. In how many ways can 9 ladies and 9 gentlemen form themselves in couples to dance ?

**Suggestion.**—The first gentleman can select a partner in 9 ways, the second in 8 ways, etc.

6. On a shelf are 6 apples, 8 pears, and 7 oranges. In how many ways may they be arranged, if all the apples, all the pears, and all the oranges are kept together ?

**Suggestion.**—The apples may be arranged among themselves in  $\boxed{6}$  ways, the pears in  $\boxed{8}$  ways, and the oranges in  $\boxed{7}$  ways, and the three groups in  $\boxed{3}$  ways; hence, the entire number of arrangements =  $\boxed{6} \times \boxed{8} \times \boxed{7} \times \boxed{3}$ .

7. In how many ways may 50 objects be divided into two classes, one of which shall contain  $\frac{2}{3}$  as many as the other ?

**Suggestion.**—The number of ways =  $\frac{\boxed{50}}{\boxed{20} \boxed{30}}$ .

8. In how many ways can two sixes, three fives, and an ace be thrown with six dice ?

**Suggestion.**  $\frac{\boxed{6}}{\boxed{3} \boxed{2} \boxed{1}} = 60 = \text{the number of ways. Why?}$

9. There are five routes to the top of a mountain. In how many ways can a person go up and down ?

10. In how many times as many ways can the letters of *anticipation* be arranged as the letters of *commencement* ?

11. In how many ways can the letters of *ubiquitous* be arranged so that *u* follows *q* ?

**Suggestion.**—Regard *qu* as a single element.

12. In how many ways can the letters of *logarithms* be arranged so that the second, fourth, and sixth places may be occupied by consonants ?

13. In how many ways can a purchaser select half a dozen handkerchiefs at a shop where seven sorts are kept ?

14. In how many ways can *a* things be given to *n* persons, allowing each person to receive any number ?

15. How many signals can be made by hoisting 4 flags of different colors one above the other, when any number of them may be hoisted at once? How many with 5 flags?

16. How many numbers ending in 5, not exceeding 8000, can be formed from the ten digits?

17. There are 8 different kinds of books, and 5 copies of each. In how many ways can a selection be made from them?

18. If the number of permutations of  $2n$  things 3 together is equal to twice the number of permutations of  $n$  things 4 together, find  $n$ .

19. In how many different ways can a party of six people form a ring?

20. How many words containing two vowels and three consonants can be formed out of 21 consonants and 5 vowels? How many will there be if the vowels are to occupy the even places?

21. How many committees of 9 Republicans and 10 Democrats can be formed out of 45 Republicans and 50 Democrats?

22. From 6 ladies and 5 gentlemen, in how many ways could you arrange sides for a game of croquet, so that there should be two ladies and one gentleman on each side?

23. Four boys are in attendance at a telegraph-office when 8 messages arrive. In how many ways can the messages be given indifferently to the boys without leaving any boy unemployed?

24. At a post-office are kept ten sorts of postage-stamps. In how many ways can a person buy 12 stamps? In how many ways can he buy 8 stamps? In how many ways can he buy 8 different stamps?

25. In how many ways can 10 copies of Homer, 6 of Virgil, and 4 of Horace, be given to 20 boys, so that each boy may receive a book?

26. A man belongs to a club of thirty members, and every day he invites five members to dine with him, making a different party each day. Show that he may do this for 118755 days.

27. Show that the greatest number of combinations that can be formed with  $2n$  things, each combination containing the same number of things, is double the greatest number that can be formed with  $2n - 1$  things.

28. Show that the number of ways in which  $p$  positive and  $n$  negative signs can be placed in a row so that no two negative signs shall be together, is equal to the number of combinations of  $p + 1$  things taken  $n$  together.

29. Show that  $2n$  persons may be seated at two round tables,  $n$  persons being seated at each, in  $\frac{2n}{n^2}$  different ways.

30. Show that, in the combinations of  $4n$  different things  $n$  together, the number of combinations in which a particular thing occurs is equal to one third of the number in which it does not occur.

31. Show that, if the figures 1, 2, 3, 4, 5 are arranged in every possible order to form numbers, ninety of the numbers will be greater than 23000.

32. Show that 200 grammars, 250 arithmetics, 150 readers, and 100 algebras, can be divided between two booksellers in 769428199 ways.

33. A signal post has 5 arms, and each arm is capable of 4 distinct positions, including the position of rest. Show that 1023 different signals may be made.

**Note.**—I am indebted to Whitworth's "Choice and Chance," Aldis's "Text-Book on Algebra," and "Higher Algebra," by Hall and Knight, for many of the above examples.

## PART SECOND.

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### CHAPTER IX.

#### *SERIAL FUNCTIONS.*

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##### 1. Definitions.

**562.** Any expression containing a variable is called a *function* of the variable.

Thus,  $ax + b$ ,  $ax^{-3}$ ,  $\sqrt{a+x^2}$ ,  $a^x$ ,  $\log. (a+x)$ , and  $a + bx + cx^2 + dx^3 + \text{etc.}$ , are functions of  $x$ .

**563.** Any series containing variable terms is called a *serial function*.

**564.** The expression  $f(x)$  represents any function of  $x$ , and is read *function x*.

**565.** When two or more functions of the same variable are used in a discussion, modified forms are used for distinction ; as,

1.  $f(x)$ ,  $F(x)$ ,  $\phi(x)$  ; read, *f minor*, *f major*, *phi* functions of  $x$ .

2.  $f'(x)$ ,  $f''(x)$ ,  $f'''(x)$  ; read, *f prime*, *f second*, *f third* functions of  $x$ .

3.  $f_1(x)$ ,  $f_2(x)$ ,  $f_3(x)$  ; read, *f one*, *f two*, *f three* functions of  $x$ .

## Development of Functions into Series.

## Theorem of Indeterminate Coefficients.

**566.** If  $A + Bx + Cx^2 + Dx^3$  etc.  $= A_1 + B_1x + C_1x^2 + D_1x^3 +$  etc., for any assigned value of  $x$  from  $-\infty$  to  $+\infty$ , and  $A, A_1, B, B_1, C, C_1, D, D_1$ , etc., are independent of  $x$ , then will  $A = A_1, B = B_1, C = C_1, D = D_1$ , etc.

**Demonstration:** Given

$A + Bx + Cx^2 + Dx^3 +$  etc.  $= A_1 + B_1x + C_1x^2 + D_1x^3 +$  etc., (A)  
for any assigned value of  $x$ . Let  $x = 0$ ; then  $A = A_1$ .

Therefore,  $A = A_1$  for every value of  $x$ . (1)

Subtract (1) from (A),

$Bx + Cx^2 + Dx^3 +$  etc.  $= B_1x + C_1x^2 + D_1x^3 +$  etc. (B)

Divide (B) by  $x$ ,

$B + Cx + Dx^2 +$  etc.  $= B_1 + C_1x + D_1x^2 +$  etc. (C)

Let  $x = 0, B = B_1$ .

Therefore,  $B = B_1$  for every value of  $x$ . (2)

etc.,                      etc.,                      etc.

**567. Corollary 1.**—If  $A + Bx + Cx^2 + Dx^3 +$  etc.  $= 0$ , for any assigned value of  $x$ , then will  $A = 0, B = 0, C = 0, D = 0$ , etc.

**568. Cor. 2.**—A function of a single variable can be developed into a series of the ascending powers of the variable in only one way.

For, if possible, let

$f(x) = a + bx + cx^2 +$  etc.; and

$f(x) = a_1 + b_1x + c_1x^2 +$  etc.; then will

$a + bx + cx^2 +$  etc.  $= a_1 + b_1x + c_1x^2 +$  etc.;

whence  $a = a_1, b = b_1, c = c_1$ , etc., and the two developments will be identical.

## 2. Applications.

## 1. Expansion of Rational Fractions.

**569.** A rational fraction of a single variable may generally be developed into a series by dividing the numerator

by the denominator, but a more expeditious method consists in the application of the principle of indeterminate coefficients.

**Illustrations.**—1. Develop  $\frac{1-x}{1+x}$  into a series of the ascending powers of  $x$ .

$$\text{Let } \frac{1-x}{1+x} = A + Bx + Cx^2 + Dx^3 + \text{etc.} \quad (\text{A})$$

Clear of fractions, and arrange the coefficients of the like powers of  $x$  into columns,

$$\begin{array}{r|l|l|l} 1-x = & A+B & x+C & x^2+D & x^3+\text{etc.} \\ & +A & +B & +C & \end{array} \quad (\text{B})$$

Equate the coefficients of the like powers of  $x$  [566],

$$A = 1; A + B = -1, B + C = 0; C + D = 0, \text{ etc.}$$

$$\therefore A = 1, B = -2, C = 2, D = -2, \text{ etc.}$$

Substitute these values of the coefficients in (A),

$$\frac{1-x}{1+x} = 1 - 2x + 2x^2 - 2x^3 + \text{etc.}$$

Let the student divide the numerator by the denominator, and show that the same result will follow.

**570.** The first term of the expansion may be obtained by dividing the first term of the numerator by the first term of the denominator, and the remaining terms by indeterminate coefficients.

2. Develop  $\frac{a}{x+bx^2}$  in the ascending powers of  $x$ .

$$\text{Put } \frac{a}{x+bx^2} = ax^{-1} + Bx^0 + Cx + Dx^2 + \text{etc.} \quad (\text{A})$$

Clear of fractions and column coefficients,

$$\begin{array}{r|l|l|l} a = & a+B & x+C & x^2+D & x^3+\text{etc.} \\ & +ab & +Bb & +Cb & \end{array} \quad (\text{B})$$

Equate coefficients,

$$(1) B + ab = 0, \quad (2) C + Bb = 0, \quad (3) D + Cb = 0, \text{ etc.}$$

$$\therefore B = -ab, C = ab^2, D = -ab^3, \text{ etc.}$$

Substitute these values in (A),

$$\frac{a}{x+bx^2} = ax^{-1} - ab + ab^2x - ab^3x^2 + \text{etc.}$$

## EXERCISE 87.

Develop to four terms :

1.  $\frac{1}{1-x}$

4.  $\frac{1+x}{1+x^2}$

7.  $\frac{a}{a+x}$

2.  $\frac{1}{3-2x}$

5.  $\frac{1-x^2}{1+x+x^2}$

8.  $\frac{5x+2x^2}{1-5x+x^2}$

3.  $\frac{1}{1-x+x^2}$

6.  $\frac{2x-3}{x+x^2+1}$

9.  $\frac{1+x+x^2}{1+x^3}$

## 2. Expansion of Irrational Functions.

**Illustrations.**—1. Expand to four terms  $\sqrt{1-x+x^2}$ .Put  $\sqrt{1-x+x^2} = 1 + Bx + Cx^2 + Dx^3 + \text{etc.}$  (A)

Square both members and column the coefficients,

$$1-x+x^2 = 1 + 2B \left| \begin{array}{c} x + B^2 \\ + 2C \end{array} \right| \left| \begin{array}{c} x^2 + 2D \\ + 2BC \end{array} \right| x^3 + \text{etc.} \quad (\text{B})$$

Equate the coefficients,

(1)  $2B = -1$ . (2)  $B^2 + 2C = 1$ . (3)  $2D + 2BC = 0$ .

$$\therefore B = -\frac{1}{2}, C = \frac{3}{8}, D = \frac{3}{16}, \text{ etc.}$$

Substitute these values in (A),

$$\sqrt{1-x+x^2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \frac{3}{16}x^3 + \text{etc.}$$

2. Expand to three terms  $\sqrt[3]{8-x^2}$ .Put  $\sqrt[3]{8-x^2} = 2 + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$  (A)

Cube both members and column coefficients,

$$8-x^2 = 8 + 12B \left| \begin{array}{c} x + 12C \\ + 6B^2 \end{array} \right| \left| \begin{array}{c} x^2 + B^3 \\ + 12D \\ + 12BC \end{array} \right| \left| \begin{array}{c} x^3 + 12E \\ + 3B^2C \\ + 6C^2 \\ + 12BD \end{array} \right| x^4 + \text{etc.}$$

Equate the coefficients,

(1)  $12B = 0$ . (2)  $12C + 6B^2 = -1$ .

(3)  $12BC + 12D + B^3 = 0$ .

(4)  $12E + 3B^2C + 6C^2 + 12BD = 0$ .

$$\therefore B = 0, C = -\frac{1}{12}, D = 0, E = -\frac{1}{288}, \text{ etc.}$$

Substitute these values in (A),

$$\sqrt[3]{8-x^2} = 2 - \frac{1}{12}x^2 - \frac{1}{288}x^4 - \text{etc.}$$

## EXERCISE 88.

Expand to four terms :

- |                      |                        |                        |
|----------------------|------------------------|------------------------|
| 1. $\sqrt{4-x}$      | 4. $\sqrt[3]{1+x}$     | 7. $\sqrt{a+x}$        |
| 2. $\sqrt{1+x-x^2}$  | 5. $\sqrt[3]{27+x^3}$  | 8. $\sqrt[3]{a-x}$     |
| 3. $\sqrt{9+x-3x^2}$ | 6. $\sqrt[3]{8+x+x^3}$ | 9. $\sqrt[3]{a^3+x^3}$ |
- 

## Convergency and Divergency of Infinite Series.

## General Definitions.

**571.** The *limit of a series* is the limit of the sum of  $n$  terms of the series, when  $n$  is indefinitely increased ; that is, when  $\lim. n = \infty$ .

**572.** A series is *convergent* when its limit is a finite constant, including *zero*.

**573.** A series is *divergent* when its limit is infinity.

**574.** A series is *indeterminate* when the sum of  $n$  terms is finite but does not approach any definite value as  $n$  is indefinitely increased.

Thus,  $1 - 1 + 1 - 1 + 1 - 1 + \dots$  is indeterminate, since, when  $n$  is even the sum is 0, and when  $n$  is odd the sum is 1, however great  $n$  be taken.

**575.** For convenience of discussion, the following notation will be adopted :

1. The terms of a series will be represented in order by  $u_1, u_2, u_3, \dots, u_n, u_{n+1}, \dots$ .

2. The sum of  $n$  terms will be represented by  $U_n$ , so that  $U_n = u_1 + u_2 + u_3 + \dots + u_n$ .

3. The limit of the series will be represented by  $U$ , so that  $U = u_1 + u_2 + u_3 + \dots + u_n + u_{n+1} + \dots$ .

## Fundamental Principles.

**576. 1.** *No series whose terms are all of the same sign can be indeterminate.*

For either the sum of  $n$  terms increases numerically without limit as  $n$  is increased indefinitely, or else it can never exceed some fixed value which it approaches as a limit. Such a series is, therefore, either convergent or divergent.

**577. 2.** *A series of finite terms whose signs are all alike is divergent.*

For, if we let  $a$  represent the numerical value of the smallest term, then, numerically,  $U > na$ , whose limit is  $\infty$ , when  $\lim. n = \infty$  and  $a$  is a finite quantity.

Thus, the series  $1 + 2 + 4 + 8 + 16 + \dots$  is divergent.

**578. 3.** *If a series is convergent it will remain convergent, and if divergent it will remain divergent, if any finite number of terms be added to or subtracted from the series.*

For, the sum of any finite number of terms is finite, and, therefore, can not change the nature of the limit of the series when combined with the series by addition or subtraction.

**579. 4.** *If a series is convergent when its terms are all positive, it is also convergent when its terms are all negative, or some positive and some negative.*

For its limit will have the same numerical value when its terms are all negative as when they are all positive, and will be numerically less when the terms do not all have the same sign as when they do.

It must not be inferred from this principle that a series

is necessarily divergent when its terms are not all of the same sign, if it is divergent when they are alike in sign. Such may or may not be the case.

### Theorems.

**580. I.** *In order that a series may be convergent, the limit of the  $(n+1)$ th term, and the limit of the sum of any number of terms beginning with the  $(n+1)$ th term must be zero, and conversely.*

**Demonstration:** If a series is convergent, then ultimately, if  $n$  is indefinitely increased,

$$(1) U - U_n = 0 \quad [498]$$

$$(2) U - U_{n+1} = 0$$

$$(3) U - U_{n+2} = 0$$

$$(4) U - U_{n+3} = 0$$

Subtract (1) from (2); (1) from (3); (1) from (4), etc.; then,

$$(a) U_n - U_{n+1} = 0; \text{ or, } u_{n+1} = 0; \text{ whence, } \lim. u_{n+1} = 0$$

$$(b) U_n - U_{n+2} = 0; \text{ or, } u_{n+1} + u_{n+2} = 0; \text{ whence,}$$

$$\lim. (u_{n+1} + u_{n+2}) = 0$$

$$(c) U_n - U_{n+3} = 0; \text{ or, } u_{n+1} + u_{n+2} + u_{n+3} = 0; \text{ whence,}$$

$$\lim. (u_{n+1} + u_{n+2} + u_{n+3}) = 0$$

etc.,

etc.,

etc.,

etc.

**581. II.** *If each term of a series whose terms are alternately positive and negative is numerically greater than the following term, the series is convergent.*

**Demonstration:** Let  $U = u_1 - u_2 + u_3 - u_4 + \dots \pm u_n \mp u_{n+1} \dots$ , in which  $u_1 > u_2 > u_3 > u_4 \dots$ , be the given series.

$$(1) U = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \text{etc.}$$

$$(2) U = u_1 - (u_2 - u_3) - (u_4 - u_5) - \text{etc.}$$

From (1) it is evident that  $U$  is positive.

From (2) it is evident that, since  $U$  is positive,  $U < u_1$ .

$\therefore U$  approaches  $u_1$  or some quantity less than  $u_1$  as a limit, and the series is, therefore, convergent.

**582. III.** *A series is convergent if after some particular term the ratio of each term to the preceding term is less than unity.*

**Demonstration:** The most unfavorable case to convergency supposable, under the conditions given, is evidently the one in which all the terms have the same sign (say plus) and all the ratios described are equal and each equal to the greatest of them. This is, therefore, the only case that needs proof.

Let  $r$  be the greatest ratio after the  $n$ th term, but  $< 1$ ; then,  
 $u_n + u_{n+1} + u_{n+2} + u_{n+3} + \text{etc.} = u_n + u_n r + u_n r^2 + \text{etc.} = \frac{u_n}{1-r}$   
 [499, P.] = a finite quantity. Therefore, the whole series is convergent [578].

**583. IV.** *A series of all positive or all negative terms is divergent, if after some particular term the ratio of each term to the preceding term is equal to or greater than unity.*

**Demonstration:** The most unfavorable case to divergency, and the only one that needs investigation, is the one in which all the ratios described are equal and each equal to the least of them.

Let  $r$  be the least ratio after the  $n$ th term, but  $=$  or  $> 1$ ; then,  
 $u_n + u_{n+1} + u_{n+2} + u_{n+3} + \text{etc.}$  is divergent [577]; and hence the whole series is divergent [578].

**584. V.** *A series of positive terms is convergent if each term is less than the corresponding term of a given convergent series of positive terms.*

**Demonstration:** Let  $U = u_1 + u_2 + \dots + u_n + u_{n+1} + \dots$  be a given convergent series; and  $V = v_1 + v_2 + \dots + v_n + v_{n+1} + \dots$  a series in which  $v_1 < u_1, v_2 < u_2, \dots, v_n < u_n, v_{n+1} < u_{n+1}, \dots$

From the nature of addition, it is evident that  $V_n < U_n$ ; and hence, too,  $\lim. V_n < \lim. U_n$ , or  $V < U$ ; therefore, if  $U$  is convergent  $V$  is convergent.

**585.** The foregoing principles and theorems will serve to test the convergency and divergency of a very large number of series, but are not of universal application, inasmuch as they do not apply to all classes of series.

**Note.**—If the terms of a series are not all of the same sign no general method can be obtained for testing their convergency or divergency.

**586.** The convergency or divergency of a series may often be determined by grouping terms, as follows:

$$\begin{aligned}
 U &= \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \dots \\
 &= 1 + \left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \left(\frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3}\right) + \left(\frac{1}{8^3} + \dots + \frac{1}{15^3}\right) + \text{etc.} \\
 \therefore U &< 1 + \frac{2}{2^3} + \frac{4}{4^3} + \frac{8}{8^3} + \text{etc.}; \text{ or } U < \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}.
 \end{aligned}$$

$\therefore$  The series is convergent.

## EXERCISE 89.

1. Is the series  $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.}$  convergent?
  2. Test the series:  $1 + 3x + 5x^2 + 7x^3 + \dots$  for convergence
    1. When  $x < 1$ .    2. When  $x > 1$ .    3. When  $x = 1$ .
  3. Test the series:  $\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \dots$  for convergence.
  4. Is  $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \text{etc.}$  convergent, when  $x < 1$ ?
  5. Test the series:  $\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \text{etc.}$  for convergence.
  6. Test the series:
 
$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \text{etc.}$$
 for convergence
    1. When  $x < 1$ .    2. When  $x > 1$ .    3. When  $x = 1$ .
- Suggestion.**—Lim.  $\frac{U_{n+1}}{U_n} = x^2$ . Why?
7. Test the series:  $1 + x + x^2 + x^3 + \text{etc.}$  for convergence
    1. When  $x = 1$ .    2. When  $x < 1$ .    3. When  $x > 1$ .
  8. Test  $1 + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \dots$  for convergence.

## The Binomial Formula.

**587.** The binomial formula is used to find the product of any number of binomial functions of the form of  $x + a$ .

**Development.**

$$(x+a)(x+b) = x^2 + (a+b)x + ab$$

Multiply both members by  $x + c$ ,

$$\begin{aligned}(x+a)(x+b)(x+c) &= x^3 + (a+b)x^2 + abx + cx^2 + (ac+bc)x + abc \\ &= x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc\end{aligned}$$

Multiply both members by  $x + d$ ,

$$\begin{aligned}(x+a)(x+b)(x+c)(x+d) &= \\ &= x^4 + (a+b+c)x^3 + (ab+ac+bc)x^2 + abcx \\ &\quad + dx^3 + (ad+bd+cd)x^2 + (abd+acd+bcd)x + abcd \\ &= x^4 + (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 + \\ &\quad (abc+abd+acd+bcd)x + abcd\end{aligned}$$

Observe the following laws in these products :

*1. The number of terms is one greater than the number of binomial factors.*

*2. The exponent of  $x$  in the first term equals the number of binomial factors, and decreases by unity in each succeeding term.*

*3. The coefficient in the first term is unity ; in the second term the sum of the second terms of the binomial factors ; in the third term the sum of the products of the second terms taken two together ; in the fourth term the sum of the products of the second terms taken three together, etc.*

*4. The last term equals the product of all the second terms.*

Are these laws true for any number of factors ?

Assume them true for  $r$  factors, so that

$$(x+a)(x+b)\dots(x+m) = x^r + p_1x^{r-1} + p_2x^{r-2} + \dots + p_{r-1}x + p_r, \text{ (A), in which}$$

$$p_1 = a + b + \dots + m$$

$$p_2 = ab + ac + \dots + am + bc + bd + \dots + bm + \text{etc.}$$

$$p_3 = abc + abd + \dots + abm + \dots$$

$$p_r = abc \dots m.$$

Multiply by  $(x + n)$ , the  $(r + 1)$ th factor, then

$$(x + a)(x + b) \dots (x + n) =$$

$$\begin{array}{r} x^{r+1} + p_1 x^r + p_2 x^{r-1} + \dots + p_r x \\ + nx^r + np_1 x^{r-1} + \dots + np_{r-1} x + np_r \\ \hline = x^{r+1} + (p_1 + n)x^r + (p_2 + np_1)x^{r-1} + \dots + np_r \end{array}$$

Laws 1 and 2 are evidently still true.

$$p_1 + n = a + b + c + \dots + n.$$

$$p_2 + np_1 = (ab + ac + \dots + am + bc + bd + \dots + bm + \text{etc.}) + (an + bn + \dots + mn),$$

which is still the product of the second terms taken two and two.

$np_r = abcd \dots n.$  Therefore, all the laws still hold true. Hence, if they are true for  $r$  factors, they are true for  $r + 1$  factors. But we found them true for four factors by multiplication; hence, they are true for five factors; and, if so, for six factors; and so on. Therefore, formula (A) is general.

**Note.**—The number of products that enter into each coefficient may be determined by the principles of combination.

### Applications.

#### Illustrations.—

1. Expand  $(x + 1)(x + 2)(x - 3)(x + 4).$

**Solution:**

$$p_1 = 1 + 2 - 3 + 4 = 4$$

$$p_2 = (1 \times 2) + (1 \times -3) + (1 \times 4) + (2 \times -3) + (2 \times 4) + (-3 \times 4) = -7$$

$$p_3 = (1 \times 2 \times -3) + (1 \times 2 \times 4) + (1 \times -3 \times 4) + (2 \times -3 \times 4) = -34$$

$$p_4 = 1 \times 2 \times -3 \times 4 = -24$$

$$\therefore (x + 1)(x + 2)(x - 3)(x + 4) = x^4 + 4x^3 - 7x^2 - 34x - 24.$$

2. Factor  $x^4 + 14x^3 + 71x^2 + 154x + 120$ , if possible.

Let  $(x+a)(x+b)(x+c)(x+d) = x^4 + 14x^3 + 71x^2 + 154x + 120$ .

Then, 1.  $a+b+c+d = +14$

2.  $ab+ac+ad+bc+bd+cd = +71$

3.  $abc+abd+acd+bcd = +154$

4.  $abcd = +120$

Resolve if possible  $+120$  into four factors whose sum is  $+14$ .  
These we find to be 2, 3, 4, 5.

$\therefore a = 2, b = 3, c = 4, \text{ and } d = 5$ .

Will these values satisfy 2 and 3?

$ab+ac+ad+bc+bd+cd = 6+8+10+12+15+20 = 71$ , correct.

$abc+abd+acd+bcd = 24+30+40+60 = 154$ , correct.

$\therefore x^4 + 14x^3 + 71x^2 + 154x + 120 = (x+2)(x+3)(x+4)(x+5)$ .

#### EXERCISE 90.

1. Expand  $(x+2)(x+3)(x+1)$

2. Expand  $(x+3)(x-2)(x-3)$

3. Expand  $(x+2)(x+3)(x-1)(x-2)$

4. Expand  $(x+3)(x+5)(x-2)(x-6)$

5. Expand  $(x+2)(x+2)(x+2)(x+2)$

6. Expand  $(x-5)(x-5)(x-5)(x-5)$

7. Expand  $(2x+1)(2x+3)(2x-5)(2x-1)$

**Suggestion.**—Put  $y$  for  $2x$ .

8. Factor  $x^3 + 9x^2 + 26x + 24$

9. Factor  $x^3 - 2x^2 - 23x + 60$

10. Factor  $x^4 + 5x^3 + 5x^2 - 5x - 6$

11. Factor  $x^4 - 2x^3 - 25x^2 + 26x + 120$

12. Factor  $x^5 + 4x^4 - 13x^3 - 52x^2 + 36x + 144$

#### The Binomial Theorem.

##### 1. For Positive Exponents.

588. If, in the binomial formula [587, A], we assume  
 $a = b = c = d$ , etc., and  $r = n$ , then will

1.  $(x+a)(x+b)(x+c)\dots = (x+a)^n$ .
2.  $x^r = x^n$ ;  $x^{r-1} = x^{n-1}$ ;  $x^{r-2} = x^{n-2}$ ; etc.
3.  $p_1 = a + a + a + \dots$  to  $n$  terms  $= na$ .
4.  $p_2 = aa + aa + aa + \dots = a^2$  taken as many times as there are combinations of 2 in  $n$ ; or,

$$p_2 = \frac{n(n-1)}{2} a^2.$$

5.  $p_3 = aaa + aaa + aaa + \dots = a^3$  taken as many times as there are combinations of 3 in  $n$ ; or,

$$p_3 = \frac{n(n-1)(n-2)}{3} a^3.$$

6.  $p^r = a \times a \times a \times a \dots$  to  $n$  factors  $= a^n$ .

$$\therefore (x+a)^n =$$

$$x^n + nx^{n-1}a + \frac{n(n-1)}{2}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{3}x^{n-3}a^3 + \frac{n(n-1)(n-2)(n-3)}{4}x^{n-4}a^4 + \dots + a^n. \quad (B)$$

**589. Cor. 1.**—If  $a$  and  $x$  be interchanged,  $(a+x)^n =$

$$a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \dots + x^n. \quad (C)$$

From (B) and (C) it will be seen that the coefficients of any two terms equidistant from the first and last terms are numerically equal.

**590. Cor. 2.**—If  $x$  be made negative in (C),  $(a-x)^n =$

$$a^n - na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 - \dots \pm x^n.$$

**591. Cor. 3.**—The sum of the coefficients in (C) equals zero.

For, put  $a=1$  and  $x=1$ ; then

$$(1-1)^n, \text{ or } 0 = 1 - n + \frac{n(n-1)}{2} - \frac{n(n-1)(n-2)}{3} + \dots \pm 1.$$

## 2. For any Rational Exponents.

**592. Lemma.**—*Lim.*  $\left(\frac{x^n - r^n}{x - r}\right)_{x=r} = n r^{n-1}$  for any rational value of  $n$ .

**Demonstration:** I. *Let*  $n$  *= any positive integer.*

$$\text{Now, } \frac{x^n - r^n}{x - r} = x^{n-1} + r x^{n-2} + r^2 x^{n-3} + \dots + r^{n-1} \quad [134].$$

$$\begin{aligned} \therefore \text{Lim. } \left(\frac{x^n - r^n}{x - r}\right)_{x=r} &= \lim. x^{n-1} + \lim. r x^{n-2} + \dots \\ &+ \lim. r^{n-1} \quad [401, 413] = r^{n-1} + r^{n-1} + r^{n-1} + \dots \\ &\text{to } n \text{ terms} = n r^{n-1}. \end{aligned}$$

II. *Let*  $n = \frac{p}{q}$ , *a positive fraction.*

$$\text{Now, } \frac{x^n - r^n}{x - r} = \frac{x^{\frac{p}{q}} - r^{\frac{p}{q}}}{x - r}. \quad (1)$$

Put  $x^{\frac{1}{q}} = y$ , or  $x = y^q$ ; and  $r^{\frac{1}{q}} = s$ , or  $r = s^q$ ; then

$$\frac{x^n - r^n}{x - r} = \frac{y^p - s^p}{y^q - s^q} = \frac{y^p - s^p}{y - s} + \frac{y^q - s^q}{y - s}.$$

Since  $x = y^q$  and  $r = s^q$ ,  $\lim. y = s$  when  $\lim. x = r$ .

$$\begin{aligned} \therefore \text{Lim. } \left(\frac{x^n - r^n}{x - r}\right)_{x=r} &= \lim. \left\{ \frac{y^p - s^p}{y - s} + \frac{y^q - s^q}{y - s} \right\}_{y=s} = \\ &= p y^{p-1} + q y^{q-1} \quad [I, 416] = \frac{p}{q} y^{p-q} = \\ &= \frac{p}{q} (x^{\frac{1}{q}})^{p-q} = \frac{p}{q} x^{\frac{p}{q}-1} = n x^{n-1}. \end{aligned}$$

III. *Let*  $n = -p$ , *a negative integer or fraction.*

$$\text{Now, } \frac{x^n - r^n}{x - r} = \frac{x^{-p} - r^{-p}}{x - r} = -x^{-p} r^{-p} \left( \frac{x^p - r^p}{x - r} \right).$$

$$\begin{aligned} \therefore \text{Lim. } \left(\frac{x^n - r^n}{x - r}\right)_{x=r} &= \lim. \left\{ -x^{-p} r^{-p} \left( \frac{x^p - r^p}{x - r} \right) \right\}_{x=r} \\ &= -r^{-2p} \cdot p r^{p-1} \quad [415] = -r^{2p} (-n r^{-n-1}) = n r^{n-1}. \end{aligned}$$

## General Demonstration of Binomial Theorem.

**593.** Let it be required to develop  $(a+x)^n$ , for any rational value of  $n$ , into a series of the descending powers of  $x$ .

$$(a+x)^n = \left\{ a \left( 1 + \frac{x}{a} \right) \right\}^n = a^n \left( 1 + \frac{x}{a} \right)^n \quad (\text{A})$$

$$\text{Put } \frac{x}{a} = z, \text{ or } \left( 1 + \frac{x}{a} \right)^n = (1+z)^n \quad (\text{B})$$

$$\text{Put } (1+z)^n = 1 + Az + Bz^2 + Cz^3 + Dz^4 + \dots \quad (\text{C})$$

Since  $z$  may have any finite value, put  $z=r$ ; then,

$$(1+r)^n = 1 + Ar + Br^2 + Cr^3 + Dr^4 + \dots \quad (\text{D})$$

Subtract (D) from (C),

$$(1+z)^n - (1+r)^n = A(z-r) + B(z^2-r^2) + C(z^3-r^3) + \dots \quad (\text{E})$$

Put  $Z$  for  $1+z$  and  $R$  for  $1+r$ ; or  $Z-R$  for  $z-r$ ; then,

$$Z^n - R^n = A(Z-R) + B(Z^2-R^2) + C(Z^3-R^3) + \dots \quad (\text{F})$$

Divide by  $Z-R = z-r$ ,

$$\frac{Z^n - R^n}{Z-R} = A + B \left( \frac{z^2 - r^2}{z-r} \right) + C \left( \frac{z^3 - r^3}{z-r} \right) + \dots \quad (\text{G})$$

Let  $\lim. z = r$ , then  $\lim. Z = R$ , since  $Z = 1+z$ , and  $R = 1+r$ ;

$$\text{and } \lim. \left( \frac{Z^n - R^n}{Z-R} \right)_{Z=R} =$$

$$\lim. \left\{ A + B \left( \frac{z^2 - r^2}{z-r} \right) + C \left( \frac{z^3 - r^3}{z-r} \right) + \dots \right\}_{z=r} \quad (\text{H})$$

$$\therefore nR^{n-1} [592] = A + 2Br + 3Cr^2 + 4Dr^3 + \dots \quad (\text{I})$$

$$\text{or, } n(1+r)^{n-1} = A + 2Br + 3Cr^2 + 4Dr^3 + \dots \quad (\text{J})$$

Multiply by  $1+r$ , and column coefficients,

$$n(1+r)^n = A + \begin{array}{c} A \\ + 2B \end{array} \left| \begin{array}{c} r + 2B \\ + 3C \end{array} \right| \begin{array}{c} r^2 + 3C \\ + 4D \end{array} \left| \begin{array}{c} r^3 + \dots \end{array} \right. \quad (\text{K})$$

Multiply (D) by  $n$ ,

$$n(1+r)^n = n + Anr + Bnr^2 + Cnr^3 + Dnr^4 + \dots \quad (\text{L})$$

Equating the coefficients of the second members in (K) and (L), we have

$$(1) \ A = n$$

$$(2) \ A + 2B = An$$

$$(3) \ 2B + 3C = Bn$$

$$(4) \ 3C + 4D = Dn; \text{ etc.}$$

$$\therefore A = n, \ B = \frac{n(n-1)}{2}, \ C = \frac{n(n-1)(n-2)}{3},$$

$$D = \frac{n(n-1)(n-2)(n-3)}{4}, \text{ etc.}$$

Substituting the values in (C),

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2} z^2 + \frac{n(n-1)(n-2)}{3} z^3 + \frac{n(n-1)(n-2)(n-3)}{4} z^4 + \text{etc.} \quad (\text{M})$$

Substitute  $\frac{x}{a}$  for  $z$  (B), and multiply by  $a^n$  (A),

$$(a+x)^n = a^n + n a^{n-1} x + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{6} x^3 + \frac{n(n-1)(n-2)(n-3)}{24} a^{n-4} x^4 + \dots \quad (N)$$

**594. Cor. 1.**  $(x+y)^n = x^n \left(1 + \frac{y}{x}\right)^n =$

$$x^n \left\{ 1 + n \cdot \frac{y}{x} + \frac{n(n-1)}{2} \cdot \frac{y^2}{x^2} + \frac{n(n-1)(n-2)}{6} \cdot \frac{y^3}{x^3} + \text{etc.} \right\}$$

$$= x^n + n x^{n-1} y + \frac{n(n-1)}{2} x^{n-2} y^2 + \frac{n(n-1)(n-2)}{6} x^{n-3} y^3 + \text{etc.} \quad (P)$$

This is the most general form of the binomial theorem, inasmuch as  $x$  and  $y$  may be both variables.

**595. Cor. 2.**—*By inspection it will be seen that*

1. The  $r$ th term of the development of  $(x+y)^n =$

$$\frac{n(n-1)(n-2)\dots(n-r+2)}{r-1} \cdot x^{n-r+1} y^{r-1}$$

2. The  $(r+1)$ th term =

$$\frac{n(n-1)(n-2)\dots(n-r+2)(n-r+1)}{r-1 \times r = r} \cdot x^{n-r} y^r$$

3. The ratio of the  $(r+1)$ th term to the  $r$ th term =

$$\frac{n-r+1}{r} \cdot \frac{y}{x}; \text{ or, } \left\{ \frac{n+1}{r} - 1 \right\} \frac{y}{x}$$

**596. Cor. 3.**—1. If  $n$  is a positive integer equal to  $r-1$ , the coefficient of the  $(r+1)$ th term, which is also the coefficient of the  $(n+2)$ th term, will reduce to zero. Therefore, the series will terminate with the  $(n+1)$ th term, which will be  $y^n$ .

2. If  $n$  is negative or fractional no factor of the  $r$ th

term ( $r$  being a positive integer) will reduce to zero, however great  $r$  be taken. Therefore, the series will be infinite.

**597. Cor. 4.—**

Since  $\lim. \left\{ \left( \frac{n+1}{r} - 1 \right) \frac{y}{x} \right\}_{r=\infty} = -1 \cdot \frac{y}{x}$ , it follows:

1. That the coefficients of all terms in the binomial theorem are finite however far the theorem be expanded.

2. That if  $y < x$  the expanded form is convergent [582].

3. That if  $y > x$ , the literal part (not coefficient) of the  $(r+1)$ th term will increase indefinitely as  $r$  increases and will ultimately become infinitely great, and as the coefficient remains finite the whole term will become infinitely great. Therefore the expanded form will be divergent [580].

4. If  $y = \pm x$  the expansion will be indeterminate; but  $(x+y)^n = (2x)^n$  or  $(0)^n = 2^n x^n$  or 0.

5. The expansions of  $(x+y)^n$  and  $(y+x)^n$  can not both be convergent for particular values of  $x$  and  $y$ ; only the one that has the greater first term.

**598. Cor. 5.—1.** The coefficient of the  $r$ th term will evidently be greatest when  $\frac{n-r+1}{r}$  is first  $< 1$ ; or when  $n-r+1$  is first  $< r$ ; or when  $2r$  is first  $> n+1$ , or when  $r$  is first  $> \frac{n+1}{2}$ .

2. The  $r$ th term when the expansion is convergent, or when  $x > y$ , is evidently greatest when  $\frac{n-r+1}{r} \cdot \frac{y}{x}$  is first  $< 1$ ; or when  $(n-r+1)y$  is first  $< rx$ ; or when  $(n+1)y$  is first  $< (x+y)r$ ; or when  $r$  is first  $> \left( \frac{n+1}{x+y} \right) y$ .

**Illustration.**—In the expansion of  $\left( 8 + \frac{2}{3} \right)^{-\frac{1}{2}}$ , the

greatest coefficient belongs to the term whose number is first greater than  $\frac{-\frac{4}{3} + 1}{2}$ , or  $\frac{1}{10}$ ; which is the first term.

The greatest term in the expansion of  $\left(8 + \frac{2}{3}\right)^{-\frac{4}{3}}$  is the one which immediately follows in number,  $\frac{-\frac{4}{3} + 1}{8\frac{2}{3}} \times \frac{2}{3}$ , or  $\frac{1}{65}$ , which is again the first term.

## EXERCISE 91.

Expand :

- |                   |                   |   |
|-------------------|-------------------|---|
| 1. $(a - 3x^2)^4$ | 3. $(x - 3a)^6$   | 5. $(x^{\frac{1}{2}} - 5)^8$                |
| 2. $(2 + 5x)^5$   | 4. $(2x^2 + 5)^7$ | 6. $(3x^{\frac{2}{3}} + a^{\frac{1}{3}})^7$ |

Expand to four terms :

- |                            |  |  |
|----------------------------|--|--|
| 7. $(1 - x)^{\frac{1}{2}}$ | 9. $(x^2 - 1)^{\frac{3}{2}}$               | 11. $(ax + b)^{-\frac{4}{3}}$                            |
| 8. $(a - x)^{\frac{2}{3}}$ | 10. $(x^{\frac{1}{2}} + a)^{-\frac{1}{2}}$ | 12. $(x^{\frac{2}{3}} - a^{\frac{1}{3}})^{-\frac{2}{3}}$ |

13. Extract the cube root of 126 to six decimal places.

Suggestion :

$$\begin{aligned}\sqrt[3]{126} &= \sqrt[3]{125 + 1} = (125 + 1)^{\frac{1}{3}} = \left\{5^3 \left(1 + \frac{1}{125}\right)\right\}^{\frac{1}{3}} = 5 \left(1 + \frac{1}{125}\right)^{\frac{1}{3}} \\ &= 5 \left\{1 + \frac{1}{3} \times \frac{1}{125} + \frac{\frac{1}{3}(\frac{1}{3} - 1)}{2} \times \left(\frac{1}{125}\right)^2 + \frac{\frac{1}{3}(\frac{1}{3} - 1)(\frac{1}{3} - 2)}{6} \times \right. \\ &\quad \left. \left(\frac{1}{125}\right)^3 + \text{etc.}\right\} = 5(1 + 0.026666 - 0.000071 + 0.000001) = 5.0132975.\end{aligned}$$

14. Find to 5 decimal places :  $\sqrt{65}$ ,  $\sqrt{80}$ ,  $\sqrt[3]{344}$ ,  $\sqrt[4]{3128}$

15. Find the 7th term of  $(2x + 3)^{10}$

16. Find the 5th term of  $\sqrt{4 + x}$

17. Find the 6th term of  $\frac{a}{\sqrt{a + x}}$

18. Find the  $r$ th term of  $(a - x)^{-5}$

19. Find the greatest coefficient of  $(2 + x)^{\frac{4}{3}}$

20. Find the coefficient of the 5th term of  $(a - x^2)^{-\frac{2}{3}}$

21. Find the numerical value of the 10th term of

$$(7 - 5y^{\frac{2}{3}})^n, \text{ when } y = 27 \text{ and } n = 8$$

## The Exponential Theorem.

**599.** The exponential theorem is the expansion of  $a^x$  in ascending powers of  $x$ , and is derived as follows :

$$\left(1 + \frac{1}{n}\right)^{nx} = \left\{ \left(1 + \frac{1}{n}\right)^n \right\}^x \quad (\text{A})$$

$$\text{But } \left(1 + \frac{1}{n}\right)^n = 1 + nx \cdot \frac{1}{n} + \frac{nx(nx-1)}{2} \cdot \frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{3} \cdot \frac{1}{n^3} + \text{etc.} \quad (\text{B})$$

$$\text{And } \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1 - \frac{1}{n}}{2} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{3} + \text{etc.} \quad (\text{C})$$

Substitute (B) and (C) in (A),

$$1 + x + \frac{x\left(x - \frac{1}{n}\right)}{2} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{3} + \text{etc.} \\ = \left\{ 1 + 1 + \frac{1 - \frac{1}{n}}{2} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{3} + \text{etc.} \right\}^x \quad (\text{D})$$

Suppose  $\lim. n = \infty$ , then

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \text{etc.} = \left(1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}\right)^x \quad (\text{E})$$

Put  $e$  for  $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}$ ; then,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \text{etc.} \quad (\text{F})$$

$$\text{Put } cx \text{ for } x, \text{ then } e^{cx} = 1 + cx + \frac{c^2 x^2}{2} + \frac{c^3 x^3}{3} + \text{etc.} \quad (\text{G})$$

Let  $e^c = a$ , and assume  $e$  as the base of a system of logarithms [465], then  $c = \log_e a$ , read logarithm  $a$  to the base  $e$ . Substitute these values in (G),

$$a^x = 1 + x \log_e a + \frac{x^2 (\log_e a)^2}{2} + \frac{x^3 (\log_e a)^3}{3} + \text{etc.}, \quad (\text{H})$$

which is convergent for all finite values of  $x$  [582].

*This is the Exponential Theorem.*

$$\mathbf{600. Scholium.} \quad e = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} = 2.7182818 \dots$$

is the base of the Napierian or natural system of logarithms, a system universally used in theoretical work instead of the system based on 10, which is used in practical work only.

## The Logarithmic Series.

**601.** The logarithmic series is the expansion of  $\log_e (1+x)$  in the ascending powers of  $x$ , and is derived as follows :

$$a^y = 1 + y \log_e a + \frac{y^2 (\log_e a)^2}{2} + \frac{y^3 (\log_e a)^3}{3} + \text{etc.} \quad [599, H]. \quad (A)$$

Transpose 1 and divide by  $y$ ,

$$\frac{a^y - 1}{y} = \log_e a + y \left\{ \frac{(\log_e a)^2}{2} + \frac{y (\log_e a)^3}{3} + \text{etc.} \right\} \quad (B)$$

Let  $\lim. y = 0$ , then

$$\log_e a = \lim. \left( \frac{a^y - 1}{y} \right)_{y=0}$$

Put  $1+x$  for  $a$ , then

$$\begin{aligned} \log_e (1+x) &= \lim. \frac{1}{y} \left\{ (1+x)^y - 1 \right\}_{y=0} \\ &= \lim. \frac{1}{y} \left\{ yx + \frac{y(y-1)}{2} x^2 + \frac{y(y-1)(y-2)}{3} x^3 + \text{etc.} \right\}_{y=0} \\ &= \lim. \left\{ x + \frac{y-1}{2} x^2 + \frac{(y-1)(y-2)}{3} x^3 + \text{etc.} \right\}_{y=0} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \text{etc.} \quad \text{Therefore,} \end{aligned}$$

$$\log_e (1+x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \text{etc.} \quad (C)$$

This is known as the *Logarithmic Series*.

**602.** The ratio of the  $(n+1)$ th term to the  $n$ th term is  $\frac{x^{n+1}}{n+1} : \frac{x^n}{n} = \frac{n}{n+1} \cdot x = \frac{1}{1 + \frac{1}{n}} \cdot x$ .

Now,  $\lim. \left( \frac{1}{1 + \frac{1}{n}} \cdot x \right)_{n=\infty} = x$ . Therefore, if  $x < 1$ , numerically the series is convergent. It is, therefore, convergent for all values of  $x$  between  $-1$  and  $+1$ .

When  $x = 1$ ,  $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{etc.}$ ; which is convergent [581].

When  $x = -1$ ,  $\log_e 0 = -1 \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} \right)$ , which is divergent, since  $\lim. \left( \frac{u_{n+1}}{u_n} \right)_{n=\infty} = \lim. \left\{ -1 \left( \frac{1}{1 + \frac{1}{n}} \right) \right\}_{n=\infty} = -1$ , and all the terms have the same sign [583].

**603.** Resume

$$\log. (1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots \quad (1)$$

Put  $-x$  for  $x$ , then

$$\log. (1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \dots \quad (2)$$

Subtract (2) from (1), then

$$\log. (1+x) - \log. (1-x), \text{ or } \log. \left( \frac{1+x}{1-x} \right) [467, \text{ P. 3}] = \\ 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right) \quad (3)$$

Put  $\frac{m-n}{m+n}$  for  $x$ , then

$$\log. \frac{m}{n} = 2 \left\{ \frac{m-n}{m+n} + \frac{1}{3} \left( \frac{m-n}{m+n} \right)^3 + \frac{1}{5} \left( \frac{m-n}{m+n} \right)^5 + \dots \right\} \quad (4)$$

Put  $m = n+1$ , then

$$\log. (n+1) - \log. n = 2 \left\{ \frac{1}{2n+1} + \frac{1}{3} \left( \frac{1}{2n+1} \right)^3 + \right. \\ \left. \frac{1}{5} \left( \frac{1}{2n+1} \right)^5 + \dots \right\} \quad (5)$$

or,

$$\log. (n+1) = \log. n + 2 \left\{ \frac{1}{2n+1} + \frac{1}{3} \left( \frac{1}{2n+1} \right)^3 + \right. \\ \left. \frac{1}{5} \left( \frac{1}{2n+1} \right)^5 + \dots \right\} \quad (D)$$

As this formula converges very rapidly for all values of  $n$ , it may be used to find the Napierian logarithm of any number from that of the next preceding number,  $n$  being regarded an integer.

### Computation of Logarithms.

**604.** The logarithms of composite numbers may be readily found, when the logarithms of primes are known, by Art. 467, P. 2.

The logarithms of prime numbers are found by formula D, Art. 603.

**Illustrations.**—

$$\log. 1 = 0 \text{ [466, P.]}$$

$$\log. 2 = \log. (1 + 1) =$$

$$0 + 2 \left( \frac{1}{3} + \frac{1}{3 \times 3^3} + \frac{1}{5 \times 3^5} + \frac{1}{7 \times 3^7} + \dots \right) \\ = 0.69314718 \dots \text{ (by actual reduction).}$$

$$\log. 3 = \log. (2 + 1) =$$

$$\log. 2 + 2 \left( \frac{1}{5} + \frac{1}{3 \times 5^3} + \frac{1}{5 \times 5^5} + \frac{1}{7 \times 5^7} + \dots \right) \\ = 1.09861228 \dots$$

$$\log. 4 = 2 \log. 2 = 1.38629436 \dots$$

$$\log. 5 = \log. (4 + 1) =$$

$$\log. 4 + 2 \left( \frac{1}{9} + \frac{1}{3 \times 9^3} + \frac{1}{5 \times 9^5} + \dots \right) \\ = 1.60943791 \dots$$

$$\log. 6 = \log. 3 + \log. 2 = 1.79175946 \dots$$

$$\log. 7 = \log. (6 + 1) =$$

$$\log. 6 + 2 \left( \frac{1}{13} + \frac{1}{3 \times 13^3} + \frac{1}{5 \times 13^5} + \dots \right) \\ = 1.94591 \dots$$

$$\log. 8 = 3 \log. 2 = 2.07944$$

$$\log. 9 = 2 \log. 3 = 2.19722$$

$$\log. 10 = \log. 5 + \log. 2 = 2.30258509$$

etc.,

etc.,

etc.

**605.** Let  $a$  and  $b$  represent the bases of two systems of logarithms and  $n$  any number.

$$\text{Let } \log_b n = x, \text{ then } b^x = n \quad (1)$$

$$\text{Let } \log_a b = m, \text{ then } a^m = b \quad (2)$$

$$\therefore a^{mx} = b^x = n, \text{ or } \log_a n = mx \quad (3)$$

$$\therefore \frac{\log_a n}{\log_a b} = \frac{mx}{x} = m; \text{ or}$$

$$\log_a n = m \log_b n. \text{ Therefore,}$$

**Principle.**—*Multiplying the  $\log_b$  of a number by the  $\log_a$  of  $b$  gives the  $\log_a$  of the number.*

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**606.**  $\text{Log}_a n = \log_{10} n \times \log_a 10$  [605, P.]

$$\begin{aligned}\therefore \log_{10} n &= \log_a n \times \frac{1}{\log_a 10} = \log_a n \times \frac{1}{2.30258509} \\ &= \log_a n \times 0.4342944 \dots\end{aligned}$$

**607.** The number 0.4342944... is called the modulus of the common system, and is represented by  $m$ .

Therefore,

**Prin. 2.**— $\text{Log}_{10} n = m \log_a n$ .

By means of this principle the Briggian or common logarithms may be derived from the Napierian or natural logarithms.

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**608.** Since  $\log_a (n+1) =$

$$\log_a n + 2 \left\{ \frac{1}{2n+1} + \frac{1}{3} \left( \frac{1}{2n+1} \right)^3 + \dots \right\},$$

$$\log_{10} (n+1) =$$

$$\log_{10} n + 2m \left\{ \frac{1}{2n+1} + \frac{1}{3} \left( \frac{1}{2n+1} \right)^3 + \dots \right\} \quad (\text{H})$$

By means of this formula the common logarithms may be computed directly.

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### Summation of Series.

**609.** No general method of summing series can be given. Series of special types may sometimes be summed by special methods. The student has already learned how to sum an arithmetical progression [486], a geometrical progression [493], an infinite series of the geometrical type [499], an arithmetico-geometrical progression [500], a series of square numbers [506], a series of cubic num-

bers [507], and series dependent upon or resolvable into these. A few additional methods will be given here.

### 610. 1. Method by Indeterminate Coefficients.

This method is applicable when the  $n$ th term is a rational integral function of  $n$ .

**Illustration.**—Find the sum  $S_n$  of  $n$  terms of the series :  
 $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \dots + n(n+1)^2$ .

**Solution :**

$$\begin{aligned}\text{Put } S_n &= 1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + 4 \times 5^2 + \dots + n(n+1)^2 \\ &= A + Bn + Cn^2 + Dn^3 + En^4 + \dots\end{aligned}$$

$$\begin{aligned}\text{and, } S_{n+1} &= 1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + 4 \times 5^2 + \dots + (n+1)(n+2)^2 \\ &= A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 + \dots\end{aligned}$$

Then, by subtraction,

$$\begin{aligned}(n+1)(n+2)^2 &= B + (2n+1)C + (3n^2+3n+1)D \\ &\quad + (4n^3+6n^2+4n+1)E + \dots, \text{ or } n^3 + 5n^2 + 8n + 4 = \\ &\quad (B+C+D+E) + (2C+3D+4E)n + (3D+6E)n^2 + 4En^3,\end{aligned}$$

since all coefficients after  $E$  are zero, there being no more than four terms in the expansion.

Equating the coefficients of the like powers of  $n$ ,

$$\begin{array}{ll} 1. 4E = 1, & 2. 3D + 6E = 5, \\ 3. 2C + 3D + 4E = 8, & 4. B + C + D + E = 4; \end{array}$$

$$\text{whence, } E = \frac{1}{4}, D = \frac{7}{6}, C = \frac{7}{4}, \text{ and } B = \frac{5}{6}.$$

$$\therefore S_n = A + \frac{5}{6}n + \frac{7}{4}n^2 + \frac{7}{6}n^3 + \frac{1}{4}n^4.$$

To find  $A$ , put  $n=1$ , then  $S_n = S_1$  = the first term  $= 1 \times 2^2$ .

$$\therefore (1 \times 2)^2 = A + \frac{5}{6} + \frac{7}{4} + \frac{7}{6} + \frac{1}{4} = A + 4;$$

whence  $A = 0$ ; and

$$\begin{aligned}S_n &= \frac{5}{6}n + \frac{7}{4}n^2 + \frac{7}{6}n^3 + \frac{1}{4}n^4 \\ &= \frac{1}{12}(8n^4 + 14n^3 + 21n^2 + 10n) \\ &= \frac{n}{12}(8n^3 + 14n^2 + 21n + 10) = \frac{n}{12}(n+1)(n+2)(3n+5)\end{aligned}$$

**611.** 2. Method by Decomposition.

This method is sometimes applicable when the  $n$ th term is a rational fractional function of  $n$ , and is resolvable into the algebraic sum of the  $n$ th terms of two or more other series of the same nature.

**Illustration.**—

Find the sum  $S_n$  of  $n$  terms of the series:  $\frac{4}{2 \times 3 \times 4} + \frac{7}{3 \times 4 \times 5} + \frac{10}{4 \times 5 \times 6} + \dots + \frac{3n+1}{(n+1)(n+2)(n+3)}$ .

**Solution:**

Let  $\frac{3n+1}{(n+1)(n+2)(n+3)} = \frac{A}{n+1} + \frac{B}{n+2} + \frac{C}{n+3}$ ; then

$A = -1$ ,  $B = 5$ , and  $C = -4$ ; whence

$$\frac{3n+1}{(n+1)(n+2)(n+3)} = \left( -\frac{1}{n+1} + \frac{5}{n+2} - \frac{4}{n+3} \right).$$

$$\therefore S_n = \Sigma \left\{ \frac{3n+1}{(n+1)(n+2)(n+3)} \right\} = \Sigma \left( -\frac{1}{n+1} \right) + \Sigma \left( \frac{5}{n+2} \right) + \Sigma \left( -\frac{4}{n+3} \right).$$

$$\Sigma \left( -\frac{1}{n+1} \right) = -\frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots - \frac{1}{n+1}$$

$$\Sigma \left( \frac{5}{n+2} \right) = \frac{5}{3} + \frac{5}{4} + \dots + \frac{5}{n+1} + \frac{5}{n+2}$$

$$\Sigma \left( -\frac{4}{n+3} \right) = -\frac{4}{4} - \dots - \frac{4}{n+1} - \frac{4}{n+2} - \frac{4}{n+3}$$

Adding the last three series, we have

$$S_n = \left( -\frac{1}{2} + \frac{4}{3} + \frac{1}{n+2} - \frac{4}{n+3} \right) = \frac{5}{6} + \frac{1}{n+2} - \frac{4}{n+3}$$

If  $\lim. n = \infty$ , then  $S_\infty = \frac{5}{6}$ .

**The Differential Method.**

**612.** If the first term of any series be taken from the second, the second from the third, the third from the fourth, and so on, a new series will be formed which is

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called the *first order of differences*. If the first order of differences be treated in the same manner as the original series, a *second order of differences* will be formed, and so on.

Thus, if we let  $a, b, c, d, e, \dots$  be any series, then  $b - a, c - b, d - c, e - d, \dots$  will be the first order of differences ;

$c - 2b + a, d - 2c + b, e - 2d + c, \dots$  the second order of differences ;

$d - 3c + 3b - a, e - 3d + 3c - b, \dots$  the third order of differences, and so on.

**613.** If we let  $a_1, b_1, c_1, d_1, \dots$  represent the first order of differences ;

$a_2, b_2, c_2, d_2, \dots$  the second order of differences ;

$a_3, b_3, c_3, d_3, \dots$  the third order of differences ;

and so on, we have the following scheme :

Series :  $a, b, c, d, e, \dots$

1st Differences :  $a_1, b_1, c_1, d_1, \dots$

2d Differences :  $a_2, b_2, c_2, \dots$

3d Differences :  $a_3, b_3, \dots$

4th Differences :  $a_4, \dots$ , and so on.

$a_1, a_2, a_3, a_4, \dots$  are the *first terms* of the successive order of differences.

**614. Problem 1.** To find the  $(n + 1)$ th term of a series.

**Solution:** Take the series  $a, b, c, d, e, \dots$  then from the above scheme,

$$1. \ b - a = a_1, \text{ whence } b = a + a_1 \quad (1)$$

$$b_1 - a_1 = a_2, \quad " \quad b_1 = a_1 + a_2 \quad (2)$$

$$b_2 - a_2 = a_3, \quad " \quad b_2 = a_2 + a_3 \quad (3)$$

$$b_3 - a_3 = a_4, \quad " \quad b_3 = a_3 + a_4 \quad (4)$$

$$2. \ c = b + b_1 = a + 2a_1 + a_2, \text{ from (1 and 2)} \quad (5)$$

$$c_1 = b_1 + b_2 = a_1 + 2a_2 + a_3, \text{ from (2 and 3)} \quad (6)$$

$$c_2 = b_2 + b_3 = a_2 + 2a_3 + a_4, \text{ from (3 and 4)} \quad (7)$$

$$3. d = c + c_1 = a + 3a_1 + 3a_2 + a_3, \text{ from (5 and 6)} \quad (8)$$

$$d_1 = c_1 + c_2 = a_1 + 3a_2 + 3a_3 + a_4, \text{ from (6 and 7)} \quad (9)$$

$$4. e = d + d_1 = a + 4a_1 + 6a_2 + 4a_3 + a_4, \text{ from (8 and 9)} \quad (10)$$

Now, since

$$b = a + a_1,$$

$$c = a + 2a_1 + a_2,$$

$$d = a + 3a_1 + 3a_2 + a_3,$$

$$e = a + 4a_1 + 6a_2 + 4a_3 + a_4,$$

it will be observed that the coefficients of any term are the same as the coefficients of a power of a binomial, whose index is one less than the number of the term. Hence, the

$$(n+1)\text{th term} = a + na_1 + \frac{n(n-1)}{2}a_2 + \frac{n(n-1)(n-2)}{3}a_3 + \dots \quad [A]$$

**615. Cor.**—The  $n$ th term =

$$a + (n-1)a_1 + \frac{(n-1)(n-2)}{2}a_2 + \dots \quad [B]$$

**Illustrative Example.**—Find the 7th term, also the  $n$ th term, of the series : 1, 3, 6, 10, ....

**Solution :** 1st differences = 2, 3, 4, ....

$$2d \quad " \quad = 1, 1, \dots$$

$$3d \quad " \quad = 0, \dots$$

$$\therefore \text{1st. } n = 7, a = 1, a_1 = 2, a_2 = 1, a_3 = 0$$

Substitute these values in [B],

$$7\text{th term} = 1 + 6 \times 2 + \frac{6 \times 5}{2} \times 1 = 28$$

$$2d. \text{ Put } n = n, a = 1, a_1 = 2, a_2 = 1, \text{ and } a_3 = 0,$$

$$\text{then } n\text{th term} = 1 + (n-1) \times 2 + \frac{(n-1)(n-2)}{2} \times 1 = \frac{n(n-1)}{2}.$$

**616. Problem 2.** To find the sum of  $n$  terms of a series.

**Solution :** Let it be required to find the sum of  $n$  terms of the series  $a, b, c, d, e, \dots$

Assume the series  $0, a, a+b, a+b+c, a+b+c+d, \dots$ , then,

1st. The first order of differences of the assumed series is the given series.

2d. The second, third, and  $n$ th orders of differences of the assumed series are the first, second, and  $(n-1)$ th orders of differences of the given series.

3d. The  $(n+1)$ th term of the assumed series is the sum of  $n$  terms of the given series.

Hence, if in formula [A] we put 0 for  $a$ ,  $a$  for  $a_1$ ,  $a_1$  for  $a_2$ ,

etc., we shall have the sum of  $n$  terms of the given series. Doing so, we shall have

$$S_n = na + \frac{n(n-1)}{2}a_1 + \frac{n(n-1)(n-2)}{3}a_2 + \dots \quad [C]$$

**Note.**—This method is applicable only when some finite order of differences will reduce to zero.

**Example.**—Find the sum of 10 terms of the series :  
 $1 + 4 + 10 + 20 + 35 + \dots$

**Solution:** First Differences = 3, 6, 10, 15, ....

Second Differences = 3, 4, 5, ....

Third Differences = 1, 1, ....

Fourth Differences = 0, ....

$\therefore$  Put  $n = 10$ ,  $a = 1$ ,  $a_1 = 3$ ,  $a_2 = 3$ ,  $a_3 = 1$ , and  $a_4 = 0$  in formula [C]; then,

$$S_n = 10 + \frac{10 \times 9}{2} \times 3 + \frac{10 \times 9 \times 8}{2 \times 3} \times 3 + \frac{10 \times 9 \times 8 \times 7}{2 \times 3 \times 4} \times 1 = 715$$

**617. Problem 3.** To interpolate terms at regular intervals between the terms of a given series.

Formula [B] may sometimes be used with advantage to interpolate terms at regular intervals between the terms of a given series.

**Illustrations.**—1. Given

$$(651)^3 = 423801, \quad (653)^3 = 426409,$$

$$(655)^3 = 429025, \text{ and } (657)^3 = 431645,$$

to find the value of  $(652)^3$ ,  $(654)^3$ , and  $(656)^3$ .

**Solution:** Series = 423801, 426409, 429025, 431649

First Differences = 2608, 2616, 2624

Second Differences = 8, 8

Third Differences = 0

Take formula  $a_n = a + (n-1)a_1 + \frac{(n-1)(n-2)}{2}a_2 + \dots$

Put  $a = 423801$ ,  $a_1 = 2608$ ,  $a_2 = 8$ , and  $n = 1\frac{1}{2}$ ,  $2\frac{1}{2}$ , and  $3\frac{1}{2}$  successively; then,

$$1. (652)^3 = 423801 + \frac{1}{2} \times 2608 - \frac{1}{8} \times 8 = 425104$$

$$2. (654)^3 = 423801 + \frac{3}{2} \times 2608 + \frac{3}{8} \times 8 = 427716$$

$$3. (656)^3 = 423801 + \frac{5}{2} \times 2608 + \frac{15}{8} \times 8 = 430336$$

$$\begin{aligned} 2. \quad \sqrt[3]{651} &= 8.666831, \quad \sqrt[3]{652} = 8.671266, \\ \sqrt[3]{653} &= 8.675697, \quad \sqrt[3]{654} = 8.680123, \text{ and} \\ \sqrt[3]{655} &= 8.684545, \text{ find } \sqrt[3]{653.75}. \end{aligned}$$

**Solution:** Here  $a = 8.666831$ ,  $a_1 = .004435$ ,  $a_2 = -0.000004$ ,  
 $a_3 = -0.000001$ ,  $a_4 = 0$ , and  $n = 3\frac{3}{4}$ . Substitute these values in  
 $a_n = a + (n-1)a_1 + \frac{(n-1)(n-2)}{2}a_2 + \frac{(n-1)(n-2)(n-3)}{6}a_3$ ; then,  
 $\sqrt[3]{653.75} = 8.666831 + \frac{11}{4} \times .004435 - \frac{77}{82} \times .000004 - \frac{231}{884} \times .000001$   
 $= 8.666831 + .012196 - .000009 - .000001 = 8.679017.$

# EXERCISE 92.

Find the  $n$ th term and the sum of  $n$  terms of the following series :

1.  $2 + 6 + 12 + 20 + \dots$
2.  $1 + 9 + 25 + 49 + \dots$
3.  $1 + 3 + 6 + 10 + \dots$
4.  $3 + 8 + 15 + 24 + \dots$
5.  $1 + 4 + 9 + 16 + \dots$
6.  $2 + 12 + 30 + 56 + \dots$
7.  $6 + 24 + 60 + 120 + 210 + \dots$
8.  $45 + 120 + 231 + 384 + 585 + \dots$
9.  $1.3.2^2 + 2.4.3^2 + 3.5.4^2 + \dots$
10.  $3.5.7 + 5.7.9 + 7.9.11 + \dots$

Sum to  $n$  terms and to infinity :

11.  $\frac{1}{1 \times 4} + \frac{1}{4 \times 7} + \frac{1}{7 \times 10} + \dots$
12.  $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \frac{1}{4.7} + \dots$
13.  $\frac{1}{2.4.6} + \frac{1}{4.6.8} + \frac{1}{6.8.10} + \dots$
14.  $\frac{1}{1.3.7} + \frac{1}{3.5.9} + \frac{1}{5.7.11} + \frac{1}{7.9.13} + \dots$
15.  $\frac{4}{1.2.3} + \frac{5}{2.3.4} + \frac{6}{3.4.5} + \frac{7}{4.5.6}$

Find the value of :

16.  $\Sigma \{n^2(3n-2)\}$       17.  $\Sigma \left\{ \frac{n}{6}(n+1)(n+2) \right\}$   
 18.  $\Sigma \left( \frac{1}{n(n+1)} \right)$       19.  $\Sigma \left( \frac{n^4+n^2+1}{n^4+n} \right)$   
 20. The  $\log. 950 = 2.977724$ ,  $\log. 951 = 2.978181$ ,  
 $\log. 952 = 2.978637$ ,  $\log. 953 = 2.979093$ ,  
 find  $\log. 952.375$ .

### Reversion of Series.

**618.** If  $y$  is a serial function of  $x$ , then may  $x$  be developed into a serial function of  $y$ , and the process is called *Reversion of Series*.

**Example.**—Revert  $y = a + bx + cx^2 + dx^3 + ex^4 + \dots$  into a serial function of  $y$ .

**Solution:** Put the series in the form

$$y-a = bx + cx^2 + dx^3 + ex^4 + \dots$$

$$\text{Put } x = A(y-a) + B(y-a)^2 + C(y-a)^3 + D(y-a)^4 + \dots$$

$$\text{Now, } bx = bA(y-a) + bB(y-a)^2 + bC(y-a)^3 + bD(y-a)^4 + \dots$$

$$cx^2 = cA^2(y-a)^2 + 2cAB(y-a)^3 + (cB^2 + 2cAC)(y-a)^4 + \dots$$

$$dx^3 = dA^3(y-a)^3 + 3dA^2B(y-a)^4 + \dots$$

$$ex^4 = eA^4(y-a)^4 + \dots$$

$$\begin{aligned} \therefore y-a &= bA(y-a) + (bB+cA^2)(y-a)^2 \\ &\quad + (bC+2cAB+dA^3)(y-a)^3 \\ &\quad + (bD+cB^2+2cAC+3dA^2B+eA^4)(y-a)^4 + \dots \end{aligned}$$

Equating the coefficients,

$$1. \ bA = 1; \text{ whence } A = \frac{1}{b}$$

$$2. \ bB + cA^2 = 0; \text{ whence } B = -\frac{c}{b^3}$$

$$3. \ bC + 2cAB + dA^3 = 0; \text{ whence } C = \frac{2c^2 - bd}{b^5}$$

$$4. \ bD + cB^2 + 2cAC + 3dA^2B + eA^4 = 0; \\ \text{whence } D = -\frac{b^2e - 5bcd + 5c^3}{b^7}$$

$$\begin{aligned} \therefore x &= \frac{1}{b}(y-a) - \frac{c}{b^3}(y-a)^2 + \frac{2c^2 - bd}{b^5}(y-a)^3 \\ &\quad - \frac{b^2e - 5bcd + 5c^3}{b^7}(y-a)^4 + \dots \end{aligned}$$

**619. Cor.**—If  $a = 0$ , then

$$x = \frac{1}{b}y - \frac{c}{b^3}y^2 + \frac{2c^2 - bd}{b^5}y^3 - \frac{b^2e - 5bcd + 5c^3}{b^7}y^4 + \dots$$

#### EXERCISE 98.

Revert the following serial functions of  $x$  into serial functions of  $y$ :

1.  $y = x + x^2 + x^3 + x^4 + \dots$

2.  $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

3.  $y = x + 2x^2 + 5x^3 + 14x^4 + \dots$

4.  $y = x + x^3 + 2x^5 + 5x^7 + \dots$

**Suggestion.**—Let  $x = Ay + By^2 + Cy^3 + Dy^4 + \dots$

5.  $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$

6.  $y = 1 + x - 2x^2 + x^3$

7. Find one value of  $x$  in the equation  $x^3 + 4x^2 + 5x = 1$

**Suggestion.**—

Put  $1 = y$ , and assume  $x = Ay + By^2 + Cy^3 + Dy^4 + \dots$

8. Find one value of  $x$  in  $x^3 + 100x = 1$

#### Recurring Series.

**620.** A series in the ascending powers of  $x$ , in which each term, after one or more fixed terms, is  $px$  times the preceding term, or  $px$  times the preceding  $+qx^2$  times the next preceding term, or  $px$  times the preceding  $+qx^2$  times the next preceding  $+rx^3$  times the next preceding term, or and so on, is a *Recurring Series*.

**621.** A recurring series is of the *first*, *second*, or *nth* order, accordingly as each term, after the law begins, is derived from *one*, *two*, or *n* preceding terms.

**622.** The forms  $px$ ,  $px + qx^2$ ,  $px + qx^2 + rx^3$ , and so on, are called the *order scales of the series*; and  $p$ ,  $p + q$ ,  $p + q + r$ , and so on, the *order scales of the coefficients*.

**Illustrations.**—If we put  $p = 3$ ,  $q = 4$ , and  $r = -2$ , then

1.  $2 + 6x + 18x^2 + 54x^3 + \dots$  is a series of the first order.

2.  $2 + 6x + 26x^2 + 102x^3 + \dots$  is a series of the second order.

3.  $2 + 6x + 26x^2 + 98x^3 + 386x^4 + \dots$  is a series of the third order.

**623. Problem 1. To determine the scale of coefficients.**

1. Let  $a + bx + cx^2 + dx^3 + \dots$  be a recurring series of the first order.

Then,  $bx = apx$ ;  $cx^2 = pbx^2$ ;  $dx^3 = pcx^3$ ; and so on.

$$\therefore p = \frac{b}{a}; \quad p = \frac{c}{b}; \quad p = \frac{d}{c}; \text{ and so on.}$$

2. Let  $a + bx + cx^2 + dx^3 + \dots$  be a series of the second order.

Then,  $aqx^2 + bpx^2 = cx^2$  (1);  $bqx^3 + cpx^3 = dx^3$  (2);  
whence,  $aq + bp = c$  (3);  $bq + cp = d$  (4).

Then, by elimination,

$$p = \frac{bc - ad}{b^2 - ac} \text{ and } q = \frac{bd - c^2}{b^2 - ac}$$

3. Let  $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \dots$  be a recurring series of the third order. Then,

$$1. ar + bq + cp = d \qquad 2. br + cq + dp = e$$

$$3. cr + dq + ep = f$$

By elimination the values of  $p$ ,  $q$  and  $r$  may be found. In the same manner the scale of coefficients of a recurring series of any order may be found.

**624. Scholium.**—In order that the scale of coefficients of any recurring series may be found, there must be given at least *twice* as many terms of the series as there are terms in the scale. In the exercises concluding this subject just twice as many terms of each series will be given as are contained in the scale of the series. This will enable the student to determine at a glance the order of the series.

When this law of notation is not followed, as when the  $n$ th term of a series only is given, it is usually best to expand the series and make trial for a scale of two terms, and, if the results thus obtained will not satisfy the series, then make trial for a scale of three terms, and so on until the proper scale is determined.

**Example.**—Find the scale of coefficients in the series  $1 + 2x + 3x^2 + 4x^3 + \dots$  and expand the series.

**Solution:** This is a series of the second order, since four terms are given. Hence,

$$1. \quad q + 2p = 3$$

$$2. \quad 2q + 3p = 4$$

whence,  $p = 2$  and  $q = -1$ , and the scale is  $2 - 1$ . The series is  $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots nx^{n-1}$ .

**625. Problem 2. To find the sum of  $n$  terms of a recurring series.**

The method of finding the sum of a recurring series is the same, whatever be the scale of the series. For the sake of simplicity we will here assume a series of the second order, whose scale is  $p + q$ , for illustration.

Let  $a_0 + a_1x + a_2x^2 + \dots a_{n-1}x^{n-1}$  be a series of the second order. Then,

$$\begin{aligned} S_n &= a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} \\ -px \, S_n &= -pa_0x - pa_1x^2 - \dots - pa_{n-2}x^{n-1} \\ &\quad - pa_{n-1}x^n \\ -qx^2 \, S_n &= -qa_0x^2 - \dots - qa_{n-3}x^{n-1} \\ &\quad - qa_{n-2}x^n - qa_{n-1}x^{n+1} \end{aligned}$$

Adding and remembering that the sum of the coefficients of each term from the third to the  $n$ th inclusive is zero,

$$(1 - px - qx^2) S_n = a_0 + (a_1 - pa_0)x - (pa_{n-1} + qa_{n-2})x^n - qa_{n-1}x^{n+1};$$

whence, 
$$S_n = \frac{a_0 + (a_1 - pa_0)x}{1 - px - qx^2} - \frac{(pa_{n-1} + qa_{n-2})x^n + qa_{n-1}x^{n+1}}{1 - px - qx^2}.$$

**626. Cor.**—If  $x < 1$  and  $\lim. n = \infty$ , then

$$S_\infty = \frac{a_0 + (a_1 - pa_0)x}{1 - px - qx^2}.$$

**Example.**—Sum  $1 + 3x + 5x^2 + 18x^3 + 48x^4 + 145x^5 + \dots$  to 7 terms and to infinity, when  $x < 1$ .

**Solution:** 1.  $5p + 3q + r = 18$       2.  $18p + 5q + 3r = 48$   
3.  $48p + 18q + 5r = 145$

$\therefore p = 2, q = 3, \text{ and } r = -1.$

$$\begin{aligned} S_7 &= 1 + 3x + 5x^2 + 18x^3 + 48x^4 + 145x^5 + 416x^6 \\ - 2x S_7 &= -2x - 6x^2 - 10x^3 - 36x^4 - 96x^5 - 290x^6 - 832x^7 \\ - 3x^2 S_7 &= -3x^2 - 9x^3 - 15x^4 - 54x^5 - 144x^6 - 435x^7 - 1248x^8 \\ + x^3 S_7 &= \phantom{-} + x^3 + 3x^4 + 5x^5 + 18x^6 + 48x^7 + 145x^8 + 416x^9 \end{aligned}$$

$$\therefore (1 - 2x - 3x^2 + x^3) S_7 = 1 + x - 4x^2 - 1219x^7 - 1103x^8 + 416x^9$$

$$S_7 = \frac{1 + x - 4x^2 - 1219x^7 - 1103x^8 + 416x^9}{1 - 2x - 3x^2 + x^3}$$

$$S_\infty = \frac{1 + x - 4x^2}{1 - 2x - 3x^2 + x^3}$$

**627.** If  $S_\infty$  is developed into a series by the method of indeterminate coefficients, the original series may be reproduced to any number of terms desired. Therefore,  $S_\infty$  is often called the *generatrix* of the series.

**628.** If the *generatrix* can be decomposed into partial fractions, the general, or  $n$ th term, of the series may easily be obtained, and hence, too, the sum of  $n$  terms.

**Illustration.**—Let it be required to find the  $n$ th term and the sum of  $n$  terms of the series

$$1 + 5x + 13x^2 + 41x^3 + \dots$$

**Solution:** It may readily be determined that  $p = 2$ ,  $q = 3$ , and the generatrix  $= \frac{1+3x}{1-2x-3x^2} = -\frac{1}{2} \cdot \frac{1}{1+x} + \frac{3}{2} \cdot \frac{1}{1-3x}$ .

$$\text{Now, } -\frac{1}{2} \cdot \frac{1}{1+x} = -\frac{1}{2} + \frac{1}{2}x - \frac{1}{2}x^2 + \dots$$

$$(-1)^n \cdot \frac{x^{n-1}}{2} = \frac{(-1)^n x^n - 1}{2x + 2} \quad [493, B];$$

$$\text{and } \frac{3}{2} \cdot \frac{1}{1-3x} = \frac{3}{2} + \frac{9}{2}x + \frac{27}{2}x^2 + \dots$$

$$\frac{3^n}{2} \cdot x^{n-1} = \frac{3^{n+1}x^n - 3}{6x - 2} \quad [493, B].$$

$$\therefore S_n = \frac{(-1)^n x^n - 1}{2x + 2} + \frac{3^{n+1}x^n - 3}{6x - 2};$$

$$n\text{th term} = (-1)^n \cdot \frac{x^{n-1}}{2} + \frac{3^n}{2} \cdot x^{n-1}.$$

#### EXERCISE 94.

Sum to infinity :

1.  $1 + 2x + 5x^2 + 13x^3 + \dots$
2.  $1 + 3x + 8x^2 + 22x^3 + \dots$
3.  $2 + 2x + 4x^2 + 14x^3 + \dots$
4.  $3 + 2x - 7x^2 - 38x^3 - \dots$
5.  $1 + 2x + 3x^2 + 11x^3 + 35x^4 + 121x^5 + \dots$
6.  $1 - 3x + 5x^2 + 5x^3 + 13x^4 + 61x^5 + \dots$
7. Sum  $1 + 2x + 9x^2 + 33x^3 + \dots$  to 6 terms.
8. Sum  $1 - 2x - 7x^2 - 8x^3 + \dots$  to 7 terms.
9. Sum  $2 - x + 6x^2 - 14x^3 + \dots$  to 8 terms.
10. Sum  $1 - 2x + 3x^2 + 6x^3 + x^4 - 2x^5 + \dots$  to 9 terms.

Find the  $n$ th term and the sum of  $n$  terms of

11.  $1 + 2x - 8x^2 + 20x^3 - \dots$
12.  $1 + 5x + 9x^2 + 13x^3 + \dots$

**Decomposition of Rational Fractional Functions.**

**629.** To decompose a rational fraction is to find two or more other fractions whose sum equals the rational fraction.

**630.** It will be only necessary to show how to decompose proper fractions, as all improper fractions may be reduced to mixed numbers, which process will already lead to a partial decomposition.

**Principles.**

**631.** 1. *Any rational fraction of the form of*

$$\frac{P}{(x+a)(x+b)\dots(x+n)} \text{ may be decomposed so that}$$

$$\frac{P}{(x+a)(x+b)\dots(x+n)} = \frac{A}{x+a} + \frac{B}{x+b} + \dots + \frac{N}{x+n}$$

**Illustration.**—Put

$$\frac{3x^2 + 14x - 29}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3} \quad (\text{A})$$

Clear of fractions and arrange the terms according to the descending powers of  $x$ ,

$$3x^2 + 14x - 29 = (A + B + C)x^2 - (A + 4B - C)x - (6A - 3B + 2C).$$

Equating the coefficients [566], we have

$$(1) A + B + C = 3 \qquad (2) A + 4B - C = -14$$

$$(3) 6A - 3B + 2C = 29$$

Finding the values of  $A$ ,  $B$ , and  $C$  by elimination, and substituting them in (A), we obtain

$$\frac{3x^2 + 14x - 29}{(x-1)(x+2)(x-3)} = \frac{2}{x-1} - \frac{3}{x+2} + \frac{4}{x-3}.$$

**632.** In a similar manner it may be shown that

$$\frac{P}{(ax+b)(cx+d)\dots(mx+n)} =$$

$$\frac{A}{ax+b} + \frac{B}{cx+d} + \dots + \frac{M}{mx+n}.$$

**633.** 2. Any rational fraction of the form of

$$\frac{P}{(x^2 + ax + b)(x^2 + cx + d) \dots (x^2 + mx + n)}$$

may be decomposed so that

$$\frac{P}{(x^2 + ax + b)(x^2 + cx + d) \dots (x^2 + mx + n)} = \frac{Ax + B}{x^2 + ax + b} + \frac{Cx + D}{x^2 + cx + d} + \dots + \frac{Mx + N}{x^2 + mx + n}.$$

**Illustration.**—Put  $\frac{4x^4 - 8x^3 - 5x^2 - 15x + 3}{(x^2 + x + 1)(x^2 - x + 1)(x^2 + x + 2)}$

$$= \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{x^2 - x + 1} + \frac{Ex + F}{x^2 + x + 2} \quad (A)$$

Clear of fractions and arrange the terms according to the descending powers of  $x$ ,

$$\begin{aligned} 4x^4 - 8x^3 - 5x^2 - 15x + 3 &= \\ (A + C + E)x^5 + (B + 2C + D + F)x^4 \\ + (2A + 4C + 2D + E)x^3 + (-A + 2B + 3C + 4D + F)x^2 \\ + (2A - B + 2C + 3D + E)x + (2B + 2D + F) \end{aligned} \quad (1)$$

Equating the coefficients [566],

- (1)  $A + C + E = 0$
- (2)  $B + 2C + D + F = 4$
- (3)  $2A + 4C + 2D + E = -8$
- (4)  $A - 2B - 3C - 4D - F = 5$
- (5)  $2A - B + 2C + 3D + E = -15$
- (6)  $2B + 2D + F = 3$

Finding by elimination the values of  $A, B, C, D, E$ , and  $F$ , and substituting them in (A), we have

$$\frac{4x^4 - 8x^3 - 5x^2 - 15x + 3}{(x^2 + x + 1)(x^2 - x + 1)(x^2 + x + 2)} = \frac{3}{x^2 + x + 1} - \frac{4}{x^2 - x + 1} + \frac{5}{x^2 + x + 2}.$$

**634.** In a similar manner it may be shown that

$$\frac{P}{(ax^2 + bx + c)(dx^2 + ex + f) \dots (mx^2 + nx + p)} = \frac{Ax + B}{ax^2 + bx + c} + \frac{Cx + D}{dx^2 + ex + f} + \dots + \frac{Mx + N}{mx^2 + nx + p}.$$

**635. 3.** A rational fraction of the form of  $\frac{P}{(x+a)^n}$  may be decomposed so that

$$\frac{P}{(x+a)^n} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \dots + \frac{N}{(x+a)^n}.$$

**Illustration.**—Put  $\frac{3x^3 - 14x^2 + 19x - 5}{(x-2)^4}$

$$= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^4} \quad (\text{A})$$

Clear of fractions and arrange the terms according to the descending powers of  $x$ ; then,

$$3x^3 - 14x^2 + 19x - 5 = Ax^3 - (6A - B)x^2 + (12A - 4B + C)x - (8A - 4B + 2C - D).$$

Equating the coefficients, we have

1.  $A = 3$

2.  $6A - B = 14$

3.  $12A - 4B + C = 19$

4.  $8A - 4B + 2C - D = 5$

Solving these equations, and substituting the values of  $A$ ,  $B$ ,  $C$ , and  $D$  in (A), we obtain

$$\frac{3x^3 - 14x^2 + 19x - 5}{(x-2)^4} = \frac{3}{x-2} + \frac{4}{(x-2)^2} - \frac{1}{(x-2)^3} + \frac{1}{(x-2)^4}.$$

**636.** In a similar manner it may be shown that

1.  $\frac{P}{(ax+b)^n} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \dots + \frac{N}{(ax+b)^n}$

2.  $\frac{P}{(x^2+ax+b)^n} = \frac{Ax+B}{x^2+ax+b} + \frac{Cx+D}{(x^2+ax+b)^2} + \dots + \frac{Mx+N}{(x^2+ax+b)^n}$

3.  $\frac{P}{(ax^2+bx+c)^n} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \dots + \frac{Mx+N}{(ax^2+bx+c)^n}$

4. Any rational fraction whose denominator may be resolved into linear and quadratic factors may be decomposed by a combination of the above methods.

**Illustration.—**

To decompose  $\frac{P}{(x-a)(x-b)^n(x^2+px+q)^n}$ , put

$$\frac{P}{(x-a)(x-b)^n(x^2+px+q)^n} = \frac{A}{x-a} + \frac{B}{x-b} + \dots$$

$$+ \frac{M}{(x-b)^n} + \frac{Px+Q}{x^2+px+q} + \dots + \frac{P'x+Q'}{(x^2+px+q)^n}$$

**EXERCISE 95.**

Decompose into partial fractions :

1.  $\frac{5x+2}{x^2-4}$
2.  $\frac{2x-5}{4x^2-1} = \frac{A}{2x+1} + \frac{B}{2x-1}$
3.  $\frac{6x^2+10x+2}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$
4.  $\frac{3x-2}{(x+1)^2}$
5.  $\frac{x^2-x+1}{(x+3)^3}$
6.  $\frac{3x^2-14x+25}{(x-3)(x^2-x-6)}$
7.  $\frac{(a-b)x+2ab}{ab+(a-b)x-x^2}$
8.  $\frac{p^2x+pq+q}{(px+q)^2}$
9.  $\frac{5x^2+x+5}{x^4+x^2+1}$
10.  $\frac{2}{x(1-4x^2)}$
11.  $\frac{3x^3-8x^2+10}{(x-1)^4}$
12.  $\frac{2x+1}{(x-1)(x^2+1)}$
13.  $\frac{x^2-1}{x^2-5x+6}$
14.  $\frac{2x-3}{(x-1)(x^2+1)^2}$
15.  $\frac{x^2+x+1}{(x+1)(x^2+1)}$
16.  $\frac{1}{x^3+x^7-x^4-x^3}$
17.  $\frac{2x^2-11x+5}{(x-3)(x^2+2x-5)}$
18.  $\frac{5x^3+6x^2+5x}{(x^2-1)(x+1)^3}$
19.  $\frac{4x^3-x^2-3x-2}{x^2(x+1)^2}$
20.  $\frac{1}{x^5-1}$
21.  $\frac{1+x+x^2}{1-x-x^4+x^5}$
22.  $\frac{x^2+4}{(x+1)^2(x-2)(x+3)}$

## CHAPTER X.

### COMPLEX NUMBERS.

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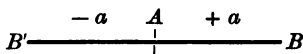
#### Graphical Treatment.

**637.** If a straight line of any assumed length be taken to represent the number *one*, then will a straight line twice as long represent the number *two*, one three times as long the number *three*, and so on. Thus, we see that any number may be represented by a line.

**638.** A line representing a number is called a *graph number*, or a *vector*. The point where a vector is supposed to begin is called the *origin*, and the point where it ends the *extremity*.

**639.** A vector is fully determined when both its length and direction are given. In a system of graphical representation of numbers, a vector running *rightward* from its origin represents a positive number and is *positive*, and one running *leftward* from its origin represents a negative number and is *negative*.

**640.** If the vector  $+a$  be made to revolve about its origin  $A$ , through an angle of  $180^\circ$ , or  $\pi$ , it will become the vector  $-a$ , or will be multiplied by  $-1$ ; and if the vector  $-a$  be revolved about its origin  $A$  through an angle of  $180^\circ$ , or  $\pi$ , it will become the vector  $+a$ , or will be multiplied by  $-1$ .



Therefore,

1. *Revolving a vector through an angle of  $180^\circ$ , or  $\pi$ , is equivalent to multiplying it by  $-1$ .*

2. *Revolving a vector about its origin through an angle of  $360^\circ$ , or  $2\pi$ , is equivalent to multiplying it twice by  $-1$ , or once by  $+1$ , which does not affect its length or direction.*

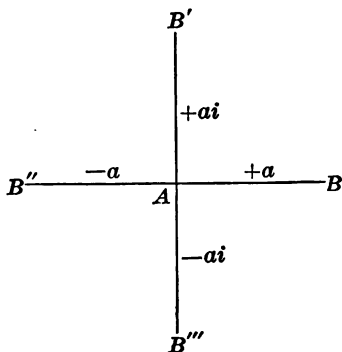
3. *Since  $-1 = \sqrt{-1} \times \sqrt{-1}$ , revolving a vector about its origin through an angle of  $90^\circ$ , or  $\frac{1}{2}\pi$ , is equivalent to multiplying it by  $\sqrt{-1}$ , or  $i$  [v. P. I. 299].*

**641.** Motion about the origin of a vector in the direction the hands of a clock go is considered negative, and counter-motion positive. The factor  $+i$  may, therefore, be taken to represent circular motion about an origin through an angle of  $90^\circ$ , or  $\frac{1}{2}\pi$ , *counter-clock-wise*; and  $-i$  through an angle of  $90^\circ$ , or  $\frac{1}{2}\pi$ , *clock-wise*.

**642.** Since  $(+i) \times (+i) \times (+i)$ , or  $(+i)^3 = -i$ , the factor symbol  $-i$  may also denote circular motion about an origin through an angle of  $270^\circ$ , or  $\frac{3}{2}\pi$ , in a positive direction.

**643.** Since  $(\pm i) \times (\pm i)$ , or  $(\pm i)^2 = -1$ , the factor symbol  $-1$  may denote circular motion about an origin through an angle of  $180^\circ$ , or  $\pi$ , in either direction.

**Illustrations.** — 1. The vector  $+a$  multiplied by  $+i = AB$  revolved about  $A$  in the positive direction through an angle of  $90^\circ = AB'$ .



x

2. The vector  $+a$  multiplied by  $-1 = AB$  revolved about  $A$  in the positive or the negative direction through an angle of  $180^\circ = AB''$ .

3. The vector  $+a$  multiplied by  $-i = AB$  revolved about  $A$  in the negative direction through an angle of  $90^\circ$ , or in the positive direction through an angle of  $270^\circ = AB'''$ .

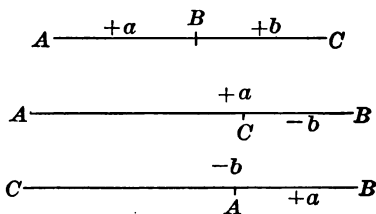
4. In a similar manner it may be shown that  $(-a) \times (+i) = AB'''$ ;  $(-a) \times (-1) = AB$ , and  $(-a) \times (-i) = AB'$ .

**644.** From what has been explained thus far it will be seen that, if a vector  $a$  units long running *rightward* from its origin represents  $+a$ , running *leftward* from its origin, it will represent  $-a$ ; running *upward* from its origin,  $+ai$ ; and running *downward*,  $-ai$ .

**645.** One vector is added to another by placing its origin to the extremity of the other and giving it the direction indicated by its factor symbol. The vector of their sum is the length and direction of the line joining the origin of the vector to which addition is made with the extremity of the vector added.

**Illustrations.—**

1. The vector  $BC$  ( $= +b$ ) added to the vector  $AB$  ( $= +a$ ) gives the vector  $AC$  ( $= +(a+b)$ ).



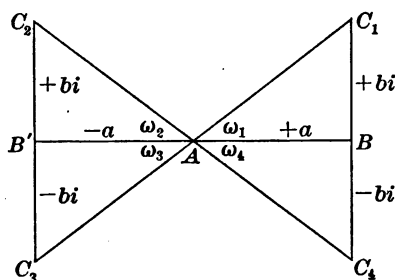
2. The vector  $BC$  ( $= -b$ ) added to the vector  $AB$  ( $= +a$ ) gives the vector  $AC$  ( $= +(a-b)$ ), when  $a > b$ .

3. The vector  $BC$  ( $= -b$ ) added to the vector  $AB$  ( $= +a$ ) gives the vector  $AC$  ( $= -(b-a)$ ), when  $a < b$ .

**Note.**—To subtract a vector is to add the vector with contrary sign.

## Representation of Complex Numbers.

**646.** Let it be required to represent graphically the complex numbers  $a + bi$ ,  $a - bi$ ,  $-a + bi$ , and  $-a - bi$ .



1.  $a + bi = +a + (+bi)$ .  
 $\therefore$  Vector  $(a + bi) = AC_1$  [645].
2.  $a - bi = +a + (-bi)$ .  
 $\therefore$  Vector  $(a - bi) = AC_4$  [645].
3.  $-a + bi = -a + (+bi)$ .  
 $\therefore$  Vector  $(-a + bi) = AC_2$  [645].
4.  $-a - bi = -a + (-bi)$ .  
 $\therefore$  Vector  $(-a - bi) = AC_3$  [645].

**647.** The vectors of the two terms of a complex number are called the *components* of the vector of the number.

Thus,  $AB$  and  $BC_1$  are the components of  $AC_1$ .

**648.** The length of the vector of a complex number is called the *modulus* of the vector, or the *modulus* of the complex number, and is equal to the square root of the sum of the squares of the lengths of the components.

Thus,  $\text{mod. } (a + bi) = \text{mod. } (a - bi) = \text{mod. } (-a + bi) = \text{mod. } (-a - bi) = \sqrt{a^2 + b^2}$ .

**649.** The direction of the vector of a complex number is determined by the angle which the vector makes with

its horizontal component, which angle is called the *amplitude* of the vector.

Thus, the amplitude of the vector  $AC_1$  is the angle  $C_1AB$ , which for distinction will be represented by  $\omega_1$ ; the amplitude of  $AC_2$  is  $C_2AB'$ , represented by  $\omega_2$ ; the amplitude of  $AC_3$  is  $C_3AB'$ , represented by  $\omega_3$ ; and the amplitude of  $AC_4$  is  $C_4AB$ , represented by  $\omega_4$ .

**650.** It is equally accurate, and sometimes more convenient, to define the amplitude of a vector as the angle included between the vector and the vector  $+a$ , measured in a positive direction from the vector  $+a$ .

Thus, the amplitude of  $AC_2$  is  $C_2AB$ . The amplitude of  $AC_3$  is the reflex angle  $BAC_3$ , described by revolving  $AB$  about  $A$  in a positive direction until it coincides with  $AC_3$ . The amplitude of  $AC_4$  is the reflex angle  $BAC_4$ .

**651.** A vector and its components may be constructed from its modulus and amplitude as follows :

1. Draw an indefinite horizontal line, and select some point in this line for the origin of the vector.

2. At the origin, deflect an angle with a protractor equal to the amplitude of the vector, and in the proper position.

3. Lay off from the origin, on the deflected side of the angle, from a scale of equal parts, the modulus of the vector. The vector is then determined.

4. At the extremity of the vector let fall a perpendicular to the horizontal line. This perpendicular and the part of the horizontal line intercepted between the origin and the foot of the perpendicular will be the components of the vector. Their lengths may be obtained by actual measurement on the scale of equal parts.

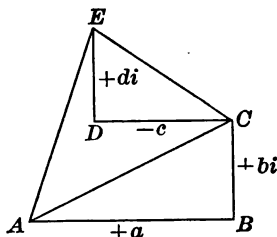
Problems.

**652.** To find the sum of two complex numbers, graphically.

Let it be required to find the sum of  $a + bi$  and  $-c + di$ .

1. Add  $-c + di$  to  $a + bi$ , graphically.

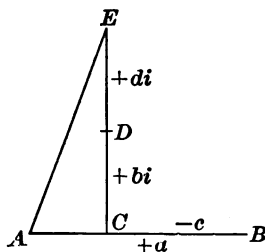
**Solution:** Construct  $a + bi$ . Its vector is  $AC$ . At its extremity,  $C$ , construct  $-c + di$ . Its vector is  $CE$ . Join  $A$  with  $E$ .  $AE$  is the vector of the sum [645].  $\sqrt{(b+d)^2 + (a-c)^2}$  is its modulus, and the angle  $EAB$  its amplitude.



**Proof:** Add  $-c + di$  to  $a + bi$  algebraically, and construct the sum. Thus,

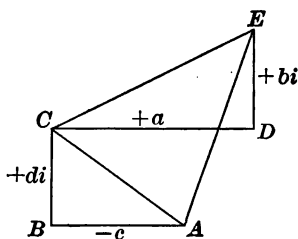
$$(a + bi) + (-c + di) = (a - c) + (b + d)i.$$

Construct  $AB = +a$ ,  $BC = -c$ ; then  $AC = a - c$ . At  $C$  erect  $CE = (b + d)i$ . Join  $A$  with  $E$ . The vector  $AE$  will be identical with  $AE$  in the preceding diagram; therefore, the solution is correct.



2. Add  $a + bi$  to  $-c + di$ .

**Solution:** Construct  $-c + di$ . Its vector is  $AC$ . At its extremity,  $C$ , construct  $+a + bi$ . Its vector is  $CE$ . Join  $A$  with  $E$ . The vector  $AE$  will be identical with the vector  $AE$  in the first case, which proves the commutative law of addition graphically as applied to complex numbers.



**Exercise.**—Find graphically the sum of :

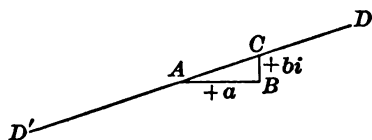
1.  $a + bi$  and  $c + di$
2.  $a - bi$  and  $c - di$
3.  $a + bi$  and  $c - di$
4.  $-a - bi$  and  $-c - di$
5.  $a + bi$  and  $a - bi$
6.  $0 + bi$  and  $0 - di$

**653. To multiply a complex number by a rational number, graphically.**

Let it be required to multiply  $a + bi$  by  $-c$ .

**Solution :**

Construct  $a + bi$ . Prolong its vector  $AC$  to  $D$ , making  $AD = c \times AC$ . Revolve  $AD$  about  $A$  through  $180^\circ$ , then  $AD'$  is the vector of  $-c$  times  $a + bi$ .



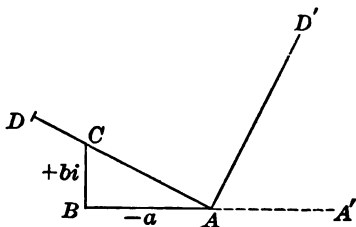
**Exercise.**—Construct the vectors of :

- |                         |                            |
|-------------------------|----------------------------|
| 1. $(a + bi) \times c$  | 4. $(-a - bi) \times c$    |
| 2. $(a - bi) \times c$  | 5. $(a - bi) \times (-c)$  |
| 3. $(-a + bi) \times c$ | 6. $(-a + bi) \times (-c)$ |

**654. To multiply a complex number by a simple imaginary number, graphically.**

Let it be required to multiply  $-a + bi$  by  $-ci$ .

**Solution :** Construct  $-a + bi$ . Prolong its vector  $AC$  to  $D$ , making  $AD = c \times AC$ .  $AD$  is the vector of  $c$  times  $-a + bi$ . Revolve  $AD$  about  $A$  through an angle of  $90^\circ$  clock-wise [641], or, which is the same, draw  $AD' = AD$  and perpendicular to  $AD$ .  $AD'$  is the vector of  $-ci$  times  $-a + bi$ .  $D'A A'$  is its amplitude.



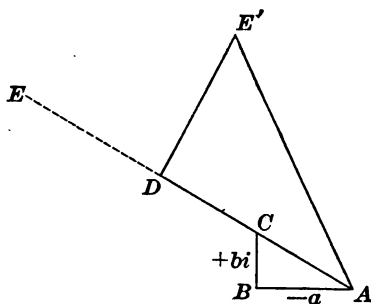
**Exercise.**—Construct the vectors of :

- |                            |                             |
|----------------------------|-----------------------------|
| 1. $(a + bi) \times ci$    | 6. $(a - bi) \times ci$     |
| 2. $(-a - bi) \times ci$   | 7. $(-a - bi) \times (-ci)$ |
| 3. $(a - bi) \times (-ci)$ | 8. $(0 - bi) \times (-ci)$  |
| 4. $(a + bi) \times (-ci)$ | 9. $(0 + bi) \times ci$     |
| 5. $(-a + bi) \times ci$   | 10. $(0 + bi) \times (-ci)$ |

**655. To multiply a complex number by a complex number, graphically.**

Let it be required to multiply  $-a + bi$  by  $c - di$ .

**Solution:** The vector of the sum of  $c$  times  $-a + bi$  and  $-di$  times  $-a + bi$  is required. Construct  $-a + bi$ . Its vector is  $AC$ . Prolong  $AC$  to  $D$ , making  $AD = c$  times  $AC$ ; then  $AD$  is the vector of  $c$  times  $-a + bi$ . Prolong  $AD$  to  $E$ , making  $DE = d$  times  $AC$ , and revolve  $DE$  about  $D$  through an angle of  $90^\circ$  clock-wise, then is  $DE'$  the vector of  $-di$  times  $-a + bi$



$+ bi$  constructed at the extremity of  $AD$ . Join  $A$  and  $E'$ .  $AE'$  is the vector of the sum of  $c$  times  $-a + bi$  and  $-di$  times  $-a + bi$ .

**Exercise.**—Multiply graphically :

- |                         |                           |
|-------------------------|---------------------------|
| 1. $a + bi$ by $c + di$ | 4. $-a + bi$ by $c + di$  |
| 2. $a - bi$ by $c + di$ | 5. $-a - bi$ by $-c + di$ |
| 3. $a - bi$ by $c - di$ | 6. $-a - bi$ by $-c - di$ |

General Principles.

**656. 1.** *The sum, the difference, the product, and the quotient of two complex numbers are, in general, complex numbers.*

For, 1.  $(a + bi) + (c + di) = (a + c) + (b + d)i$ .

2.  $(a + bi) - (c + di) = (a - c) + (b - d)i$ .

3.  $(a + bi)(c + di) = ac + bci + adi - bd$   
 $= (ac - bd) + (bc + ad)i$ .

4.  $\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)}$   
 $= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \left( \frac{bc - ad}{c^2 + d^2} \right)i$ .

**657. 2.** *The sum and the product of two conjugate complex numbers are real.*

For, 1.  $(a + bi) + (a - bi) = 2a$ .

$$2. (a + bi)(a - bi) = a^2 + b^2.$$

*Schottum.*  $a^2 + b^2$  is the square of the modulus of  $\pm a + bi$  and of  $\pm a - bi$ , and is called the **norm** of each. Therefore,

*Cor.*—*The product of two conjugate complex numbers equals their norm.*

---

**658. 3.** *The norm of the product of two complex numbers equals the product of their norms.*

For, norm  $(a + bi)(c + di)$

$$= \text{norm } \{(ac - bd) + (ad + bc)i\}$$

$$= (ac - bd)^2 + (ad + bc)^2 \text{ [657, Sch.]}$$

$$= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$$

$$= (a^2 + b^2) \times (c^2 + d^2)$$

$$= \text{norm } (a + bi) \text{ multiplied by norm } (c + di).$$

*Cor.*—*The modulus of the product of two complex numbers equals the product of their moduli.*

---

**659. 4.** *If  $a + bi = 0$ , then  $a = 0$  and  $b = 0$ .*

For, if  $a + bi = 0$ ,  $bi = -a$  and  $-b^2 = a^2$ ;

whence,  $a^2 + b^2 = 0$ , which is possible only when  $a = 0$  and  $b = 0$ .

*Cor.*—*If a complex number vanishes, its modulus vanishes; and conversely, if the modulus vanishes, the complex number vanishes.*

---

**660. 5.** *If  $a + bi = c + di$ , then  $a = c$  and  $b = d$ .*

For, if  $a + bi = c + di$ ,  $(a - c) + (b - d)i = 0$ ;

whence,  $a - c = 0$  and  $b - d = 0$  [P. 4],

and  $a = c$  and  $b = d$ .

**661. Problem.** To find the value of  $e^{x+yi}$ .

**Solution:** Assuming the exponential law of multiplication [275, P.], and Formula (G), Art. 599, sufficiently general to include imaginary exponents; then

$$e^{x+yi} = e^x \times e^{yi} = e^x \left\{ 1 + yi + \frac{y^2 i^2}{2} + \frac{y^3 i^3}{3} + \frac{y^4 i^4}{4} + \text{etc.} \right\} \\ = e^x \left\{ \left( 1 - \frac{y^2}{2} + \frac{y^4}{4} - \frac{y^6}{6} + \text{etc.} \right) + \left( y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \text{etc.} \right) i \right\} \quad (\text{A})$$

**662.** The expression  $1 - \frac{y^2}{2} + \frac{y^4}{4} - \frac{y^6}{6} + \text{etc.}$  is called *cosine y*, and is written *cos. y*.

The expression  $y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \text{etc.}$  is called *sine y*, and is written *sin. y*. Therefore,

$$e^{yi} = \cos. y + i \sin. y. \quad (\text{B})$$

**663.** Resume the equations

$$\cos. y = 1 - \frac{y^2}{2} + \frac{y^4}{4} - \frac{y^6}{6} + \text{etc.} \quad (1)$$

$$\sin. y = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \text{etc.} \quad (2)$$

$$e^{yi} = \cos. y + i \sin. y. \quad (3)$$

Put  $-y$  for  $y$  in (1), (2), and (3); then

$$\cos. (-y) = 1 - \frac{y^2}{2} + \frac{y^4}{4} - \frac{y^6}{6} + \text{etc.} = \cos. y \quad (4)$$

$$\sin. (-y) = -y + \frac{y^3}{3} - \frac{y^5}{5} + \frac{y^7}{7} - \text{etc.} = -\sin. y \quad (5)$$

$$e^{-yi} = \cos. (-y) + i \sin. (-y) = \cos. y - i \sin. y \quad (6)$$

Multiply (3) by (6),

$$1 = (\cos. y)^2 - i^2 (\sin. y)^2 = \cos.^2 y + \sin.^2 y \quad (\text{C})$$

**Note.**  $\cos.^2 y$  denotes  $(\cos. y)^2$  and  $\sin.^2 y$  denotes  $(\sin. y)^2$ .

**Cor.**  $\sin. y = \sqrt{1 - \cos.^2 y}$ ;

$$\cos. y = \sqrt{1 - \sin.^2 y}.$$

664. Put  $ny$  for  $y$  in (B); then

$$e^{ny} = \cos. ny + i \sin. ny.$$

Raise (B) to the  $n$ th power; then

$$e^{ny} = (\cos. y + i \sin. y)^n.$$

$$\therefore \cos. ny + i \sin. ny = (\cos. y + i \sin. y)^n \quad (D)$$

665. Let  $n = 2$  in (D); then

$$\begin{aligned} \cos. 2y + i \sin. 2y &= (\cos. y + i \sin. y)^2 \\ &= \cos.^2 y - \sin.^2 y + 2 (\sin. y \cos. y) i. \end{aligned}$$

$$\therefore \cos. 2y = \cos.^2 y - \sin.^2 y \quad [660] \quad (E)$$

$$\sin. 2y = 2 \sin. y \cos. y \quad [660] \quad (F)$$

666. Put  $x + y$  for  $y$  in (B); then

$$e^{(x+y)i} = e^{xi} \cdot e^{yi} = \cos. (x + y) + i \sin. (x + y).$$

$$\begin{aligned} \text{But } e^{xi} \cdot e^{yi} &= (\cos. x + i \sin. x) (\cos. y + i \sin. y) \quad (B) \\ &= \cos. x \cos. y - \sin. x \sin. y + i (\sin. x \cos. y + \cos. x \sin. y) \end{aligned}$$

$$\therefore \cos. (x + y) = \cos. x \cos. y - \sin. x \sin. y \quad [660] \quad (G)$$

$$\sin. (x + y) = \sin. x \cos. y + \cos. x \sin. y \quad [660] \quad (H)$$

### Graphical Representation of $\sin. y$ and $\cos. y$ .

667. It is evident that all the conditions expressed in the equation  $\sin.^2 y + \cos.^2 y = 1$  will be satisfied by assuming 1 as the modulus of a vector whose amplitude is the variable angle  $y$  and whose components are  $\sin. y$  and  $\cos. y$ . But, to make this expression conform to the numerical values of  $\sin. y$  and  $\cos. y$  as expressed in Art. 662,  $y$  must be taken to represent the number of vector units in the arc which measures the amplitude, and  $\sin. y$  as the vertical and  $\cos. y$  as the horizontal component of the vector; for in this way only would  $\sin. y = 0$  and  $\cos. y = 1$  when  $y = 0$ .



## CHAPTER XI.

### THEORY OF FUNCTIONS.

---

#### Definitions.

**669.** A quantity whose value changes, or is supposed to change, according to a definable law, is a *definite variable*, or simply a *variable*.

**670.** A variable whose law of change is not dependent upon that of another variable is an *independent variable*.

**671.** A variable whose law of change is dependent upon that of another variable is a *dependent variable*, and is called a *function* of that variable.

Hence it is, that any expression containing a variable is a function of that variable [562].

**672.** Any law of change may be imposed upon an independent variable ; but, when it is once imposed, the law of change of any function of the variable becomes determined.

**673.** The simplest treatment of functions of a single variable is that in which the variable is supposed to increase or decrease uniformly by equal increments, finite or infinitely small.

**674.** A function is said to be *continuous* so long as an infinitely small change in the independent variable produces an infinitely small change in the function, and *dis-*

*continuous* when an infinitely small change in the independent variable produces a finite or infinitely great change in the function.

**Illustration.**—Thus, the function  $\frac{1}{1-x}$  assumes all values between  $+1$  and  $+\infty$  as  $x$  assumes all values between  $0$  and  $+1$ , and is, therefore, *continuous* from  $+1$  to  $+\infty$ ; but, as the value of  $x$  continues to increase from a quantity infinitesimally less than  $+1$  to a quantity infinitesimally greater than  $+1$ , or takes an infinitely small step across  $+1$ , the function takes a leap through the whole gamut of numbers from  $+\infty$  to  $-\infty$ , and is, therefore, *discontinuous* between these values.

**675.** So long as a function increases in value as the independent variable increases in value, and hence, too, decreases in value as the independent variable decreases in value, it is an *increasing function*; but when it decreases in value as the independent variable increases in value, and, hence, increases in value as the independent variable decreases in value, it is a *decreasing function*.

**Illustration.**—Let  $y = f(x) = x^2 - 4x + 3$ .

Assign values to  $x$  and calculate the corresponding values of  $y$  by synthetic division [106], you will obtain results as follows:

For  $x = -3, -2, -1, 0, +1, +2, +3, +4, +5$   
 $y = +24, +15, +8, +3, 0, -1, 0, +3, +8$

Here  $y$  decreases from  $+24$  to  $-1$  as  $x$  increases from  $-3$  to  $+2$ , and is, therefore, a decreasing function between these values of  $x$ ; and it increases from  $-1$  to  $+8$  as  $x$  increases from  $+2$  to  $+5$ , and is, therefore, an increasing function between these values of  $x$ .

**676.** The *maximum value* of a function is the value at which the function changes from an increasing to a decreasing function.

**677.** The *minimum value* of a function is the value at which the function changes from a decreasing to an increasing function.

**678.** The maxima and minima values of a function are often called the *turning values* of the function.

**679.** A turning value of a function may be a finite constant, zero, or infinity.

**Illustrations.**—1. Take  $y = f(x) = 3 + (4 - x)^2$ .

As  $x$  increases from 0 to 4,  $y$  decreases from 19 to 3; and as  $x$  continues to increase from 4 to  $\infty$ ,  $y$  increases from 3 to  $\infty$ . Therefore, 3 is a turning value (a minimum) of  $y$ .

2. Take  $y = (a - x)^2$ .

As  $x$  increases from 0 to  $a$ ,  $y$  decreases from  $a^2$  to 0; and as  $x$  continues to increase from  $a$  to  $\infty$ ,  $y$  increases from 0 to  $\infty$ . Therefore, 0 is a turning value of  $y$ .

3. Take  $y = f(x) = \frac{1}{(1 - x)^2}$ .

As  $x$  increases in value from 0 to  $+1$ ,  $(1 - x)^2$  decreases from 1 to 0, and  $y$  increases from 1 to  $\infty$ ; and as  $x$  continues to increase from 1 to  $\infty$ ,  $(1 - x)^2$  increases from 0 to  $\infty$ , and  $y$  decreases from  $\infty$  to 0. Therefore,  $\infty$  is a turning value (a maximum) of  $y$ .

**680.** The *limit* of a function is the value of the function at which it ceases to be continuous.

**Note.**—Notice the distinction between the meaning of the word *limit* as here used and as used in Art. 398. In the latter sense,  $\infty$  would be the limit of  $y$  in illustration 3, Art. 679, instead of a maximum.

**681.** The limit of a function may be a finite constant, zero, or infinity.

**Illustrations.**—1. Take  $y = f(x) = 2 - \frac{2}{2^x}$ .

As  $x$  increases from 0 to  $\infty$ ,  $\frac{2}{2^x}$  decreases from 2 to 0,

and  $y$  increases from 0 to 2; and, as  $x$  can not be supposed greater than  $\infty$ ,  $y$  can not become greater than 2, neither can  $y$  begin to decrease at 2. Therefore, 2 is the limit of  $y$ .

2. Take  $y = f(x) = x^2(1 + x^{\frac{1}{2}})$ .

As  $x$  decreases from 1 to 0,  $y$  decreases from 2 to 0; and as  $x$  can not be taken less than 0 (negative) without making  $y$  imaginary,  $y$  can not become less than 0, neither can  $y$  change from a decreasing to an increasing function at 0. Therefore, 0 is the limit of  $y$ .

3. We have already seen [674] that  $y = f(x) = \frac{1}{1-x}$  increases from 1 to  $\infty$  as  $x$  increases from 0 to  $+1$ , and thereafter becomes discontinuous. Therefore,  $\infty$  is the limit of  $y$ .

**682.** A function may have two sets of values approaching the same or different limits for the same set of values of the independent variable.

**Illustrations.**—1. Take  $y^2 = f(x) = 16 - x^2$ ;  
then  $y = \pm \sqrt{16 - x^2}$ .

Here are two values of  $y$  for each value of  $x$ , numerically equal but opposed in sign. As  $x$  increases from 0 to 4 one value of  $y$  decreases from 4 to 0, and the other increases from  $-4$  to 0. If  $x$  becomes infinitesimally greater than 4 both values of  $y$  become imaginary. Therefore, 0 is the limit of both values of  $y$ .

2. Take  $y^2 = f(x) = 4x$ ;  
then  $y = \pm 2\sqrt{x}$ .

Here, again, are two values of  $y$  for each value of  $x$ . As  $x$  increases from 0 to  $+\infty$ , one value of  $y$  increases from 0 to  $+\infty$  and the other decreases from 0 to  $-\infty$ ; and as  $x$  can not be supposed greater than  $+\infty$ ,  $+\infty$  is the limit of one value of  $y$  and  $-\infty$  the limit of the other value.

**683.** The limit of an increasing function is a *superior*

or maximum limit; that of a decreasing function an *inferior* or minimum limit.

### Graphical Representation of Functions of a Single Variable.

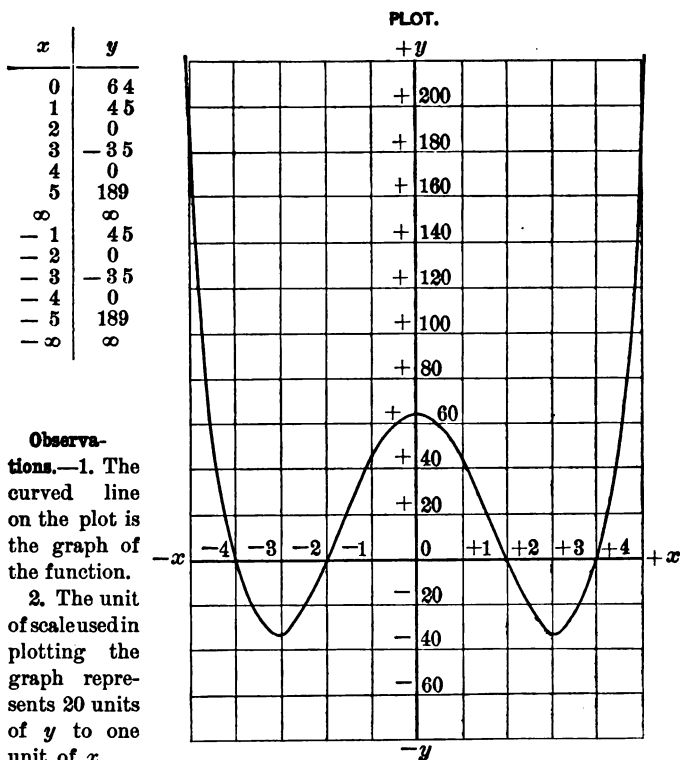
**684.** Every function of a single variable may be approximately represented by a line, straight or curved, called the *graph* of the function.

*Method.*—Let  $y = f(x)$ . Assign successive values to  $x$  and calculate the corresponding values of  $y$ . Construct two indefinite straight lines intersecting each other at right angles, one running right and left and the other up and down from their intersection. These are the *axes of reference*. The first is the  $x$ -axis and the second the  $y$ -axis, and their intersection the *origin*. Regard distance rightward from the  $y$ -axis *positive*, and distance leftward *negative*; distance upward from the  $x$ -axis *positive*, and distance downward *negative*.

Assume a fixed length as a unit of scale, and lay off on the  $x$ -axis from the origin the successive values of  $x$  based on this scale, and at the extremity of each  $x$  value, and on a line parallel to the  $y$ -axis, lay off the corresponding values of  $y$ . Thus will be located a series of successive points; draw a continuous line through these points; it will be the graph of the function, and its accuracy will depend upon the nearness to each other of the successive values of  $x$  taken, the relation of the unit of scale to that of  $x$  and  $y$ , and the correctness of the instruments used in plotting.

**Illustrations.**—1. Take  $y = f(x) = x^4 - 20x^2 + 64$ .

Assign special values to  $x$  and calculate the corresponding values of  $y$  by synthetic division [106]. You will readily derive the following table of values and make the following plot :



**Observations.**—1. The curved line on the plot is the graph of the function.

2. The unit of scale used in plotting the graph represents 20 units of  $y$  to one unit of  $x$ .

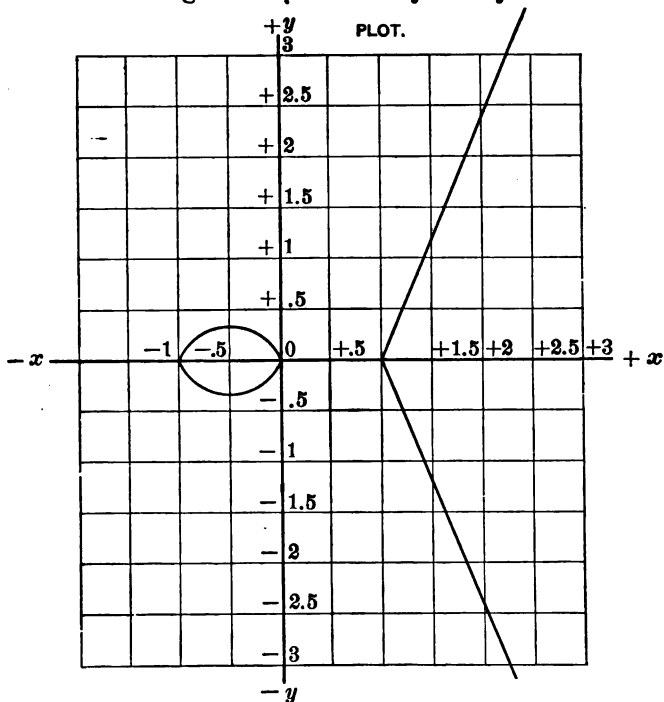
3. The graph exhibits *three* turning values of the function; two *minima* at the points  $(x=3, y=-35)$  and  $(-3, -35)$ , and one *maximum* at the point  $(0, 64)$ .

4. When  $x = +\infty$ ,  $y = +\infty$ , and when  $x = -\infty$ ,  $y = +\infty$ . The graph, like the function, descends from  $(-\infty, +\infty)$  to  $(-3, -35)$ , then ascends from  $(-3, -35)$  to  $(0, 64)$ , then descends from  $(0, 64)$  to  $(3, -35)$ , then again ascends from  $(3, -35)$  to  $(+\infty, +\infty)$ . It is a continuous graph from beginning to end.

5. At  $x = 2, 4, -2$ , and  $-4$ , the graph crosses the  $x$ -axis, exhibiting the fact that for these values of  $x$ ,  $y = f(x) = x^4 - 20x^2 + 64 = 0$ . The values of  $x$  that render  $f(x) = 0$  are, however, the roots of the equation  $f(x) = 0$ ; therefore, the values of the roots of  $f(x) = 0$  may be approximately found even if incommensurable, by plotting  $f(x) = y$  and determining with a scale of equal parts where the graph crosses the  $x$ -axis.

2. Plot  $y = \pm \sqrt{x^3 - x} = \pm \sqrt{x(x+1)(x-1)}$ .

The following table of values may readily be obtained :



**Observations.**—1. The graph consists of two branches between the points  $(-1, 0)$  and  $(0, 0)$ , symmetrical with respect to the  $x$ -axis. These branches are *confuent* at the points mentioned.

2. The graph is discontinuous for all values of  $x$  antecedent to  $-1$ , counting from  $x = -\infty$ , and also for all values of  $x$  between  $0$  and  $+1$ .

3. The graph again consists of two branches, symmetrical with respect to the  $x$ -axis, for all positive values of  $x$  greater than  $+1$ .

4. The limits of the first branch are at  $(-1, 0)$  and  $(0, 0)$ ; the limits of the second branch are also at  $(-1, 0)$  and  $(0, 0)$ ; the limits of the third branch are at  $(+1, 0)$  and  $(+\infty, +\infty)$ ; and the limits of the fourth branch are at  $(+1, 0)$  and  $(+\infty, -\infty)$ .

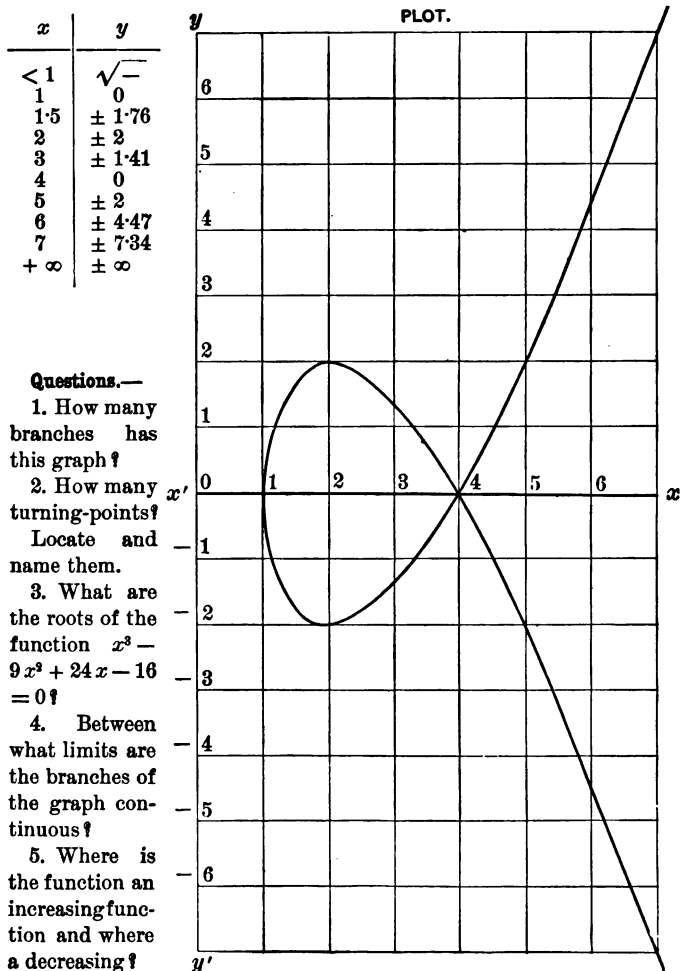
$x$	$y$
0	0
$+ < +1$	$\sqrt{-}$
1	0
1.5	$\pm 1.4$
2	$\pm 2.4$
2.5	$\pm 3.6$
3	$\pm 4.9$
-1	0
$- < -1$	$\sqrt{-}$
-2	$\pm .44$
-4	$\pm .58$
-5	$\pm .61$
-6	$\pm .62$
-8	$\pm .53$

5. The first branch has a turning-point (maximum) somewhere between  $(- \cdot 5, + \cdot 61)$  and  $(- \cdot 8, + \cdot 53)$ . The second branch also has a turning-point (minimum) between  $(- \cdot 5, - \cdot 61)$  and  $(- \cdot 8, - \cdot 53)$ .

6. The branches of the graph meet the  $x$ -axis when  $x = 0, +1$ , and  $-1$ . These values are, therefore, the roots of  $f(x) = x^3 - x = 0$ .

3. Plot  $y^2 = x^3 - 9x^2 + 24x - 16$ .

$$\text{or } y = \pm \sqrt{x^3 - 9x^2 + 24x - 16}.$$



**Questions.—**

1. How many branches has this graph?

2. How many turning-points?

Locate and name them.

3. What are the roots of the function  $x^3 - 9x^2 + 24x - 16 = 0$ ?

4. Between what limits are the branches of the graph continuous?

5. Where is the function an increasing function and where a decreasing function?

## EXERCISE 96.

Plot and discuss the following functions :

(Use paper ruled in squares, called plotting-paper.)

1.  $y = 4x + 6$

6.  $y^2 = x^3$

2.  $y = 8x$

7.  $y^2 = x^2(x - 1)$

3.  $y = 81x^{-3}$

8.  $y = x^3 - 8x^2 + 20x - 10$

4.  $y^2 = 4x$

9.  $y = 3x + 18x^2 - 2x^3$

5.  $y^2 = 16 - x^2$

10.  $y^2 = x^3 + 3x^2 - 5x - 20$

## Differentials and Derivatives of Functions.

## Definitions.

**685.** The limit of the ratio of the increment of a function to the increment of the independent variable producing the increment of the function, when the limit of the increment of the independent variable is zero, is called the *derivative* of the function.

Thus, if we let  $y = f(x)$ , and represent the increment of  $x$  by  $\Delta x$  and the corresponding increment of  $y$  by  $\Delta y$ , then will  $\lim. \left( \frac{\Delta y}{\Delta x} \right)_{\Delta x = 0} =$  the derivative of the function.

**686.** The limit of the increment of the independent variable is called the *differential of the independent variable*, and is represented by  $dx$ ; and the limit of the increment of the function is called the *differential of the function*, and is represented by  $dy$ .

Therefore,  $\lim. \left( \frac{\Delta y}{\Delta x} \right)_{\Delta x = 0} = \frac{dy}{dx}$ .

**Notice.**  $dy$  and  $dx$  represent single quantities (differentials) and are not equivalent to  $d \times y$  and  $d \times x$ .

**Illustration.**—

Let  $y = x^2$  (1)

then,  $y + \Delta y = (x + \Delta x)^2 = x^2 + 2x(\Delta x) + (\Delta x)^2$  (2)

Subtract (1) from (2),  $\Delta y = 2x(\Delta x) + (\Delta x)^2$  (3)

Divide by  $\Delta x$ ,  $\frac{\Delta y}{\Delta x} = 2x + \Delta x$  (4)

$\therefore \lim. \left( \frac{\Delta y}{\Delta x} \right)_{\Delta x=0} = \lim. (2x + \Delta x)_{\Delta x=0}$  [401, P.] (5)

$\therefore \frac{dy}{dx} = 2x$  = the derivative of  $x^2$ ,

and  $dy = 2x dx$  = the differential of  $x^2$ .

**687.** *The differential of a function equals the derivative of the function multiplied by the differential of the independent variable.*

**688.** *The derivative of a function equals the differential of the function divided by the differential of the independent variable.*

**689.** *If the differential of a function, and hence, too, the derivative of a function, is positive, the function is an increasing one; if negative, a decreasing one.*

**Principles.**

**690.** Let  $y = f(x) = x^n$ , (1)

then,  $y + \Delta y = (x + \Delta x)^n$   
 $= x^n + nx^{n-1} \cdot \Delta x + A \cdot (\Delta x)^2$  (2)

in which  $A =$

$\frac{n(n-1)}{2} x^{n-2} + \frac{n(n-1)(n-2)}{3} x^{n-3} \cdot \Delta x + \text{etc.}$  [593].

Subtracting (1) from (2),  $\Delta y = nx^{n-1} \cdot \Delta x + A \cdot (\Delta x)^2$  (3)

Dividing by  $\Delta x$ ,  $\frac{\Delta y}{\Delta x} = nx^{n-1} + A \cdot \Delta x$  (4)

$\therefore \lim. \left( \frac{\Delta y}{\Delta x} \right)_{\Delta x=0} = \lim. (nx^{n-1} + A \cdot \Delta x)_{\Delta x=0}$

$\therefore \frac{dy}{dx} = nx^{n-1}$ , since  $\lim. A = \text{a finite constant}$  [582], and

$\lim. \Delta x = 0$ . (5)

$\therefore dy = nx^{n-1} dx$ . Therefore,

**Prin. 1.**—*The differential of a variable with a constant exponent equals the continued product of the exponent, the variable with its exponent diminished by unity, and the differential of the variable.*

**Illustrations.**—

$$\begin{aligned} 1. \quad d(x^4) &= 4x^3 dx & 2. \quad d(x)^{-\frac{2}{3}} &= -\frac{2}{3}x^{-\frac{2}{3}} dx \\ 3. \quad d(a+bx)^p &= p(a+bx)^{p-1} d(a+bx) \end{aligned}$$

$$691. \text{ Let } y = ax \quad (1)$$

$$\text{then, } y + \Delta y = a(x + \Delta x) = ax + a(\Delta x) \quad (2)$$

$$\text{Subtracting, } \Delta y = a(\Delta x) \quad (3)$$

$$\text{Dividing, } \frac{\Delta y}{\Delta x} = a$$

$$\therefore \text{ Lim. } \frac{\Delta y}{\Delta x} = a$$

$$\text{whence, } \frac{dy}{dx} = a, \text{ and } dy = a dx. \text{ Therefore,}$$

**Prin. 2.**—*The differential of a constant times a variable equals the constant times the differential of the variable.*

$$\text{Thus, } d(3x^5) = 3 \cdot d(x^5) = 3 \times 5x^4 dx = 15x^4 dx.$$

$$692. \text{ Let } y = ax + b \quad (1)$$

$$\begin{aligned} \text{then, } y + \Delta y &= a(x + \Delta x) + b = \\ &ax + a(\Delta x) + b \end{aligned} \quad (2)$$

$$\text{Subtracting, } \Delta y = a(\Delta x)$$

$$\text{Dividing, } \frac{\Delta y}{\Delta x} = a; \text{ whence, } \frac{dy}{dx} = a,$$

$$\text{and } dy = a dx. \text{ Therefore,}$$

**Prin. 3.**—*The differential of a constant term is zero.*

$$693. \text{ Let } v = f(x), w = f'(x), \text{ and } z = f''(x); \text{ and let } y = v + w - z \quad (1)$$

$$\begin{aligned} \text{then, } y + \Delta y &= v + \Delta v + w + \Delta w - (z + \Delta z) \\ &= v + w - z + \Delta v + \Delta w - \Delta z \end{aligned} \quad (2)$$

$$\text{Subtracting,} \quad \Delta y = \Delta v + \Delta w - \Delta z \quad (8)$$

$$\text{Dividing by } \Delta x, \quad \frac{\Delta y}{\Delta x} = \frac{\Delta v}{\Delta x} + \frac{\Delta w}{\Delta x} - \frac{\Delta z}{\Delta x} \quad (4)$$

$$\text{whence,} \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{\Delta w}{\Delta x} - \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta x} \quad (5)$$

[413, P.]

$$\therefore \quad \frac{dy}{dx} = \frac{dv}{dx} + \frac{dw}{dx} - \frac{dz}{dx} \quad (6)$$

$$\text{whence,} \quad dy = dv + dw - dz. \quad \text{Therefore,}$$

**Prin. 4.**—The differential of a polynomial whose terms are functions of the same independent variable equals the algebraic sum of the differentials of its terms.

$$\begin{aligned} \text{Illustration.} \quad d(x^3 + 3x^2 - 2x + 5) \\ = d(x^3) + d(3x^2) + d(-2x) + d(5) \\ = 3x^2 dx + 6x dx - 2dx = (3x^2 + 6x - 2) dx. \end{aligned}$$

$$\begin{aligned} 694. \text{ Let} \quad v = f(x) \text{ and } z = f'(x), \\ \text{and} \quad y = vz \end{aligned} \quad (1)$$

$$\begin{aligned} \text{then, } y + \Delta y &= (v + \Delta v)(z + \Delta z) \\ &= vz + v \cdot \Delta z + z \cdot \Delta v + \Delta v \cdot \Delta z \end{aligned} \quad (2)$$

$$\text{Subtracting,} \quad \Delta y = v \cdot \Delta z + z \cdot \Delta v + \Delta v \cdot \Delta z \quad (3)$$

$$\text{Dividing by } \Delta x, \quad \frac{\Delta y}{\Delta x} = v \cdot \frac{\Delta z}{\Delta x} + z \cdot \frac{\Delta v}{\Delta x} + \frac{\Delta v}{\Delta x} \cdot \Delta z$$

$$\therefore \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left( v \cdot \frac{\Delta z}{\Delta x} \right) + \lim_{\Delta x \rightarrow 0} \left( z \cdot \frac{\Delta v}{\Delta x} \right) + \lim_{\Delta x \rightarrow 0} \left( \frac{\Delta v}{\Delta x} \cdot \Delta z \right)$$

$$\therefore \quad \frac{dy}{dx} = v \cdot \frac{dz}{dx} + z \cdot \frac{dv}{dx} + 0 \quad [413, P.]$$

$$\text{whence,} \quad dy = v dz + z dv. \quad \text{Therefore,}$$

**Prin. 5.**—The differential of the product of two continuous functions of the same independent variable equals the sum of the products obtained by multiplying each function by the differential of the other.

**Illustration.**—

$$\begin{aligned} d(-3x^3 \times 5x^{\frac{2}{3}}) &= -3x^3 \times d(5x^{\frac{2}{3}}) + 5x^{\frac{2}{3}} \times d(-3x^3) \\ &= \{-3x^3 \times \frac{2}{3} \times 5x^{-\frac{1}{3}} + 5x^{\frac{2}{3}} \times 3 \times (-3x^2)\} dx \\ &= -55x^{\frac{2}{3}} dx. \end{aligned}$$

**Cor.**  $d(vwz) = v \cdot d(wz) + wz \cdot dv$  [694, P.]  
 $= vw dz + v z dw + wz dv$ . [694, P.]; etc.

**695.** Let  $v = f(x)$ , and  $z = f'(x)$ ; and

$$y = \frac{v}{z} = vz^{-1};$$

$$\begin{aligned}\text{then } dy &= v \cdot d(z^{-1}) + z^{-1} dv \text{ [694, P.]} \\ &= -vz^{-2} dz + z^{-1} dv \\ &= \frac{dv}{z} - \frac{v dz}{z^2} = \frac{z dv - v dz}{z^2}.\end{aligned}$$

Therefore,

**Prin. 6.**—*The differential of a fraction whose terms are continuous functions of the same independent variable equals the denominator into the differential of the numerator minus the numerator into the differential of the denominator, all divided by the square of the denominator.*

$$\begin{aligned}\text{Thus, } d\left(\frac{x^2}{y^3}\right) &= \frac{y^3 \cdot d(x^2) - x^2 \cdot d(y^3)}{y^6} \\ &= \frac{2xy^3 dx - 3x^2 y^2 dy}{y^6}.\end{aligned}$$

**696.** Let  $y = \log. x$

$$\begin{aligned}\text{then } y + \Delta y &= \log. (x + \Delta x) = \log. x \left(1 + \frac{\Delta x}{x}\right) \\ &= \log. x + \log. \left(1 + \frac{\Delta x}{x}\right) \text{ [467, P. 2]} \\ &= \log. x + \frac{\Delta x}{x} - \frac{1}{2} \cdot \frac{(\Delta x)^2}{x^2} + \frac{1}{3} \cdot \frac{(\Delta x)^3}{x^3} - \text{etc. [601, C.]} \\ \Delta y &= \frac{\Delta x}{x} - \frac{1}{2} \cdot \frac{(\Delta x)^2}{x^2} + \frac{1}{3} \cdot \frac{(\Delta x)^3}{x^3} - \text{etc.} \\ \frac{\Delta y}{\Delta x} &= \frac{1}{x} + B \cdot \Delta x; \text{ in which } B = -\frac{1}{2} \cdot \frac{1}{x^2} + \frac{1}{3} \cdot \frac{\Delta x}{x^3} - \text{etc.;} \\ \text{which for very small values of } \Delta x &\text{ is convergent [582].}\end{aligned}$$

$$\therefore \text{Lim.} \left(\frac{\Delta y}{\Delta x}\right)_{\Delta x=0} = \lim. \left(\frac{1}{x} + B \cdot \Delta x\right)_{\Delta x=0}$$

$$\text{whence, } \frac{dy}{dx} = \frac{1}{x}; \text{ and } dy = \frac{dx}{x}. \text{ Therefore,}$$

**Prin. 7.**—*The differential of the log. of a quantity equals the differential of the quantity divided by the quantity itself.*

**Cor.**—Since  $\log_{10} x = m \log. x$  [607, P.]

$$d. (\log_{10} x) = \frac{m dx}{x}.$$

**Illustration.**—

$$\begin{aligned} \text{Thus, } d \left( \log. \frac{x+a}{x} \right) &= d \left( \frac{x+a}{x} \right) \div \frac{x+a}{x} \\ &= \frac{x}{x+a} \left\{ \frac{x dx + (x+a) dx}{x^2} \right\} = \frac{1}{x+a} \left( \frac{2x+a}{x} \right) dx. \end{aligned}$$

**697.** Let  $y = a^x$ , in which  $a$  is a constant.

then,  $\log. y = x \log. a$  [468, P.]

$$d (\log. y) = \log. a . dx$$

$$\text{or, } \frac{dy}{a^x} = \log. a . dx$$

whence,  $dy = a^x \log. a . dx$ . Therefore,

**Prin. 8.**—*The differential of a constant with a variable exponent equals the continued product of the original quantity, the logarithm of the constant, and the differential of the variable exponent.*

$$\begin{aligned} \text{Thus, } d(a+b)^{\sqrt{x}} &= d(a+b)^{x^{\frac{1}{2}}} = (a+b)^{x^{\frac{1}{2}}} \log. (a+b) \\ &\times d(x^{\frac{1}{2}}) = \frac{1}{2} x^{-\frac{1}{2}} (a+b)^{x^{\frac{1}{2}}} \log. (a+b) dx. \end{aligned}$$

**698. Problem.** Find the differential of  $x^x$ .

$$\text{Let } y = x^x$$

$$\text{then, } \log. y = x \log. x$$

$$\text{and } d (\log. y) = x . d (\log. x) + \log. x . dx$$

$$\text{or, } \frac{dy}{x^x} = x . \frac{dx}{x} + \log. x . dx$$

$$\text{whence, } dy = x^x (1 + \log. x) dx.$$

## EXERCISE 97.

Differentiate :

1.  $y = 5ax^3 - 3bx^2 + 2cx - d$
2.  $y = 5x^3x^2 + z$
3.  $y = x^3 + 3x^2 + 4x + 5$
4.  $y = (a + bx)^{n^2}$
5.  $y = \frac{x^3 + 3x^2 + 2}{2x}$
6.  $y = \sqrt{\frac{ax+b}{c}}$
7.  $y = \sqrt[3]{5x+6}$
8.  $y^2 = 2px$
9.  $y^3 = 3ax^3$
10.  $a^2y^2 + b^2x^2 = a^2b^2$
11.  $y = ax^{\frac{3}{2}} + bx^{\frac{1}{2}} + c$
12.  $y = \frac{2x^4}{(a+x)^3}$
13.  $y = \frac{x}{\sqrt{x+a}}$
14.  $y = \frac{a}{(b^2 + x^2)^3}$
15.  $y = (x+a)^2(x+b)^3$
16.  $y = (x+a)^n(x-b)^p$
17.  $y = \log_e(x^2 + x)$
18.  $y = (\log_e x)^3$
19.  $y = 3x^{3^2}$
20.  $y = ax^x$
21.  $y = x^4(a+x^2)^{-\frac{3}{2}}$
22.  $y = \log_e(a+x)^a$
23.  $y = c^x \div d^x$
24.  $y = \frac{\sqrt{1+x}}{\sqrt{1-x}}$
25.  $y = a^{\log_e x}$

## Applications.

## EXERCISE 98.

1. At what rate is the area of a circle increasing when the radius is 6 inches and is increasing at the rate of 3 inches per second?

**Solution:** Let  $y$  = the area, and  $x$  = the radius; then,  
 $y = \pi x^2$   
 and  $dy = 2\pi x dx$ .

This denotes that at any instant the rate of increase of the area is  $2\pi x$  times as great as the rate of increase of the radius at the same instant. But when the radius is 6 inches, it increases at the rate of 3 inches per second; or, when  $x = 6$  inches,  $dx = 3$  inches.

$\therefore dy = 2\pi \times 6 \text{ inches} \times 3 \text{ inches} = 36\pi$  square inches; that is, the area is increasing at such a rate that, if kept uniform for one second, the increase would amount to  $36\pi$  square inches.

2. At what rate is the area of a square increasing when the side of the square is 4 inches and is increasing at the rate of 2 inches per second?

3. The volume of a sphere increases how many times as fast as its radius? When its radius is 6 inches and increases at the rate of 1 inch per second, at what rate is the volume increasing?

4. At what rate is the diagonal of a square increasing when the side of the square is 8 inches and is increasing at the rate of 2 inches per second?

5. The radius of a circle is 4 inches and its circumference is increasing at the rate of  $2\pi$  inches per second. At what rate is the radius increasing at the same instant?

6. A boy approaches a tree 90 feet high standing on a level road at the rate of 3 miles an hour. At what rate is he approaching the top of the tree when he is 220 feet from the base?

7. The diagonal of a cube is increasing at the rate of 36 inches per second, when the side of the cube is 5 inches long. At what rate is the side increasing at the same time?

8. If  $x$  increases at the rate of .5 per instant, at what rate is  $\log_{10} x$  increasing when  $x = 42$ ?

9. The  $\log_{10} 42 = 1.62325$ . What, then, would be the  $\log_{10} 42.5$ , if the increase were uniform? How does the result compare with  $\log_{10} 42.5$  as found in the table?

### Successive Derivatives.

699. If the derivative of  $f(x)$  be treated as a new function of  $x$  [ $f_1(x)$ ], there may be found from it a second derivative of  $f(x)$  [ $f_2(x)$ ] in the same way as  $f_1(x)$  was derived from  $f(x)$ , and so on, until a derivative is found that is independent of  $x$  [ $f(x_0)$ ].



3. In general, if  $f(x)$  contains the factor  $x + a$   $p$  times,  $(x + b)$   $q$  times,  $(x + c)$   $r$  times . . . then will  $(x + a)^{p-1} (x + b)^{q-1} (x + c)^{r-1}$  . . . be the H. C. D. of  $f(x)$  and  $f_1(x)$ .

4. The H. C. D. of  $f(x)$  and  $f_1(x)$  contains one factor less of each kind than does  $f(x)$ .

**701. Theorem.**—*Every polynomial composed of binomial factors of the first degree, some of which are equal, may be decomposed into factors containing no equal binomial factors of the first degree.*

For, let  $f(x)$  be a polynomial composed of binomial factors of the first degree, some of which are equal,  $f_1(x)$  its first derivative,  $f'(x)$  the H. C. D. of  $f(x)$  and  $f_1(x)$ , and  $\phi(x)$  the other factor of  $f(x)$ ; then,

1.  $\phi(x)$  will be devoid of equal factors of the first degree [700, 4].

2. If  $f'(x)$  still contains equal factors of the first degree it may be resolved into two factors,  $f''(x)$  and  $\phi'(x)$ , in which  $\phi'(x)$  is devoid of equal factors [700, 4].

3. This process may be continued until no factor is left that contains equal factors of the first degree, which will be when the last H. C. D. found is unity.

**Illustration.**—Resolve  $x^7 + x^6 - 12x^5 - 12x^4 + 48x^3 + 48x^2 - 64x - 64$  into factors devoid of equal binomial factors of the first degree.

**Solution:**

$$f(x) = x^7 + x^6 - 12x^5 - 12x^4 + 48x^3 + 48x^2 - 64x - 64$$

$$f_1(x) = 7x^6 + 6x^5 - 60x^4 - 48x^3 + 144x^2 + 96x - 64$$

$$f_1(x) = x^4 - 8x^2 + 16 [158] = (x^2 - 4)^2 = (x + 2)(x + 2)(x - 2)(x - 2)$$

$$\phi(x) = f(x) + f_1(x) = x + 1$$

$$\therefore f(x) = (x + 2)(x + 2)(x - 2)(x - 2)(x + 1).$$

#### EXERCISE 100.

Factor :

1.  $x^4 + 2x^3 - 11x^2 - 12x + 36$

2.  $x^6 - 5x^5 + x^4 + 37x^3 - 86x^2 + 76x - 24$

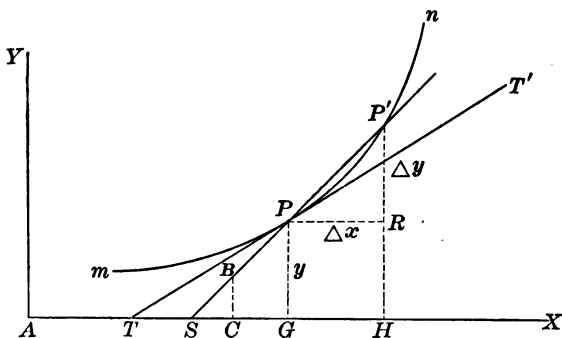
$$3. x^8 + x^7 - 26x^6 - 27x^5 + 216x^4 + 243x^3 - 486x^2 - 729x - 729$$

$$4. x^{10} - 30x^8 + 345x^6 - 1900x^4 + 5040x^2 - 5184$$

$$5. x^{10} - 13x^8 + 42x^6 - 58x^4 + 37x^2 - 9$$

### Graphical Significance of $f_1(x)$ .

702. Let  $mn$  be the graph of  $y = f(x)$ .



Let  $P$  be a point on the graph whose co-ordinates are  $x$  and  $y$ .  
Let  $GH = PR = \Delta x$ ; then will  $P'R = \Delta y$ .

Draw the secant line  $P'PS$ , also the tangent line  $T'PT$ .

Take  $SB = 1$ , and draw  $BC = \sin. S$  and  $SC = \cos. S$ .

Now, the triangles  $P'PR$  and  $BSC$  are similar.

$$\therefore \frac{P'R}{PR} = \frac{BC}{SC} = \tan. S \quad [668] \quad (1)$$

$$\therefore \frac{\Delta y}{\Delta x} = \tan. S \quad (2)$$

Let the point  $P'$  approach the point  $P$  on the graph so as to make  $\Delta x$  diminish uniformly; then will the secant line  $P'PS$  revolve about  $P$  and approach the tangent line  $T'PT$  as its limit, and the angle  $S$  will approach the angle  $T$  as its limit.

$$\therefore \text{Lim.} \left( \frac{\Delta y}{\Delta x} \right)_{\Delta x = 0} = \text{lim.} (\tan. S)_{\Delta x = 0}$$

$$\text{or,} \quad \frac{dy}{dx} = \tan. T. \quad \text{Therefore,}$$

*The first derivative of a function is equivalent to the tangent of the angle which a tangent line to the graph of the function makes with the axis of abscissas.*

# Maxima and Minima of Functions.

**703.** The maximum or minimum value of a quadratic function may readily be found, as follows :

**Example 1.**—What is the maximum or minimum value of  $x^2 + 8x + 6$ , and what value of  $x$  will render it a maximum or minimum ?

**Solution :** Let  $f(x) = x^2 + 8x + 6 = m$

Complete the square,  $x^2 + 8x + 16 = m + 10$

Extract the  $\sqrt{\phantom{x}}$ ,  $x + 4 = \pm \sqrt{m + 10}$

Transpose,  $x = -4 \pm \sqrt{m + 10}$

Now,  $m < -10$ , else would  $x$  be imaginary.

$\therefore m = -10$  is the minimum value of  $f(x)$ .

But when  $m = -10$ ,  $x = -4$ ; then,  $x = -4$  renders  $f(x) = x^2 + 8x + 6 = -10$ , a minimum.

**Example 2.**—What is the maximum or minimum value of  $8x - 3x^2 + 9$ , and what value of  $x$  will render it a maximum or a minimum ?

**Solution :** Let  $f(x) = 8x - 3x^2 + 9 = m$

Complete the square,  $9x^2 - 24x + 16 = 43 - 3m$

Extract the  $\sqrt{\phantom{x}}$ ,  $3x - 4 = \pm \sqrt{43 - 3m}$

Transpose and divide,  $x = \frac{4}{3} \pm \frac{1}{3} \sqrt{43 - 3m}$

Now,  $3m > 43$ , or  $m > 14\frac{1}{3}$ , else would  $x$  be imaginary.

$\therefore m = 14\frac{1}{3}$  is the maximum value of  $f(x)$ .

But, when  $m = 14\frac{1}{3}$ ,  $x = 1\frac{1}{3}$ ; therefore,  $x = 1\frac{1}{3}$  renders  $f(x) = 8x - 3x^2 + 9 = 14\frac{1}{3}$ , a maximum.

**Example 3.**—Divide 36 into two parts whose product shall be the greatest possible.

**Solution :** Let  $x$  and  $36 - x$  = the two parts,

and  $x(36 - x)$ , or  $36x - x^2 = m$ .

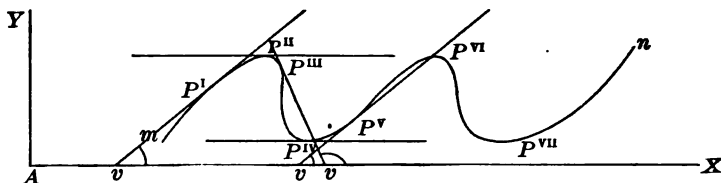
Then,  $x = 18 \pm \sqrt{324 - m}$ .

Now,  $m = 324$  is a maximum ;

$\therefore x = 18$  and  $36 - x = 18$ .

**704. General Method.**—

Let  $mn$  be the graph of  $y = f(x)$ .



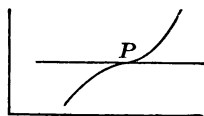
Conceive a point,  $P$ , to move along the graph, carrying with it a tangent line to the graph, in such a manner as to cause the abscissa ( $x$ ) of the point to increase uniformly. Let  $v$  be the value of the variable angle which the tangent line makes with the  $x$ -axis. At  $P^I$   $v < 90^\circ$ ; hence,  $\tan. v$ , or  $f_1(x)$ , is positive [668, Sch.]. This is true, however near  $P^I$  is to  $P^{II}$ . At  $P^{II}$ , the tangent line is parallel to the  $x$ -axis; hence,  $v = 0$ , and  $\tan. v$ , or  $f_1(x) = 0$ . At  $P^{III}$ ,  $v > 90^\circ$ ; hence,  $\tan. v$ , or  $f_1(x)$ , is negative [668, Sch.]. This is true, however near  $P^{III}$  is to  $P^{II}$ . Again, just before  $P$  arrives at  $P^{IV}$ ,  $v > 90^\circ$ , and  $\tan. v$  is negative; when  $P$  is at  $P^{IV}$ ,  $v = 0$  and  $\tan. v = 0$ ; when  $P$  has just passed  $P^{IV}$ ,  $v < 90^\circ$  and  $\tan. v$  is positive. Therefore,

**705. Prin. 1.**  $f_1(x) = 0$  at turning values of  $f(x)$ .

**Prin. 2.** Immediately before a maximum value of  $f(x)$ ,  $f_1(x)$  is positive, and immediately after, negative.

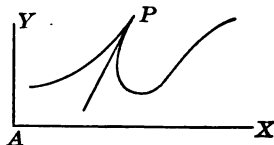
**Prin. 3.** Immediately before a minimum value of  $f(x)$ ,  $f_1(x)$  is negative, and immediately after, positive.

**706. Caution 1.**—A root of  $f_1(x) = 0$  is not necessarily the abscissa of a turning point. For a tangent line to a graph may be parallel to the  $x$ -axis where there is no turning point, as where two branches tangent to the same line coalesce at the point of tangency. (See diagram.)



It is only when Prin. 2 or Prin. 3 is satisfied, as well as  $f_1(x) = 0$ , that a turning point is established.

**Caution 2.**—There may be turning points under peculiar conditions when  $f_1(x) \neq 0$ . For there may be turning points where the tangent line to the graph is not parallel to the  $x$ -axis; as where two branches coalesce and cease. (See diagram.)



**707. Observations.**—1. So long as  $f(x)$  remains continuous, its maxima and minima values succeed each other alternately.

2. If two successive turning values of  $f(x)$  have the same sign, the graph of  $f(x)$  between these values can not cross the  $x$ -axis, or  $f(x) \neq 0$  between these values.

3. If two successive turning values of  $f(x)$  have opposite signs, the graph of  $f(x)$  must cross the  $x$ -axis between these values, or  $f(x) = 0$  somewhere between these values.

4. If  $x = a$  and  $x = b$  render  $f(x) = 0$ , and  $a \neq b$ , there must be a turning value of  $f(x)$  between  $x = a$  and  $x = b$ .

**Example.**—Find the turning values of

$$f(x) = x^3 - 9x^2 + 24x + 16.$$

**Solution:**  $f(x) = x^3 - 9x^2 + 24x + 16$

$$f_1(x) = 3x^2 - 18x + 24 = 0;$$

or,  $f_1(x) = x^2 - 6x + 8 = 0;$

whence,  $x = 4$  or  $2$ , critical values.

$$f_1(x - \Delta x) \left\{ \begin{array}{l} x = 4 \\ \Delta x = 0 \end{array} \right\} = (4 - \Delta x)^2 - 6(4 - \Delta x) + 8 = -$$

$$f_1(x + \Delta x) \left\{ \begin{array}{l} x = 4 \\ \Delta x = 0 \end{array} \right\} = (4 + \Delta x)^2 - 6(4 + \Delta x) + 8 = +$$

$\therefore f(x)$  is a minimum when  $x = 4$

But  $f(x)_{x=4} = 4^3 - 9 \times 4^2 + 24 \times 4 + 16 = 32.$

$\therefore$  Minimum value of  $f(x) = 32$

$$f_1(x - \Delta x) \left\{ \begin{array}{l} x = 2 \\ \Delta x = 0 \end{array} \right\} = (2 - \Delta x)^2 - 6(2 - \Delta x) + 8 = +$$

$$f_1(x + \Delta x) \left\{ \begin{array}{l} x = 2 \\ \Delta x = 0 \end{array} \right\} = (2 + \Delta x)^2 - 6(2 + \Delta x) + 8 = -$$

$\therefore f(x)_{x=2}$  is a maximum

But  $f(x)_{x=2} = 2^3 - 9 \times 2^2 + 24 \times 2 + 16 = 36$

$\therefore$  Maximum value of  $f(x) = 36.$

The value of  $f(x)_{x=a}$  is best obtained by synthetic division, as in Art. 106.

#### EXERCISE 101.

Find the maxima and minima values of :

1.  $4x^3 - 15x^2 + 12x - 1$

5.  $x^3 - 3x^2 - 9x + 5$

2.  $2x^3 - 21x^2 + 36x - 20$

6.  $(x-1)^4(x+2)^3$

3.  $x^2 + 6x + 5$

7.  $(x-a)^2(x+b)^2$

4.  $x^2 - 6x + 5$

8.  $x^3 - 3x^2 + 3x + 7$

9.  $x^2 + 6x - 5$

11.  $x^4 - 8x^2 + 16$

10.  $x^2 - 6x - 5$

12.  $x^4 + x^3 + x^2 - 16$

13. Show where a line  $a$  feet long must be divided so that the rectangle of the two parts may be the greatest possible.

14. Find the altitude of the maximum cylinder that can be inscribed in a sphere whose radius is  $r$ .

**Suggestion.**—Let  $BC = x$ ,  $BD = r - x$ , and  $AB = y$ , then,  $y^2 = (r + x)(r - x) = r^2 - x^2$ , and

$$f(x) = V = \pi y^2 \times 2x = 2\pi x(r^2 - x^2)$$

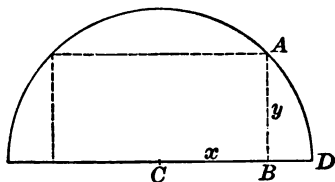
$$f_1(x) = 2\pi x \times (-2x)$$

$$+ (r^2 - x^2) \times 2\pi = 0$$

whence,  $x = \frac{r}{3} \sqrt{3}$

and,  $y^2 = r^2 - x^2 = \frac{2}{3} r^2$

$$y = \frac{2}{3} r \sqrt{3}$$



15. Find the altitude of the maximum cylinder that can be inscribed in a cone whose altitude is  $a$  and whose radius is  $b$ .

16. Find the volume of the maximum cone that can be inscribed in a given sphere.

17. Find the area of the maximum rectangle that can be inscribed in a square whose side is  $a$ .

18. What is the maximum convex surface of a cylinder the sum of whose altitude and diameter is a constant  $a$ ?

19. Find the volume of the maximum cone that may be generated by revolving a right triangle, the sum of whose legs is  $a$ , about one of the legs as an axis.

20. Required the area of the maximum rectangle that can be inscribed in a given circle.

21. Required the greatest right triangle which can be constructed upon a given line as hypotenuse.

## CHAPTER XII.

### *THEORY OF EQUATIONS.*

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#### Introduction.

**708.** Equations of the first and second degree have already been treated, and need no further attention here.

**709.** Jerome Cardan, an Italian mathematician (1501–1576), published in 1545 a method of solving cubic equations, now known as “Cardan’s Formula.” But, as this formula is not finally reducible when the roots of an equation are real and unequal, it is not of much practical value.

**710.** René Descartes, a French mathematician (1596–1650), transformed the general bi-quadratic equation so as to make its solution depend upon that of the cubic equation; but, as he invented no new method of solving the latter, the same difficulties are encountered in the application of his rule as are met in Cardan’s.

**711.** Nicholas Henry Abel, a Norwegian mathematician (1802–1829), demonstrated, in 1825, the impossibility of a general solution of an equation of a higher degree than the fourth. Previous to that date many such solutions were attempted.

**712.** The real roots of *numerical equations* of any degree are, however, attainable through laws and principles to be developed in this chapter.

## Normal Forms.

**713. Theorem I.**—Every equation of one unknown quantity with real and rational coefficients can be transformed into an equation of the form of

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \dots + L = 0,$$

in which  $A$  and all the exponents of  $x$  are positive integers, and each of the remaining coefficients, including  $L$ , is either an integer or zero.

**Note.**  $L$  may be regarded the coefficient of  $x^0$ .

**Demonstration.**—1. If the equation contains fractional terms, it may be cleared of fractions.

2. If there are any terms in the second member, they may be transposed to the first member.

3. All terms containing like exponents of  $x$  may be collected into one term by addition.

4. If  $A$  is negative, both members may be divided by  $-1$ .

5. If  $x$  contains negative exponents, both members may be multiplied by  $x$  with a positive exponent numerically equal to the greatest negative exponent.

6. If  $x$  contains fractional exponents,  $x^m$  may be substituted for  $x$ , in which  $m$  is the L. C. M. of the denominators of the fractional exponents.

The roots of the transformed equation will be the  $m$ th root of the roots of the original equation.

7. The terms may now be arranged according to the descending powers of  $x$ .

**714.** The equation

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \dots + L = 0,$$

is known as the *first normal form* of an equation of one unknown quantity, and will hereafter be represented by  $F_n(x) = 0$ .

**Example.**—Transform  $3x^{\frac{3}{2}} + \frac{4}{x^{\frac{1}{2}}} - 8 + 7x^{-\frac{3}{2}} = \frac{4}{3} + \frac{3}{x^{\frac{1}{2}}}$

into the first normal form, and compare the corresponding roots of the two equations.

**Solution:** Given  $3x^{\frac{3}{2}} + \frac{4}{x^{\frac{1}{2}}} - 8 + 7x^{-\frac{1}{2}} = \frac{4}{3} + \frac{8}{x^{\frac{1}{2}}}$ . (A)

Clear of fractions,  $9x + 12 - 24x^{\frac{1}{2}} + 21x^{-\frac{1}{2}} = 4x^{\frac{1}{2}} + 9x^{\frac{1}{2}}$  (B)

Transpose and collect terms,

$$9x + 21x^{-\frac{1}{2}} - 28x^{\frac{1}{2}} - 9x^{\frac{1}{2}} + 12 = 0 \quad (C)$$

Multiply by  $x^{\frac{1}{2}}$ ,

$$9x^{\frac{3}{2}} + 21 - 28x^{\frac{1}{2}} - 9x^{\frac{1}{2}} + 12x^{\frac{1}{2}} = 0 \quad (D)$$

Put  $x = x^2$ ,  $9x^2 + 21 - 28x^{\frac{1}{2}} - 9x^{\frac{1}{2}} + 12x^{\frac{1}{2}} = 0$  (E)

Rearrange terms,

$$9x^2 + 0x^{\frac{3}{2}} + 0x^{\frac{1}{2}} + 0x^0 - 28x^{\frac{1}{2}} - 9x^{\frac{1}{2}} + 12x^{\frac{1}{2}} + 0x^2 + 0x + 21 = 0 \quad (F)$$

The roots of (A) =  $\sqrt[3]{\text{of the roots of (F)}}$ .

**715.** An equation that contains all the powers of  $x$ , from the highest to the lowest, is called a *Complete Equation*. An incomplete equation may be written in the form of a complete equation by supplying the wanting terms with coefficients of *zero*.

Thus,  $x^5 - 4x^3 + 2x - 5 = 0$  may be written  $x^5 \pm 0x^4 - 4x^3 \pm 0x^2 + 2x - 5 = 0$ .

**716. Theorem II.**—The equation  $F_n(x) = 0$  may be transformed into an equation of the form of

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0,$$

in which the coefficient of  $x^n$  is unity, and each of the remaining coefficients is either an integer or zero.

**Demonstration.**—

Take  $F_n(x) = Ax^n + Bx^{n-1} + Cx^{n-2} + \dots + L = 0$

Put  $x = \frac{x}{A}, \frac{Ax^n}{A^n} + \frac{Bx^{n-1}}{A^{n-1}} + \frac{Cx^{n-2}}{A^{n-2}} + \dots + L = 0$

Multiply by  $A^{n-1}$ ,

$$x^n + Bx^{n-1} + ACx^{n-2} + \dots + A^{n-1}L = 0$$

Put  $p_1$  for  $B$ ,  $p_2$  for  $AC$ ,  $\dots$   $p_n$  for  $A^{n-1}L$ ,

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0$$

This is the *second normal form* of an equation of one unknown quantity, and will hereafter be represented by  $f_n(x) = 0$ .

**717. Cor. 1.**—Each root of  $f_n(x) = 0$  is  $A$  times as great as the corresponding root of  $F_n(x) = 0$ .

**718. Cor. 2.**—The coefficient of the second term of  $f_n(x)$  is the same as the coefficient of the second term of  $F_n(x)$ , and the succeeding coefficients of  $f_n(x)$  are obtained by multiplying the succeeding coefficients of  $F_n(x)$ , in order, by  $A, A^2, A^3, \dots, A^{n-1}$ .

**Note.**—If terms are wanting, supply them with coefficients of 0.

**Example.**—Transform the equation  $4x^5 - 3x^4 + 2x^2 - 7 = 0$  into an equation of the form of  $f(x) = 0$ .

**Solution:**

Given  $F(x) = 4x^5 - 3x^4 + 0x^3 + 2x^2 + 0x - 7 = 0$ ,

then will  $f(x) = x^5 - 3x^4 + 4 \times 0x^3 + 4^2 \times 2x^2 + 4^3 \times 0x - 4^4 \times 7 = 0$  [718]

or,  $f(x) = x^5 - 3x^4 + 32x^2 - 1792 = 0$ .

The roots of  $f(x) = 0$  are 4 times as great as those of  $F(x) = 0$ .

#### EXERCISE 102.

Transform the following equations into equations of the form of  $f(x) = 0$ . Compare the roots of the transformed equation with the roots of the original equation.

1.  $3x^4 + 2x^3 - 3x^2 + 7x - 5 = 0$

2.  $2x^5 + 4x^4 - x^3 + x^2 - 7 = 0$

3.  $4x^6 + 3x^4 - 5x^2 + 7x - 1 = 0$

4.  $3x^{\frac{5}{2}} - 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}} - 2x^{\frac{1}{2}} + 4x^{\frac{1}{2}} - 2 = 0$

5.  $x^{-\frac{5}{2}} + 2x^{-\frac{3}{2}} + 3x^{-\frac{1}{2}} - x^{-\frac{1}{2}} + x^{-\frac{1}{2}} - 2x^{-\frac{1}{2}} + 2 = 0$

6.  $\frac{3}{4}x + \frac{5}{6}x^{-\frac{2}{3}} + \frac{2}{3}x^{\frac{1}{3}} - \frac{1}{3}x^{-2} + 3 = 0$

7.  $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 2x^{\frac{1}{3}} + 3x^{-\frac{1}{3}} + 2 = 0$

8.  $\frac{5}{6}x^{\frac{1}{2}} + \frac{3}{4}x^{\frac{3}{2}} - \frac{1}{3}x + 1 = 0$

9.  $\frac{2}{3} + \frac{5}{6}x - \frac{3}{4}x^{-\frac{1}{2}} + 5 = 0$

## Divisibility of Equations.

**719. Theorem III.**—If  $a$  is a root of  $F_n(x) = 0$ , then  $x - a$  is a factor of  $F_n(x)$ .

For, let  $F_n(x) \div (x - a) = F_{n-1}(x) + \frac{r}{x - a}$

then,  $\{F_{n-1}(x)\}(x - a) + r = F_n(x) = 0$

but,  $x - a = 0$ , since  $x = a$ .

$\therefore r = 0$ ;

whence,  $F_n(x) \div (x - a) = F_{n-1}(x)$ .

**720. Cor. 1.**—If  $a$  is an integral root of  $F_n(x) = 0$ , it is a divisor of the absolute term of  $F_n(x)$  [163].

**721. Cor. 2.**—If  $x - a$  is a factor of  $F_n(x)$ , then  $a$  is a root of  $F_n(x) = 0$ .

For,  $F_n(x) = \{F_{n-1}(x)\}(x - a) = 0$ ;

whence,  $x - a = 0$ , and  $x = a$ .

**722. Cor. 3.**—If  $x$  is a factor of  $F_n(x)$ , then zero is a root of  $F_n(x) = 0$ .

## Number of Roots.

**723. Theorem IV.**  $F_n(x) = 0$  has at least one root.

The demonstration of this theorem may be found in special treatises on the Theory of Equations. It is too long and tedious to be introduced here.

**724. Theorem V.**  $F_n(x) = 0$  has  $n$  roots and only  $n$ .

For,  $F_n(x) = 0$  has at least one root. [T. IV.]

Let  $a =$  one root of  $F_n(x) = 0$ ;

then,  $F_n(x) = \{F_{n-1}(x)\}(x - a) = 0$  [T. III.]

$\therefore F_{n-1}(x) = 0$ .

Let  $b =$  one root of  $F_{n-1}(x) = 0$ ; [T. IV.]

then,  $F_{n-1}(x) = \{F_{n-2}(x)\}(x - b) = 0$  [T. III.]

$\therefore F_{n-2}(x) = 0$ .

Now, as  $F_n(x) = 0$  is of the  $n$ th degree, and each time a root is removed by division the degree is lowered by unity, it follows that  $n$  roots and only  $n$  can be removed before  $F_n(x)$  reduces to an absolute factor. Therefore,  $F_n(x) = 0$  has  $n$  roots and only  $n$ .

**725. Cor.**  $F_n(x) = 0$  may be written

$$A(x-a)(x-b)(x-c)\dots(x-l) = 0;$$

or simply  $(x-a)(x-b)(x-c)\dots(x-l) = 0$ , in which there are  $n$  factors of the form of  $x-r$ , the second terms of which are the roots of  $F_n(x) = 0$  with their signs changed, and may be positive or negative, fractional or integral, rational, irrational, or imaginary, subject only to restrictive conditions explained hereafter.

### Relation of Roots to Coefficients.

**726. Theorem VI.**—If  $F_n(x) = 0$  be put in the form of  $x^n + B_1x^{n-1} + C_1x^{n-2} + \dots + L_1 = 0$ , by dividing both members of the equation by  $A$ , the coefficient of  $x^n$ , then will

1.  $B_1$  = the sum of the roots with their signs changed.
2.  $C_1$  = the sum of the products of the roots taken two together.
3.  $D_1$  = the sum of the products of the roots with their signs changed, taken three together.
4.  $E_1$  = the sum of the products of the roots taken four together. And so on to
5.  $L_1$  = the product of all the roots with their signs changed.

**Demonstration:** Let the  $n$  roots of the equation be  $a, b, c, \dots, l$ ; then,  $F_n(x) = x^n + B_1x^{n-1} + C_1x^{n-2} + \dots + L_1$

$$= (x-a)(x-b)(x-c)\dots(x-l) \text{ [725].}$$

After which the theorem is a direct inference from the binomial formula [587], and the principle that "changing the signs of an even

number of factors does not change the sign of their product" [page 26, Ex. 3].

**Cor.**—*Changing the signs of the alternate terms of  $F_n(x) = 0$  changes the signs of its roots.*

### Imaginary Roots.

**727. Theorem VII.**—*Imaginary roots can enter  $F_n(x) = 0$  only in conjugate pairs.*

For in this way only will their sum and the sum of their products be real [657], as they must be [713].

**728. Cor. 1.**—*The product of the imaginary roots of  $F_n(x) = 0$  is positive.*

For the product of each pair is positive.

Thus,  $(a + bi)(a - bi) = a^2 + b^2$ .

**729. Cor. 2.**—*When all the roots of  $F_n(x) = 0$  are imaginary the absolute term is positive.*

**Suggestion.**—For the equation is then of an even degree.

**730. Cor. 3.**  $F_n(x) = 0$  *has at least one real root opposite in sign to the absolute term, when  $n$  is odd.*

**731. Cor. 4.**  $F_n(x) = 0$  *has at least two real roots, one positive and the other negative, if  $n$  is even and the absolute term is negative.*

**732. Cor. 5.**—*The sign of  $F_n(x)$  for any real value of  $x$  depends on the real roots of  $F_n(x) = 0$ .*

For the product of  $x - (a + bi)$  and  $x - (a - bi) = (x - a)^2 + b^2$ , a positive quantity; and this is true of every pair of factors containing conjugate imaginary terms.

**733. Cor. 6.**—*Every entire function of  $x$  with real and rational coefficients may be divided into real factors of the first or second degree.*

### Fractional Roots.

**734. Theorem VIII.**—No root of  $f_n(x) = 0$  can be a rational fraction.

Take  $f_n(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0$  [716]. If possible, let  $x = \frac{a}{b}$ , a rational fraction in its lowest terms. Then, by substitution,

$$\frac{a^n}{b^n} + \frac{p_1 a^{n-1}}{b^{n-1}} + \frac{p_2 a^{n-2}}{b^{n-2}} + \dots + p_n = 0.$$

Multiplying by  $b^{n-1}$ , and transposing terms, we have

$$\frac{a^n}{b} = -p_1 a^{n-1} - p_2 a^{n-2} b - \dots - p_n b^{n-1} =$$

an integer, which is impossible.

*Scholium.*—From this theorem it follows that the rational fractional roots of  $F_n(x) = 0$  may be obtained by transforming  $F_n(x) = 0$  into  $f_n(x) = 0$  and dividing the roots of the latter equation by  $A$ , the coefficient of  $x^{n-1}$  in the former.

### Relations of Roots to Signs of Equation.

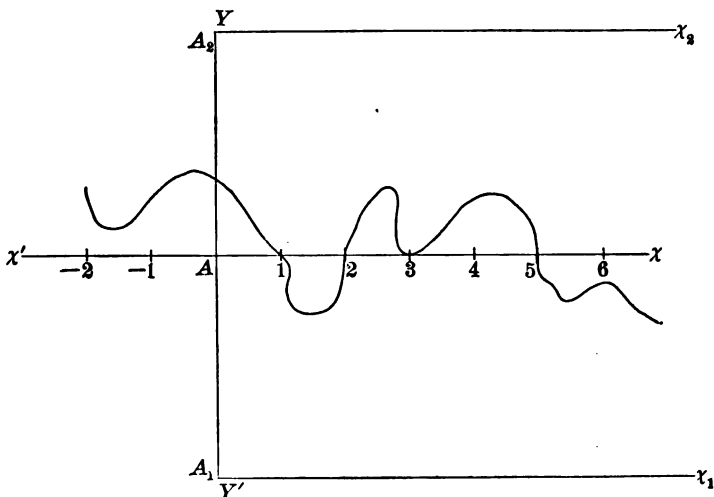
**735. Theorem IX.**—If  $F_n(x) = 0$  has no equal roots, then  $F_n(x)$  will change sign if  $x$  passes through a real root.

For, take  $F_n(x) = (x-a)(x-b)(x-c) \dots (x-l) = 0$  [725]; conceive  $x$  to start with a value less than the least root and continually increase until it becomes greater than the greatest root. At first, every factor of  $F_n(x)$  is negative, but, at the instant it becomes greater than the least root, the sign of the factor containing that root will become plus, while the others remain minus; whence,  $F_n(x)$  will change sign. It will, moreover, retain its new sign until it passes over the next greater root, when it will again change sign, and so on.

**736. Cor. 1.**—If for any two assigned values of  $x$ ,  $F_n(x)$  has different signs, one, or, if more than one, an odd number of roots of  $F_n(x) = 0$  lie between these values.

**737. Cor. 2.**—If for any two assigned values of  $x$ ,  $F_n(x)$  has the same sign, either no root or an even number of roots of  $F_n(x) = 0$  lie between these values.

**738.** Some of the properties of  $F_n(x) = 0$ , already discussed, are beautifully illustrated by the following graph.



1. It is seen that  $y = F_n(x) = 0$  when  $x = 1, 2, 3$ , and  $5$ . Therefore, these values of  $x$  are roots of  $F_n(x) = 0$ .

2. Immediately before  $x = 1$ ,  $y$  is positive, and immediately after  $x = 1$ ,  $y$  is negative; immediately before  $x = 2$ ,  $y$  is negative, and immediately after  $x = 2$ ,  $y$  is positive, etc.; illustrating that when  $x$  passes over a real root,  $F_n(x)$  changes sign.

3. At  $x = 3$  two values of  $y$  become zero; therefore, two roots become identical, or, in other words, 3 is twice a root. Were the absolute term of  $F_n(x)$  so changed as to make  $y$  somewhat less, the  $x$ -axis would cross the graph twice between  $x = 2$  and  $x = 4$ , once before  $x = 3$ , and once after, thus proving conclusively the duality of the root 3, when  $y = 0$ .

4. Immediately before  $x = -2$  and  $x = 6$ , the graph approaches

the  $x$ -axis, but in each case makes a turn before reaching it, preventing, thereby, equal roots or unequal real roots. These turns locate the position of imaginary roots. The truth of this statement becomes manifest when we suppose the absolute term of  $F_n(x)$  to so change as to cause  $y$  to gradually decrease, the  $x$ -axis will gradually arise and finally touch the graph at  $x = -2$ , thereby making two equal roots, and, if  $y$  continues to decrease, the  $x$ -axis will cross both branches above the turn at  $x = -2$ , making two unequal real roots.

The student will be interested in observing the changes in the roots if the absolute term of the equation so changes as to cause the  $x$ -axis to gradually move from the position  $A_1\chi_1$  to the position  $A_1\chi_2$ .

5. It must not be assumed, however, that imaginary roots always denote a turning point in the graph of the equation. Such may or may not be the case.

**739.** If any two successive terms in a complete equation have like signs, there is a *permanence* of sign; if unlike signs, a *variation* of sign. Thus, in the equation

$$x^6 - 5x^5 + 8x^4 + 7x^3 - 3x^2 + 2x - 5 = 0$$

there are five variations and one permanence.

**740. Theorem X.**—No complete equation has a greater number of positive roots than there are variations of sign, nor a greater number of negative roots than there are permanences of sign.

**Demonstration:** Let the following be the successive signs of a complete equation:

+   -   -   +   +   -

There are here two permanences and three variations. To introduce another positive root, the equation must be multiplied by  $x - a$ .

The signs of the product will readily appear from the following work:

+	-	-	+	+	-	
+	-					
+	-	-	+	+	-	
		-	+	+	-	+
+	-	±	+	±	-	+

The double sign denotes a doubt, growing out of an ignorance of the relative numerical magnitudes of the terms added.

Now, a careful inspection will show that, whether we regard both doubtful signs negative, both positive, or one negative and the other

positive, the number of permanences will not be increased, but the number of terms is increased by one; therefore, the number of variations must be increased by *at least one*. Since the introduction of a positive root introduces at least one variation, it follows that the number of positive roots can not exceed the number of variations.

In a similar manner, by introducing the factor  $x + a$ , it may be shown that the number of negative roots can not exceed the number of permanences of sign.

This is Descartes' celebrated rule of signs.

**741. Cor. 1.**—*If all the roots of an equation are real, the number of variations equals the number of positive roots, and the number of permanences equals the number of negative roots.*

**742. Cor. 2.**—*An equation whose terms are all positive can have no positive roots.*

**743. Cor. 3.**—*An equation whose terms are alternately positive and negative can have no negative roots.*

### Limits of Roots.

**744.** A number known to be equal to or larger than the largest root of an equation is called a *superior limit* to the roots of the equation.

**745.** A number known to be equal to or smaller than the smallest root of an equation is called an *inferior limit* to the roots of the equation.

**746. Theorem XI.**—*If the first  $h$  coefficients of  $F_n(x)$  are positive, and  $P$  is the smallest of them, then, if  $Q$  is numerically the largest subsequent coefficient,  $\sqrt[n]{\frac{Q}{P}} + 1$  is a superior limit to the roots of  $F_n(x) = 0$ .*

**Demonstration:** It is evident that the case in which  $x$  must have the greatest value to make  $F_n(x) = 0$  when the first  $h$  coefficients are positive, is the one in which these coefficients are all equal to the least one of them ( $P$ ), and the remaining  $n + 1 - h$  coefficients are all

negative and each equal to the greatest among them ( $Q$ ). Therefore, the value of  $x$  is a superior limit to the roots of  $F_n(x) = 0$ , if

$$Px^n + Px^{n-1} + \dots + Px^{n+1-k} = Qx^{n-k} + Qx^{n-k-1} + \dots + Q;$$

$$\text{or,} \quad \frac{Px^{n+1} - Px^{n+1-k}}{x-1} = \frac{Qx^{n+1-k} - Q}{x-1}$$

$$\text{or,} \quad Px^{n+1-k}(x^k - 1) = Q(x^{n+1-k} - 1)$$

$$\text{or,} \quad P(x^k - 1) = Q\left(1 - \frac{1}{x^{n+1-k}}\right)$$

$$\text{or if,} \quad P(x^k - 1) = Q, \text{ since } 1 > \left(1 - \frac{1}{x^{n+1-k}}\right)$$

$$\text{or} \quad x = \sqrt[k]{\frac{Q}{P} + 1}.$$

**747. Cor.**—If the signs of the alternate terms of an equation be changed, then will the superior limit to the roots of the transformed equation, with its sign changed, be the inferior limit to the roots of the original equation [726, Cor.].

### Equal Roots.

**748. Theorem XII.**—If  $F_n(x) = 0$  has equal roots, it may be separated into two or more equations with unequal roots.

This is a direct inference from Art. 701.

### Commensurable Roots.

**749.** The integral and rational fractional roots of  $F_n(x) = 0$  are called its *commensurable roots*.

**750. Problem 1.** To find the commensurable roots of  $F_n(x) = 0$ .

**Solution:** Pursue the following line of investigation:

1. Determine the number of roots the equation has [724].
2. Determine how many roots may be positive and how many negative [740].
3. Determine the limit to the positive and the negative roots [746, 747].

4. Determine what integral numbers may be roots [720].
5. Find and remove the integral roots by synthetic division [719, 105].
6. Determine whether there are any equal roots [701], and if so, remove them by synthetic division.
7. Find the rational fractional roots from the equation resulting from the removal of the integral roots, and according to Theorem VIII, Scholium.

**Illustrations.**—1. Find the commensurable roots of

$$F_4(x) = 24x^4 + 122x^3 + 5x^2 - 26x - 5 = 0.$$

**Solution :**

1. This equation has four roots, all real, or two real [724, 731].
2. There are one variation and three permanences of sign; therefore, there can not be more than one positive nor more than three negative roots [740].
3. The only integral roots possible are +1, -1, +5, and -5 [720].

4. The largest positive root  $< \sqrt[3]{\frac{26}{5}} + 1$ , or  $< 2$  [746].

5. Neither +1 nor -1 is a root, since  $F_4(x)$  is not divisible by either  $x-1$  or  $x+1$ , as witness:

$$\begin{array}{r} -1) 24 + 122 + \quad 5 - 26 - \quad 5 \\ \quad - 24 - \quad 98 + \quad 93 - \quad 67 \\ \hline \quad \quad 98 - \quad 93 + \quad 67 - \quad 72^* \end{array}$$

$$\begin{array}{r} +1) 24 + 122 + \quad 5 - 26 - \quad 5 \\ \quad 24 + 146 + 151 + 125 \\ \hline \quad \quad 146 + 151 + 125 + 120^* \end{array}$$

**Note.**—It is evident that when +1 is a root of  $F_n(x) = 0$ , the sum of the positive coefficients must equal the sum of the negative coefficients; and, if -1 is a root, +1 is a root if the signs of the alternate terms are changed. These facts determine a more expeditious method of testing whether either +1 or -1 is a root.

6. -5 is a root, as witness:

$$\begin{array}{r} -5) 24 + 122 + \quad 5 - 26 - \quad 5 \\ \quad - 120 - 10 + 25 + \quad 5 \\ \hline \quad \quad + \quad 2 - \quad 5 - \quad 1 \end{array}$$

7. The resulting equation, after removing the root -5, is  $F_3(x) = 24x^3 + 2x^2 - 5x - 1 = 0$ , which has no integral roots.

Transform  $F_3(x) = 0$  into an equation of the form of  $f_3(x) = 0$ ,  $f_3(x) = x^3 + 2x^2 - 120x - 576 = 0$  [734, Sch.].

8.  $f_2(x) = 0$  has three roots [724], only one of which can be positive, and the largest positive root possible is  $\sqrt{577} = 24$ .

9. The divisors of 576 not exceeding 24 are 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24. From the relative values of the positive and negative coefficients it will be seen at a glance that  $x > 6$ .

10. + 8 and + 9 are not roots, but + 12 is a root, as witness:

$$\begin{array}{r}
 + 8) 1 + 2 - 120 - 576 \\
 \quad + 8 + 80 - 320 \\
 \hline
 \quad + 10 - 40 - 896 * \\
 + 9) 1 + 2 - 120 - 576 \\
 \quad + 9 + 99 - 189 \\
 \hline
 \quad + 11 - 21 - 765 * \\
 + 12) 1 + 2 - 120 - 576 \\
 \quad + 12 + 168 + 576 \\
 \hline
 \quad 14 + 48
 \end{array}$$

11. The resulting equation, after removing the root + 12 from  $f_2(x) = 0$ , is  $f_3(x) = x^2 + 14x + 48 = 0$ ; whose roots are found to be - 8 and - 6 [331].

12. The four roots of  $F_4(x) = 0$  are, therefore, - 5, +  $\frac{12}{24}$ , -  $\frac{8}{24}$ , and -  $\frac{6}{24}$  [734, Sch.], or - 5, +  $\frac{1}{2}$ , -  $\frac{1}{3}$ , and -  $\frac{1}{4}$ .

2. Find the commensurable roots of

$$\begin{aligned}
 f_7(x) &= x^7 + 3x^6 - 12x^5 - 36x^4 + 48x^3 + \\
 &\quad 144x^2 - 64x - 192 = 0.
 \end{aligned}$$

**Solution:** It may readily be found by synthetic division that + 2, - 2, and - 3 are the only integral roots of this equation.

The resulting equation, after removing these roots, is  $f_4(x) = x^4 - 8x^2 + 16 = 0$ . If any of the roots of this equation are integral, they must be equal to one or more of the roots already found. They may, however, all be incommensurable or imaginary.

Factoring  $f_4(x)$ , we have  $(x^2 - 4)(x^2 - 4) = 0$ ;

whence,  $x = + 2, - 2, + 2, - 2$ .

Therefore, all the roots of  $f_7(x) = 0$  are  $\pm 2, \pm 2, \pm 2$ , and - 3.

#### EXERCISE 103.

Find the commensurable roots of:

1.  $x^3 - 3x^2 + 7x - 5 = 0$
2.  $x^3 - 6x^2 + 10x - 8 = 0$
3.  $x^3 - 11x^2 + 41x - 55 = 0$

4.  $x^3 + 6x^2 + 14x + 12 = 0$
5.  $x^4 - 3x^3 - 2x^2 + 12x - 8 = 0$
6.  $12x^3 + 8x^2 - 3x - 2 = 0$
7.  $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$
8.  $x^4 - 8x^3 + 10x^2 + 24x + 5 = 0$
9.  $x^5 + 3x^4 - 3x^3 - 9x^2 - 4x - 12 = 0$
10.  $x^4 - 5x^3 + 3x^2 + 2x + 8 = 0$
11.  $8x^3 - 16x^2 - 8x + 21 = 0$
12.  $16x^4 - 48x^3 + 32x^2 + 12x - 9 = 0$
13.  $3x^5 + 2x^4 - 21x^3 - 14x^2 + 36x + 24 = 0$
14.  $9x^5 + 81x^4 + 203x^3 + 99x^2 - 92x - 60 = 0$
15.  $18x^5 + 9x^4 + 22x^3 + 11x^2 - 96x - 48 = 0$
16.  $x^4 + 4x^3 - 13x^2 - 28x + 60 = 0$
17.  $x^4 + 2x^3 - 11x^2 - 12x + 36 = 0$
18.  $x^5 + 4x^4 + x^3 - 10x^2 - 4x + 8 = 0$
19.  $3x^5 + 14x^4 + 8x^3 - 42x^2 - 79x - 52 = 0$

### Incommensurable Roots.

**751.** The *incommensurable roots* of an equation are best sought for after all the commensurable roots have been removed by division and the resulting equation transformed into an equation of the form of  $f_*(x) = 0$ .

**752.** The first step necessary in the search for the values of the incommensurable roots of an equation is to find the number and situation of such roots.

Jacques Charles François Sturm, a Swiss mathematician (1803-1855), discovered a method of doing this in 1829, known as *Sturm's method*.

**753. Sturm's Series of Functions.**—Assuming that  $f_n(x) = 0$  has no equal roots, this eminent mathematician formed a series of functions, as follows :

The first two terms of the series are  $f_n(x)$ , and its first derivative, which we will now represent by  $f_{n-1}(x)$ .

The other functions, and which are called Sturmiian functions, are derived as follows : Divide  $f_n(x)$  by  $f_{n-1}(x)$ , and represent the remainder *with its sign* changed by  $f_{n-2}(x)$ . Divide  $f_{n-1}(x)$  by  $f_{n-2}(x)$ , and represent the remainder with its sign changed by  $f_{n-3}(x)$  ; continue this process until the last remainder with its sign changed is an absolute term. Represent this remainder  $f_0(x)$ . There will then be  $n + 1$  of these functions, as follows :

$$f_n(x), f_{n-1}(x), f_{n-2}(x) \dots f_0(x).$$

**Caution.**—Care must be taken in the operation of successive division not to reject any negative factors except in the remainders.

**754. Relation of the terms of Sturm's series of functions.**—If we put  $q_1, q_2, q_3 \dots$  as the successive quotients obtained in finding the Sturmiian functions, it is evident that

$$f_n(x) = f_{n-1}(x) q_1 - f_{n-2}(x) \quad (1)$$

$$f_{n-1}(x) = f_{n-2}(x) q_2 - f_{n-3}(x) \quad (2)$$

$$f_{n-2}(x) = f_{n-3}(x) q_3 - f_{n-4}(x) \quad (3)$$

$$f_{n-3}(x) = f_{n-4}(x) q_4 - f_{n-5}(x) \quad (4)$$

$$f_{n-4}(x) = f_{n-5}(x) q_5 - f_{n-6}(x) \quad (5)$$

$$\text{etc.,} \quad \text{etc.,} \quad \text{etc.}$$

### 755. Fundamental Principles.

1. No two consecutive functions can vanish, i. e., become 0, for the same value of  $x$ .

For, if possible, let  $x = a$  make  $f_{n-2}(x) = 0$  and  $f_{n-3}(x) = 0$ ; then will  $f_{n-4}(x) = 0$  [754, 3], and hence, too,  $f_{n-5}(x) = 0$

[754, 4], and so on until lastly  $f_0(x) = 0$ ; but  $f_0(x)$  is the absolute term and can not be zero. Therefore, etc.

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2. *If any one of the functions intervening between  $f_n(x)$  and  $f_0(x)$  vanishes for any value of  $x$ , the two adjacent functions have opposite signs for this value.*

Thus, if  $x = a$  causes  $f_{n-3}(x)$  to vanish,  $f_{n-2}(x) = -f_{n-4}(x)$  [754, 3].

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3. *If any value of  $x$ , as  $x = a$ , causes any intervening function to vanish, then will the number of variations and the number of permanences in the signs of the functions be the same for the immediately preceding and the immediately succeeding values of  $x$ , i. e., for  $x = a - 0$  and  $x = a + 0$ .*

For the two adjacent functions will have opposite signs when  $x = a$  [755, 2], and will not change their signs for any value of  $x$  from  $x = a - 0$  and  $x = a + 0$ , since no root of either can lie between these values [755, 1]. But the function in question does change its sign, since  $x$  passes over a root of the function in going from  $x = a - 0$  to  $x = a + 0$ . If the signs of the three functions for  $x = a - 0$  are  $+$ ,  $+$ ,  $-$ , for  $x = a + 0$  they will be  $+$ ,  $-$ ,  $-$ , which in either case form one permanence and one variation. Similarly,  $+$ ,  $-$ ,  $-$  will change to  $+$ ,  $+$ ,  $-$ ;  $-$ ,  $+$ ,  $+$  will change to  $-$ ,  $-$ ,  $+$ ; and  $-$ ,  $-$ ,  $+$  will change to  $-$ ,  $+$ ,  $+$ .

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4. *If any value of  $x$  causes  $f_n(x)$  to vanish, then will one variation in the signs of the functions be lost in passing from the immediately preceding value of  $x$  to the immediately succeeding value.*

Let  $f_n(x) =$

$$(x - a_1)(x - a_2)(x - a_3) \dots (x - a_n);$$

then  $f_{n-1}(x) =$

$$(x - a_2)(x - a_3)(x - a_4) \dots (x - a_n) + \quad (1\text{st term})$$

$$(x - a_1)(x - a_3)(x - a_4) \dots (x - a_n) + \quad (2\text{d term})$$

$$(x - a_1)(x - a_2)(x - a_4) \dots (x - a_n) + \quad (3\text{d term})$$

$$(x - a_1)(x - a_2)(x - a_3) \dots (x - a_n) + \quad (4\text{th term})$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$(x - a_1)(x - a_2)(x - a_3) \dots (x - a_{n-1}) \quad (n\text{th term})$$

Now, if  $x$  equals, say  $a_3$ , then will  $f_n(x)$  and all the terms of  $f_{n-1}(x)$ , except the third, vanish.

Now, the third term of  $f_{n-1}(x)$  contains all the factors of  $f_n(x)$  except  $x - a_3$ . Therefore, if  $x$  is infinitesimally less than  $a_3$ ,  $x - a_3$  will be negative and  $f_n(x)$  and  $f_{n-1}(x)$  will have opposite signs or will form a variation; but, if  $x$  is infinitesimally greater than  $a_3$ ,  $x - a_3$  will be positive and  $f_n(x)$  and  $f_{n-1}(x)$  will have like signs or will form a permanence. Therefore, a variation is lost in passing from  $a_3 - 0$  to  $a_3 + 0$ .

**756.** These principles are true if  $f_n(x)$  contains imaginary roots as well as when all the roots are real, since the signs of the functions depend wholly upon the real factors they contain [732].

### Sturm's Theorem.

**757.** *The number of variations of sign lost in the terms of the Sturmian series, as the value of  $x$  continuously changes from  $a$  to  $b$ ,  $a$  being less than  $b$ , equals the number of real roots of  $f_n(x) = 0$  lying between  $a$  and  $b$ .*

**Demonstration.**—For each time the value of  $x$ , in ascending from  $a$  to  $b$ , passes over a root of  $f_n(x) = 0$ , there is lost one variation of sign [755, 4] and only one [755, 3].

**758. Cor. 1.**—The theorem is equally true for  $F_n(x) = 0$ , there being nothing in the demonstration of it to restrict its application to  $f_n(x) = 0$ .

**759. Cor. 2.**—The difference between the number of variations when  $+\infty$  and  $-\infty$  are substituted for  $x$  in the series is the number of real roots in the equation.

**760. Cor. 3.**—The difference between the number of variations when 0 and  $+\infty$  are substituted for  $x$  is the number of positive roots, and, when 0 and  $-\infty$  are substituted for  $x$ , the number of negative roots.

**761. Remark 1.**—It is evident that the sign of the absolute term of a function is the sign of the value of the function, when  $x = 0$ .

**762. Remark 2.**—The sign of the first term of a function is the sign of the value of the function, when  $x = \pm\infty$ .

For,  $Ax^n = Ax^{n-1} \cdot x > Bx^{n-1} + Cx^{n-1} + Dx^{n-1} + \dots + Lx^{n-1} > Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + \dots + L$ , when  $x = \pm\infty$ .

**763. Remark 3.**—The sign of the value of a function for any integral or decimal value of  $x$  is best determined by the method explained in Art. 106.

**Illustration.**—Find the sign of  $F_4(x) = 3x^4 - 2x^3 + 7x^2 - 3x - 8$  when  $x = 1.2$ .

**Solution:** The value of  $F_4(x)$  when  $x = 1.2$  is  $+1.2448$ , as witness:

$$\begin{array}{r}
 1.2) 3 - 2 \quad + 7 \quad - 3 \quad - 8 \\
 \underline{3.6 + 1.92 + 10.704 + 9.2448} \\
 1.6 + 8.92 + 7.704 + 1.2448 *
 \end{array}$$

$\therefore$  The sign of  $F_4(x)$  is  $+$ .

**Note.**—In practice it is usually not necessary to make the last multiplication and addition to determine the *sign* of the value.

**764. Remark 4.**—Though it is not usually best to apply Sturm's method of solution to equations before the commensurable roots have been removed by division, on account of the great labor involved in deriving and evaluating the different functions when the equation is of a high degree, yet such a course may be pursued. If there are equal roots, the fact will appear in deriving the functions, and if there are integral or fractional roots they will be discovered in evaluating the functions to determine their signs.

**Example.**—Determine the number and situation of the real roots in  $f_3(x) = x^3 - 12x^2 + 57x - 94 = 0$ .

**Solution :**  $f_3(x) = x^3 - 12x^2 + 57x - 94$

$$f_2(x) = 3x^2 - 24x + 57$$

$$f_1(x) = -x + 3$$

$$f_0(x) = -$$

Substituting in these functions as follows, we shall have :

For  $x = +\infty$ ,                    +   +   -   -        one variation.

For  $x = 0$ ,                        -   +   +   -        two variations.

For  $x = -\infty$ ,                   -   +   +   -        two variations.

There is, therefore, one real root between 0 and  $+\infty$ . There is no negative root. Therefore, there are two imaginary roots.

To find the situation of the real root, we proceed as follows :

For  $x = 1$ , we have           -   +   +   -        two variations.

For  $x = 2$ , we have           -   +   +   -        two variations.

For  $x = 3$ , we have           -   +   ±   -        two variations.

For  $x = 4$ , we have           +   +   -   -        one variation.

Therefore, there is one real root between 3 and 4, or the first figure of the real root is 3.

To find the next figure, we proceed as follows :

For  $x = 3.1$ , we have       -   +   -   -        two variations.

For  $x = 3.2$ , we have       -   +   -   -        two variations.

For  $x = 3.3$ , we have       -   +   -   -        two variations.

For  $x = 3.4$ , we have       +   +   -   -        one variation.

Therefore, the root lies between 3.3 and 3.4, or the first two figures of the root are 3.3.

By a continuation of this process the root might be extended to any number of figures. A more expeditious method, however, is known, and will be explained hereafter, for extending a root after a sufficient number of figures have been found to distinguish the root from any other root lying near it. Thus, if an equation had the two roots 3.1256... and 3.1234..., the first four figures of each root only would be found by Sturm's theorem.

When it is known, as in the above example, that only one real root lies between two numbers, it becomes necessary only to study the signs of  $f_n(x)$ , since passing over the roots of the intermediate functions does not cause a change in the number of variations.

The same conclusion will be reached by the simple application of Art. 735, since  $f_2(x)$  changes sign between  $x = 3$  and  $x = 4$ .

## EXERCISE 104.

Find the number and situation of the real roots in the following equations :

1.  $x^3 - 4x^2 - 6x + 8 = 0$
2.  $x^3 + 6x^2 - 3x + 9 = 0$
3.  $x^4 + 3x^3 - 6x + 2 = 0$
4.  $x^5 - 10x^3 + 6x + 1 = 0$
5.  $2x^4 - 11x^2 + 8x - 16 = 0$
6.  $x^4 - 12x^2 + 12x - 3 = 0$

## Horner's Method of Root Extension.

**765.** In 1819, W. G. Horner, an English mathematician, published an elegant method of extending a root of an equation to any desired number of places, after a sufficient number of initial figures have been found by other methods to distinguish the root from other roots of the equation. This method is based upon the following principle :

**766. Principle.**—If  $F_n(x)$  be continuously divided by  $x - a$ , the successive remainders will be the coefficients in inverse order of an equation whose roots are  $a$  less than the roots of  $F_n(x) = 0$ .

**Demonstration :**

$$\text{Take } F_n(x) = Ax^{n-1} + Bx^{n-2} + \dots + Jx^2 + Kx + L = 0 \quad (\text{A})$$

Put  $x_1 + a = x$ , or  $x_1 = x - a$ ;

$$F_n(x_1 + a) = A(x_1 + a)^{n-1} + B(x_1 + a)^{n-2} + \dots + J(x_1 + a)^2 + K(x_1 + a) + L = 0 \quad (\text{B})$$

Expand terms, bracket coefficients of like powers of  $x_1$ , and represent the coefficients of the transformed equation by  $A_1, B_1, \dots, J_1, K_1, L_1$ ; then,

$$F_n(x_1) = A_1x_1^{n-1} + B_1x_1^{n-2} + \dots + J_1x_1^2 + K_1x_1 + L_1 = 0 \quad (\text{C})$$

Now, the roots of (C) are evidently  $a$  less than those of (A).

Substitute  $x_1 = x - a$  in (C),

$$F_n(x - a) = A_1(x - a)^n + B_1(x - a)^{n-1} + \dots + J_1(x - a)^2 + K_1(x - a) + L_1 = 0 \quad (\text{D})$$

Now,  $F_n(x-a)$  is evidently equivalent to  $F_n(x)$ , and will leave the same remainder when divided by  $x-a$  as will  $F_n(x) + (x-a)$ . But, if  $F_n(x-a)$  is continuously divided by  $x-a$ , the successive remainders will be  $L_1, K_1, J_1, \dots, B_1$ , and  $A_1$ , or the coefficients of  $F_n(x_1)$  in inverse order. Therefore the theorem.

### Applications.

1. Transform  $x^4 + x^3 + x^2 + 3x - 100 = 0$  into an equation whose roots are 2 less than those of the given equation.

	Form.			
1	+ 1	+ 1	+ 3	- 100 (+ 2
	<u>+ 2</u>	<u>+ 6</u>	<u>+ 14</u>	<u>+ 34</u>
	+ 3	+ 7	+ 17	- 66 *
	<u>+ 2</u>	<u>+ 10</u>	<u>+ 34</u>	
	+ 5	+ 17	+ 51 *	
	<u>+ 2</u>	<u>+ 14</u>		
	+ 7	+ 31 *		
	<u>+ 2</u>			
	+ 9 *			

The transformed equation is  $x^4 + 9x^3 + 31x^2 + 51x - 66 = 0$ .

**Explanation.**—Dividing by  $x-2$ , by synthetic division, the coefficients of the first quotient are  $1 + 3 + 7 + 17$ , and the first remainder is  $-66$ , the absolute term of the transformed equation.

Dividing  $1 + 3 + 7 + 17$  again by  $+2$ , the second quotient is  $1 + 5 + 17$ , and the second remainder, or the coefficient of  $x$  in the transformed equation, is  $+51$ .

Dividing  $1 + 5 + 17$  again by  $+2$ , the third quotient is  $1 + 7$ , and the third remainder, or the coefficient of  $x^3$  in the transformed equation, is  $+31$ .

Dividing  $1 + 7$  again by  $+2$ , the fourth quotient is  $1$ , and the fourth remainder, or the coefficient of  $x^3$  in the transformed equation, is  $+9$ .

Therefore, the transformed equation is  $x^4 + 9x^3 + 31x^2 + 51x - 66 = 0$ .

**Query.**—Could you tell by inspection that the roots of the transformed equation are less than those of the original equation?

**Query.**—Since  $1 + 9 + 31 + 51 > 66$ , can  $x^4 + 9x^3 + 31x^2 + 51x - 66 = 0$  have a positive root equal to or greater than unity? Why, or why not?

2. Transform  $x^4 + 9x^3 + 31x^2 + 51x - 66 = 0$  into an equation whose roots are  $\cdot 8$  less than those of the given equation.

1	+ 9	+ 31	+ 51	- 66	( $\cdot 8$
	- 8	7.84	31.072	65.6576	
	9.8	38.84	82.072	- 0.3424 *	
	- 8	8.48	37.856		
	10.6	47.32	+ 119.928 *		
	- 8	9.12			
	11.4	+ 56.44 *			
	- 8				
	12.2 *				

The coefficients of the first quotient are  $1 + 9.8 + 38.84 + 82.072$ , and the first remainder is  $- 0.3424$ , which is the absolute term of the transformed equation. The second remainder, or the coefficient of  $x$ , is  $119.928$ . The third remainder, or the coefficient of  $x^2$ , is  $56.44$ . The fourth remainder, or the coefficient of  $x^3$ , is  $12.2$ .

The transformed equation is

$$x^4 + 12.2x^3 + 56.44x^2 + 119.928x - .3424 = 0.$$

3. Transform

$$x^4 + 12.2x^3 + 56.44x^2 + 119.928x - .3424 = 0$$

into an equation whose roots are  $\cdot 002$  less than those of the given equation.

1	12.2	56.44	119.928	- .3434. ....	( $\cdot 002$
	- .002	.024404	.112928808	+ .240081857616	
	12.202	56.464404	120.040928808	- .103318142384 *	
	- .002	.024408	.112977624		
	12.204	56.488812	120.153906432 *		
	- .002	.024412			
	12.206	56.513224 *			
	- .002				
	12.208 *				

The coefficients of the first quotient are  $1 + 12.202 + 56.464404 + 120.040928808$ , and the first remainder, or the absolute term of the transformed equation, is  $- .103318142384$ . The second, third, and fourth remainders, or the coefficients of  $x$ ,  $x^2$ , and  $x^3$ , are  $120.153906432$ ,  $56.513224$ , and  $12.208$ .

The transformed equation is

$$x^4 + 12.208x^3 + 56.513224x^2 + 120.153906432x - .103318142384 = 0.$$

4. The integral part of one of the roots of  $x^4 + x^3 + x^2 + 3x - 100 = 0$  is 2. Extend the root.

Form.			
1 + 1	+ 1	+ 3	- 100
+ 2	+ 6	+ 14	+ 34
+ 3	+ 7	+ 17	- 66 *
+ 2	+ 10	+ 34	
+ 5	+ 17	+ 51 *	65·6576
+ 2	+ 14		- 0·3424 *
+ 7	+ 31 *	31·072	
+ 2		82·072	
+ 9 *	7·84	37·856	·240081857616
	38·84	119·928 *	- ·102318142384 *
·8	8·48		
+ 9·8	47·32	·112928808	
·8	9·12	120·040928808	
10·6	56·44 *	·112977624	
·8		120·153906432 *	
11·4	·024404		
·8	56·464404		
12·2 *	·024408		
	56·488812		
·002	·024412		
12·202	56·513224 *		
·002			
12·204			
·002			
12·206			
·002			
12·208 *			

**Explanation.**—1. Transform the equation into one whose roots are less by 2. The new equation is  $x^4 + 9x^3 + 31x^2 + 51x - 66 = 0$ . The roots corresponding to the one we are considering will now be a decimal.

2. Since  $(\cdot 1)^3 = \cdot 01$ ;  $(\cdot 1)^2 = \cdot 001$ ; and  $(\cdot 1)^4 = \cdot 0001$ , the first three terms are small in comparison to  $51x$ ; therefore,  $51x = 66$  nearly, whence 51 may be taken as a trial divisor to find the next figure of the root, considerable allowance being made for the omitted terms. At first we would be tempted to try  $\cdot 9$  for the value of  $x$ . But, upon transforming the equation into one whose roots are less by  $\cdot 9$ , we shall find that the absolute term will become positive, which shows that  $\cdot 9$

is a superior limit. We therefore use .8 for the next term of the root, and transform the equation into one whose roots are .8 less. The transformed equation is  $x^4 + 12.2x^3 + 56.44x^2 + 119.928x - .3424 = 0$ . The root of this equation is now less than .1.

3. Omitting the first three terms of the equation on account of their smallness, and using the coefficient of  $x$  as a trial divisor, we see that the root is less than .01 and is above .002. The next figure of the root is therefore 0, and the following one 2. Transform the equation into one whose roots are less by .002; the resulting equation is  $x^4 + 12.208x^3 + 56.513224x^2 + 120.153906432x - .102318142384 = 0$ .

The work may be extended as far as we please.

**767. Remark 1.**—When the number of decimal places in the absolute term becomes equal to the number of such places desired in the root, we may begin to drop one figure in the preceding term (trial divisor), two in the next preceding term, and so on toward the left. When all the figures of the first term are exhausted, the remaining figures of the root may be found by simply dividing by the trial divisor.

**768. Remark 2.**—The absolute term after each transformation must have a sign opposite to that of the next trial divisor.

**769. Remark 3.**—The method may be applied with equal facility to extending an integral root after a sufficient number of initial figures have been obtained by trial or by Sturm's Theorem to distinguish the root from others of the equation. It may be used with exactness whenever there is an exact root; hence, the incorrectness of the title "Horner's Method of Approximation" given the method by most authors.

**770. Remark 4.**—The negative roots are the numerical equivalents of the positive roots of the equation resulting from changing the signs of the alternate terms, and may be found accordingly.

5. Solve  $x^3 - 1728 = 0$ , or find the  $\sqrt[3]{1728}$ .

Solution.			
1	+ 0	+ 0	- 1728 (12
	<u>10</u>	<u>100</u>	<u>1000</u>
	10	100	- 728 *
	<u>10</u>	<u>200</u>	<u>+ 728</u>
	20	300 *	0
	<u>10</u>	64	
	30 *	<u>364</u>	
	32		

6. Extract the 5th root of 4312345 to thousandths, i. e., solve approximately  $x^5 - 4312345 = 0$ .

Solution.

1	0	0	0	0	- 4312345	21·229
	20	400	8000	160000	8200000	
	20	400	8000	160000	- 1112345 *	
	20	800	24000	640000		
	40	1200	32000	800000 *		
	20	1200	48000	84101	884101	
	60	2400	80000 *	884101	- 228244 *	
	20	1600	4101	88304		
	80	4000 *	84101	972405 *	198220·84832	
	20	101	4203		- 30023·15168 *	
	100 *	4101	88304	18699·2416		
	1	102	4306	991104·2416		
	101	4203	92610 *	18877·3264		
	1	103	886·208	1009981·5680 *		
	102	4306	93496·208			
	1	104	890·424			
	103	4410 *	94386·632			
	1	21·04	894·648			
	104	4431·04	95281·280 *			
	1	21·08				
	105 *	4452·12				
	·2	21·12				
	105·2	4473·24				
	·2	21·16				
	105·4	4494·40 *				
	·2					
	105·6					
	·2					
	105·8					
	·2					
	106·0 *					

The number of decimal places in the second remainder is greater than the number required in the root; therefore, the remaining figures may be found by dividing the remainder by 1009981·568 [767, Rem. 1].

## EXERCISE 108.

Solve :

1.  $x^2 - 704x - 55025 = 0$
2.  $x^3 - 15252992 = 0$
3.  $x^3 + 3x^2 - 3x - 7 = 0$
4.  $x^4 - 4x^3 - 6x^2 + 32x - 26 = 0$
5.  $x^4 - 19x^3 + 24x^2 + 712x - 40 = 0$
6.  $x^5 + 12x^4 + 59x^3 + 150x^2 + 201x + 94 = 0$
7.  $3x^4 + 24x^3 + 68x^2 + 82x - 964 = 0$
8. Find the cube root of 2
9. Find the fifth root of 5
10.  $x^3 + 11x^2 - 102x + 181 = 0$
11.  $x^4 + 9x^3 + 31x^2 + 51x - 66 = 0$
12.  $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321 = 0$
13. One root of the equation  $x^3 + 2x^2 + 3x - 13089030$  is 235. Find a cubic equation whose root is 225.

## Cubic Equations.

**771.** A cubic equation containing an integral root may be readily factored.

Let  $-a$  be a root of a cubic equation, then  $x + a$  is a factor of the equation [719]. Let  $x^2 + mx + n$  be the other factor; then,

$$(x + a)(x^2 + mx + n) = 0 \quad (\text{A})$$

$$\text{or, } x^3 + (a + m)x^2 + (am + n)x + an = 0 \quad (\text{B})$$

We now observe that if we subtract the factor  $a$  of the absolute term from the coefficient of  $x^2$ , and the factor  $n$  from the coefficient of  $x$ , the latter remainder divided by the former will give  $a$ , the root with the sign changed. This, then, is the condition under which a factor of the absolute term is a root with the sign changed.

**Illustration.**—Solve  $x^3 - x^2 - 4x + 4 = 0$ .

**Solution:** 1. The factors of  $+4$  are  $+2$  and  $+2$ ;  $-2$  and  $-2$ ;  $+4$  and  $+1$ ; and  $-4$  and  $-1$ .

Try whether  $+4$  is a root with the sign changed.

Take  $-1$  and  $-4$ , the coefficients of  $x^2$  and  $x$ .

Subtract,  $+4$  and  $+1$ , the factors of the absolute term.

Divide,  $-5$   $-5$  (which  $\neq 4$ ).

$\therefore 4$  is not a root with the sign changed.

2. Try whether  $-2$  is a root.

Take  $-1$  and  $-4$

Subtract,  $+2$  and  $+2$

Divide,  $-3$   $-3$  ( $+2$

$\therefore -2$  is a root.

Now,  $x^3 - x^2 - 4x + 4 = (x + 2)(x^2 - 3x + 2) = 0$ ;

whence,  $x = -2, 2, \text{ and } 1$ .

### Cardan's Formula.

**772.** I. The general cubic equation  $x^3 + ax^2 + bx + c = 0$  may be transformed into an equation of the form of  $y^3 + py + q = 0$ , by putting  $x = y - \frac{1}{3}a$ .

**Demonstration:** Take  $x^3 + ax^2 + bx + c = 0$ . (A)

Put  $x = y - \frac{1}{3}a$ ; then,

$$x^3 = y^3 - ay^2 + \frac{1}{3}a^2y - \frac{1}{27}a^3$$

$$ax^2 = ay^2 - \frac{2}{3}a^2y + \frac{1}{9}a^3$$

$$bx = by - \frac{1}{3}ab$$

$$c = c$$

$$\therefore x^3 + ax^2 + bx + c = y^3 + \left(b - \frac{1}{3}a^2\right)y + \left(\frac{2}{27}a^3 - \frac{1}{3}ab + c\right)$$

Put  $p$  for  $b - \frac{1}{3}a^2$ , and  $q$  for  $\frac{2}{27}a^3 - \frac{1}{3}ab + c$ ; then,

$$x^3 + ax^2 + bx + c = y^3 + py + q = 0. \quad (\text{A})$$

**773. II.** The equation  $y^3 + py + q = 0$  may be transformed into a quadratic by putting  $y = z - \frac{p}{3z}$ .

**Demonstration:** Take  $y^3 + py + q = 0$ . (B)

Put  $y = z - \frac{p}{3z}$ ; then,

$$y^3 = z^3 - pz + \frac{p^2}{3z} - \frac{p^3}{27z^3}$$

$$py = pz - \frac{p^2}{3z}$$

$$q = q$$

$$\therefore y^3 + py + q = z^3 - \frac{p^3}{27z^3} + q = 0;$$

$$\text{whence, } 27z^6 + 27qz^3 - p^3 = 0;$$

$$\text{or, } z^6 + qz^3 - \frac{1}{27}p^3 = 0. \quad (C)$$

**774. III.** The roots of  $z^6 + qz^3 - \frac{1}{27}p^3 = 0$  are

$$z = \left( -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{\frac{1}{3}}; \text{ whence,}$$

$$y = z - \frac{p}{3z} =$$

$$\left( -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{\frac{1}{3}} + \left( -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{\frac{1}{3}} \quad (D)$$

This is Cardan's formula.

The values of  $x$  may be obtained by subtracting  $\frac{1}{3}a$  from the values of  $y$ .

**775. IV.** Cardan's formula fails when all the roots are real and unequal.

For, let the roots be  $a + \sqrt{b}$ ,  $a - \sqrt{b}$ , and  $c$ . Then, since the coefficient of  $y^2$  is 0,

$$(-a - \sqrt{b}) + (-a + \sqrt{b}) - c = 0 \quad [726, 1];$$

whence,  $c = -2a$ .

The equation whose roots are  $a + \sqrt{b}$ ,  $a - \sqrt{b}$ , and  $-2a$ , is  $y^3 - (3a^2 + b)y + 2(a^3 - ab) = 0$ .

$$\therefore p = -(3a^2 + b), \text{ and } q = 2(a^3 - ab),$$

whence,  $\sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = \left( a^2 - \frac{1}{9}b \right) \sqrt{-3b}$ , which is im-

aginary, and, therefore, irreducible when  $b$  is positive. or when the roots are all real and unequal.

**Illustrations.**—1. Solve  $x^3 - 3x^2 + 4 = 0$ .

Here  $a = -3$ ; hence,  $x = y - \left(\frac{-3}{3}\right) = y + 1$ .

Substitute,  $(y + 1)^3 - 3(y + 1)^2 + 4 = 0$ .

Reduce,  $y^3 - 3y + 2 = 0$ . (1)

Here  $p = -3$ ; hence,  $y = z - \frac{p}{3z} = z + \frac{1}{z}$ .

Substitute,  $\left(z + \frac{1}{z}\right)^3 - 3\left(z + \frac{1}{z}\right) + 2 = 0$ .

Reduce,  $z^6 + 2z^3 = -1$ .

Complete the square,  $z^6 + 2z^3 + 1 = 0$ .

Extract the  $\sqrt{\phantom{x}}$ ,  $z^3 + 1 = 0$ .

Factor,  $(z + 1)(z^2 - z + 1) = 0$ ;

whence,  $z = -1$  or  $\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$ .

$x = y + 1 = z + \frac{1}{z} + 1 = -1$ , or  $2$ , or  $2$ .

**776.** Sometimes an integral root can only be approximately found.

2. Solve  $x^3 + 3x^2 + 9x - 13 = 0$ . (A)

Here  $a = 3$ ; hence,  $x = y - \left(\frac{3}{3}\right) = y - 1$ .

Substitute,

$(y - 1)^3 + 3(y - 1)^2 + 9(y - 1) - 13 = 0$ .

Reduce,  $y^3 + 6y - 20 = 0$ . (1)

Here  $p = 6$ ; hence,  $y = z - \frac{p}{3z} = z - \frac{2}{z}$ .

Substitute in (1),

$\left(z - \frac{2}{z}\right)^3 + 6\left(z - \frac{2}{z}\right) - 20 = 0$ .

Reduce,  $z^6 - 20z^3 = 8$ . (2)

Complete the square,  $z^6 - 20z^3 + 100 = 108$ .

Extract the  $\sqrt{\phantom{x}}$ ,  $z^3 - 10 = \pm 10.392304$ .

Transpose,  $z^3 = 20.392304$  or  $-3.92304 +$

Extract the  $\sqrt[3]{\phantom{x}}$ ,  $z = 2.73 +$  or  $-.73 +$

$x = y - 1 = z - \frac{2}{z} - 1 = 2.73 - .73 - 1 = 1$ , or  $-.73 + 2.73 - 1 = 1$ .

These two values are identical. The other two roots are found by dividing equation (A) by  $x - 1$ .

## EXERCISE 108.

Solve :

- |                                 |                                |
|---------------------------------|--------------------------------|
| 1. $x^3 - 3x^2 + 7x - 5 = 0$    | 8. $x^3 + x^2 - 8x - 12 = 0$   |
| 2. $x^3 - 6x^2 + 10x - 8 = 0$   | 9. $x^3 - x^2 - 8x + 12 = 0$   |
| 3. $x^3 - 11x^2 + 41x - 55 = 0$ | 10. $x^3 - 11x - 20 = 0$       |
| 4. $x^3 + 6x^2 + 14x + 12 = 0$  | 11. $x^3 - 26x + 60 = 0$       |
| 5. $x^3 - 4x^2 + 5x - 6 = 0$    | 12. $x^3 - 4x^2 + 3 = 0$       |
| 6. $x^3 + 5x^2 + 6x + 8 = 0$    | 13. $x^3 - 4x^2 - 7x + 10 = 0$ |
| 7. $x^3 + 7x^2 + 16x + 12 = 0$  | 14. $x^3 + 4x^2 - 7x - 10 = 0$ |

## Recurring Equations.

**777.** A *Recurring Equation* is one in which the coefficients of the first and last terms, and of those equidistant from the first and last terms, are numerically equal, and the signs of the corresponding terms are either alike throughout or unlike throughout ; as,

1.  $x^5 - 4x^4 + 5x^3 + 5x^2 - 4x + 1 = 0.$
2.  $x^5 + 3x^4 - 2x^3 + 2x^2 - 3x - 1 = 0.$
3.  $x^6 + 4x^5 - 5x^4 + 3x^3 - 5x^2 + 4x + 1 = 0.$

**778.** In a recurring equation of an even degree in which the corresponding terms have unlike signs, the middle term is wanting.

For, according to definition, it is both positive and negative.

**779.** A *Reciprocal Equation* is one such that, if  $a$  is a root,  $\frac{1}{a}$  is also a root.

**780. Theorem I.**—A recurring equation is also a reciprocal equation.

**Demonstration:** Let  $\alpha$  be a root of

$$f_n(x) = x^n + Ax^{n-1} + Bx^{n-2} + \dots \pm Bx^2 \pm Ax \pm 1 = 0 \quad (A)$$

$$\text{then, } \alpha^n + A\alpha^{n-1} + B\alpha^{n-2} + \dots \pm B\alpha^2 \pm A\alpha \pm 1 = 0 \quad (B)$$

Substitute  $\frac{1}{\alpha}$  for  $x$  in  $f_n(x) = 0$ ,

$$\frac{1}{\alpha^n} + \frac{A}{\alpha^{n-1}} + \frac{B}{\alpha^{n-2}} + \dots \pm \frac{B}{\alpha^2} \pm \frac{A}{\alpha} \pm 1 = 0 \quad (C)$$

$$\text{whence, } 1 + A\alpha + B\alpha^2 + \dots \pm B\alpha^{n-2} \pm A\alpha^{n-1} \pm \alpha^n = 0 \quad (D)$$

$$\text{or, } \alpha^n + A\alpha^{n-1} + B\alpha^{n-2} + \dots \pm B\alpha^2 \pm A\alpha \pm 1 = 0 \quad (E)$$

Now, (E) is identical with (B); therefore, if  $\alpha$  is a root of (A),  $\frac{1}{\alpha}$  is also a root.

**781. Theorem II.**—*A recurring equation of an odd degree has  $+1$  for a root when the signs of the corresponding terms are unlike.*

**Demonstration:**

$$\text{Let } x^{2n+1} + Ax^{2n} + Bx^{2n-1} + \dots - Bx^2 - Ax - 1 = 0 \quad (A)$$

$$\text{then, } (x^{2n+1} - 1) + Ax(x^{2n-1} - 1) + Bx^2(x^{2n-2} - 1) + \dots = 0 \quad (B)$$

Now, each term of (B) is divisible by  $x - 1$  [184, P.];

$$\therefore x - 1 = 0, \text{ or } x = 1.$$

**782. Theorem III.**—*A recurring equation of an odd degree has  $-1$  for a root when the signs of the corresponding terms are alike.*

**Demonstration:**

$$\text{Let } x^{2n+1} + Ax^{2n} + Bx^{2n-1} + \dots + Bx^2 + Ax + 1 = 0 \quad (A)$$

$$\text{then, } (x^{2n+1} + 1) + Ax(x^{2n-1} + 1) + Bx^2(x^{2n-2} + 1) + \dots = 0 \quad (B)$$

Now, each term of (B) is divisible by  $x + 1$  [135, P.];

$$\therefore x + 1 = 0, \text{ or } x = -1.$$

**783. Theorem IV.**—*A recurring equation of an even degree has  $+1$  and  $-1$  for roots when the signs of the corresponding terms are unlike.*

**Demonstration:**

$$\text{Let } x^{2n} + Ax^{2n-1} + Bx^{2n-2} + \dots - Bx^2 - Ax - 1 = 0 \quad (A).$$

$$\text{then, } (x^{2n} - 1) + Ax(x^{2n-2} - 1) + Bx^2(x^{2n-4} - 1) + \dots = 0 \quad (B)$$

Now, (B) is divisible by both  $x - 1$  and  $x + 1$  [134, 136, P.];

$$\therefore x - 1 = 0 \text{ and } x + 1 = 0; \text{ whence, } x = \pm 1.$$

**784. Theorem V.**—*A recurring equation of an even degree may be transformed into an equation of one half the degree when the signs of the corresponding terms are alike.*

**Demonstration :**

$$\text{Let } x^{2n} + Ax^{2n-1} + Bx^{2n-2} + \dots + Bx^2 + Ax + 1 = 0 \quad (\text{A})$$

Divide by  $x^n$ , and collect terms,

$$\left(x^n + \frac{1}{x^n}\right) + A\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + B\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \dots + P = 0 \quad (\text{B})$$

$$\text{Put } x + \frac{1}{x} = z ; \text{ then will}$$

$$x^2 + \frac{1}{x^2} = z^2 - 2$$

$$x^3 + \frac{1}{x^3} = z^3 - 3z,$$

and, in general, each term of (B) may be transformed into a term of only half the degree.

**Illustration.**—

$$\text{Take } x^6 + 4x^5 - 3x^4 + 2x^3 - 3x^2 + 4x + 1 = 0. \quad (\text{A})$$

$$\text{Divide by } x^3, \quad x^3 + 4x^2 - 3x + 2 - \frac{3}{x} + \frac{4}{x^2} + \frac{1}{x^3} = 0 \quad (\text{B})$$

Rearrange the terms and factor,

$$x^3 + \frac{1}{x^3} + 4\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) + 2 = 0 \quad (\text{C})$$

$$\text{Put } x + \frac{1}{x} = y; \quad (1)$$

$$\text{then, } x^2 + 2 + \frac{1}{x^2} = y^2; \quad (2)$$

$$\text{or, } x^3 + \frac{1}{x^3} = y^2 - 2;$$

$$\text{and, } x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = y^3;$$

$$\text{or, } x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = y^3;$$

$$\text{or, } x^3 + \frac{1}{x^3} = y^3 - 3y. \quad (3)$$

Substitute (1), (2), and (3) in (C),

$$(y^3 - 3y) + 4(y^2 - 2) - 3y + 2 = 0;$$

$$\text{whence, } y^3 + 4y^2 - 6y - 6 = 0.$$

## EXERCISE 107.

Solve :

1.  $x^3 - 2x^2 + 2x - 1 = 0$
2.  $x^3 - 3x^2 - 3x + 1 = 0$
3.  $x^4 - 3x^3 + 3x - 1 = 0$
4.  $x^4 + 3x^3 - 3x - 1 = 0$
5.  $2x^4 - 5x^3 + 4x^2 - 5x + 2 = 0$
6.  $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$
7.  $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$
8.  $5x^5 + 11x^4 - 88x^3 - 88x^2 + 11x + 5 = 0$

## Reduction of Binomial Equations.

**785.** A *Binomial Equation* is an equation of two terms, one of which is absolute ; as,  $x^a \pm a = 0$ .

**786.** Every binomial equation can be reduced to the form  $y^a \pm 1 = 0$ .

**Demonstration :** Take the general binomial equation  $x^a \pm a = 0$ .

Put  $\frac{\sqrt[a]{a}}{y}$  for  $x$ ,  $\frac{a}{y^a} \pm a = 0$ ; whence,  $y^a \pm 1 = 0$ .

**787.**  $y^a \pm 1 = 0$  is a recurring equation, and may be so solved.

**Illustrative Solutions.**—1. Solve  $x^4 + 1 = 0$ . (A)

Divide by  $x^2$ ,  $x^2 + \frac{1}{x^2} = 0$  (1)

Put  $x + \frac{1}{x} = y$  (2)

Square  $x^2 + 2 + \frac{1}{x^2} = y^2$

Transpose,  $x^2 + \frac{1}{x^2} = y^2 - 2$

Substitute in (1),  $y^2 - 2 = 0$

Factor,  $(y + \sqrt{2}), (y - \sqrt{2}) = 0$ ;

whence,  $y = \pm \sqrt{2}$ .

Substitute in (2),  $x + \frac{1}{x} = \pm \sqrt{2}$  (3)

whence,  $x = \frac{1}{2}(\sqrt{2} \pm \sqrt{-2})$ , or  $-\frac{1}{2}(\sqrt{2} \mp \sqrt{-2})$ .

## CHAPTER XIII.

### DETERMINANTS AND PROBABILITIES.

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#### Introduction.

788. In the polynomial

$a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1,$  (A)  
it will be seen :

1. That the letters  $a$ ,  $b$ , and  $c$  of each term are arranged in natural order.

2. That the subscript figures, 1, 2, and 3, are distributed among the letters in the six different terms in as many ways as possible, using all in each term and making no repetitions.

3. That the first term contains no *inversions* of subscript figures, they advancing in natural order from left to right ; the second term contains *one inversion*, 3 standing before 2 ; the third term contains *two inversions*, 2 and 3 both standing before 1 ; the fourth term contains *one inversion*, 2 standing before 1 ; the fifth term contains *two inversions*, 3 standing before 1 and 2 ; and the sixth term contains *three inversions*, 3 and 2 both standing before 1, and 3 standing before 2.

4. That in the positive terms there is an even number of inversions (zero being regarded an even number), and in the negative terms there is an odd number of inversions.

**789.** If we now arrange the nine different quantities found in (A) in a square, as follows :

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}; \quad (B)$$

form all the possible products of them taken three together, using in each product one and only one from each row, and one and only one from each column ; arrange the factors of the products in the natural literal order ; consider those products positive which have an even number of inversions of subscript figures, and those negative which have an odd number ; and take the algebraic sum of these products, we will have :

$$a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1. \quad (A)$$

Therefore, form (B) may be taken as the representative of form (A), and when so taken it is called a *determinant*, and (A) its *development*.

**790.** Definition.—A *Determinant* is any  $n^2$  quantities arranged in a square, as follows :

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ b_1 & b_2 & b_3 & \dots & b_n \\ c_1 & c_2 & c_3 & \dots & c_n \\ . & . & . & \dots & . \\ p_1 & p_2 & p_3 & \dots & p_n \end{vmatrix}, \quad (C)$$

and interpreted to denote the algebraic sum of all the products that may be formed by taking one and only one quantity from each row, and one and only one from each column ; arranging the letters of each product in the natural literal order, and regarding all products positive that have an even number of inversions of subscript figures, and all negative that have an odd number of such inversions.

**791.** The quantities contained in a determinant are called the *elements* of the determinant.

**792.** Determinants are divided into orders, named *second, third, .... nth*, accordingly as they contain  $2^2, 3^2, \dots n^2$  elements.

Thus,  $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$  is a determinant of the *second order*. Form (B) is a determinant of the *third order*, and form (C) a determinant of the *nth order*.

**793.** The diagonal joining the upper left-hand element with the lower right-hand element is called the *principal diagonal*; and the one joining the upper right-hand element with the lower left-hand element the *secondary diagonal*.

**794.** The product of all the elements along the principal diagonal is called the *principal term* of the development.

**795.** If the elements on the principal diagonal are known in order, the entire determinant may be written; hence it is that a determinant is often expressed by a modified form of the principal term of its development; as,  $[a_1 b_2 c_3 \dots p_n]$ , or  $\Sigma (\pm a_1 b_2 c_3 \dots p_n)$ .

**796.** It is evident that there are as many terms in the development of a determinant of the *nth order* as there are permutations of *n* things taken all together, or  $[n]$ .

---

### Properties of Determinants.

**797.** If we rearrange the factors of the terms in form (A) so as to place the subscripts in natural order, we shall have

$$a_1 b_2 c_3 - a_1 c_2 b_3 + c_1 a_2 b_3 - b_1 a_2 c_3 + b_1 c_2 a_3 - c_1 b_2 a_3. \quad (A_1)$$

It will be observed that in form  $(A_1)$ ,

1. The value of each term and of the entire polynomial is the same as in form  $(A)$ .

2. The first term contains no literal inversion; the second term contains *one*,  $c$  standing before  $b$ ; the third term contains *two*,  $c$  standing before both  $a$  and  $b$ ; the fourth term contains *one*,  $b$  standing before  $a$ ; the fifth term contains *two*,  $b$  and  $c$  both standing before  $a$ ; and the sixth term contains *three*,  $c$  and  $b$  both standing before  $a$ , and  $c$  before  $b$ .

3. The terms which contain an *even* number of literal inversions are *positive*, and those which contain an *odd* number *negative*.

798. If we now interchange the rows and columns in  $(B)$ , giving us the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \quad (D)$$

make all the possible products of three elements, using, each time, one and only one from each row, and one and only one from each column; arrange the factors of the products so that the subscripts stand in natural order; consider those products positive which have an even number of literal inversions, and those negative which have an odd number; and take the algebraic sum of these products, we shall have

$$a_1 b_2 c_3 - a_1 c_2 b_3 + c_1 a_2 b_3 - b_1 a_2 c_3 + b_1 c_2 a_3 - c_1 b_2 a_3. \quad (A_1)$$

This shows that in a determinate of the third order an interchange of rows and columns does not change the value.

Is this law true for a determinant of the  $n$ th order?

1. It is evident that the number of terms in the development of both forms is the same, each being  $|n|$ .

2. Each term in the development of either form has a corresponding term of equal numerical value in the development of the other form, because both developments contain all the possible products of  $n$  elements that can be formed from the  $n^2$  elements by taking one and only one from each row and one and only one from each column.

3. *The signs of the corresponding terms will be the same.* For the number of literal inversions in a term of the second development is equal to the number of subscript inversions in the corresponding term of the first development, as will readily appear from the fact that, if a subscript in any term of the first development follows  $r$  subscripts greater than itself, then, in the second development, the letter containing this subscript must precede  $r$  letters antecedent to it in the natural order. Therefore,

*Prin. 1.—An interchange of rows and columns in a determinant of any order does not change the value of the determinant.*

799. In form (A) and in form (A<sub>1</sub>) the second term equals *minus* the first term with the subscripts of  $b$  and  $c$  interchanged; the third term equals *minus* the second term with the subscripts of  $a$  and  $c$  interchanged; and so on, showing that any term in the development of a determinant of the third order equals *minus* some other term in the development with the subscripts of two factors interchanged.

*Is this law true for the development of a determinant of the  $n$ th order?*

1. It is evident that, if  $Pc, k_m$  be any term in the development of a determinant of the  $n$ th order, then will  $Pc_m k$  be numerically another term of the development; because  $P$  in both instances is the product of  $n - 2$  ele-

ments, none of which are taken from rows  $c$  and  $k$ , and none from columns  $r$  and  $m$ ; and  $c, k_m$  and  $c_m k_r$  are different elements taken from these rows and columns and combined with  $P$ . Therefore, the products are not identical.

2. *The signs of the original and the derived terms are always opposite.* For,

(1) Suppose the two subscripts interchanged to be consecutive. Let the original term be  $P c_m k_n Q$ , and the derived term  $P c_n k_m Q$ . Since  $m$  and  $n$  follow all the subscripts contained in  $P$  and precede all contained in  $Q$ , an interchange of them can not affect the number of inversions they make with the subscripts of either  $P$  or  $Q$ ; but such an interchange will either change a *natural* into an *inversion* or an *inversion* into a *natural*, either of which will evidently cause a change of sign.

(2) Suppose the two subscripts interchanged to be non-consecutive. Let the original term be  $P c_n Q k_m R$ , and the derived term  $P c_m Q k_n R$ . Suppose  $Q$  to contain  $q$  subscripts. Let  $m$ , in the original term, interchange consecutively with each of the subscripts in  $Q$  and with the subscript of  $c$ , then will it make  $q + 1$  interchanges before it becomes the subscript of  $c$ .  $n$  will now be the subscript of the first element in  $Q$ . Let it now interchange consecutively with each of the remaining subscripts in  $Q$  and with the subscript of  $k$ ; then will it make  $q$  interchanges before it becomes the subscript of  $k$ . Therefore, for the two subscripts of  $c$  and  $k$  in the original term to interchange there must be made  $2q + 1$ , or an odd number of consecutive interchanges, each one of which will cause a change of sign (1) in the entire term. Therefore, the sign of the term will be changed. Therefore,

*Prin. 2.*—If two subscripts be interchanged in any term of the development of a determinant, another term of the development will be obtained whose sign is opposite to that of the original term.

800. If we let  $Ph_k Qk_n R$  be a term in the development of a determinant, then will  $Pk_h Qh_n R$  be the term formed by the elements which occupy the same places, if rows  $h$  and  $k$  be interchanged, and will have the same sign. But  $Pk_h Qh_n R$  is also a term of the development of the original determinant, and has there an opposite sign to  $Ph_k Qk_n R$  [P. 2]. Therefore,

*Prin. 3.*—Interchanging two rows in a determinant changes the sign of the determinant.

*Cor. 1.*—Interchanging two columns in a determinant changes the sign of the determinant.

801. It is evident that if two columns or two rows of a determinant are in every respect alike, an interchange of them would not affect either the form or value of the determinant. But, according to Principle 3, the sign of the value would be changed. Now, both these statements can be true only when the value of the determinant is zero. Therefore,

*Prin. 4.*—A determinant that has two rows or two columns identical equals zero.

802. Since every term in the development of a determinant contains one factor and only one from each row and one and only one from each column, it follows that,

*Prin. 5.*—Multiplying or dividing all the elements of one row or one column of a determinant by any quantity multiplies or divides the determinant by that quantity.

**Cor. 1.**—Changing the signs of all the elements in any row or column changes the sign of the determinant.

**Cor. 2.**—If two rows or two columns of a determinant differ only by a common factor, the value of the determinant is zero.

**803. Definitions.**—If any number of rows and the same number of columns be *deleted* (stricken out) of a determinant, the remaining elements, taken in order, form a determinant called a *minor*, and the elements common to the deleted rows and columns form another *minor*. These minors are said to be *complementary*.

Thus, in the following determinant of the fourth order,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

the complementary minors are

$$\begin{vmatrix} b_1 & d_1 \\ b_3 & d_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_2 & c_2 \\ a_4 & c_4 \end{vmatrix}.$$

**804.** If a single row and a single column be deleted, the remaining minor is called the *principal minor*, and it, together with its complementary minor, which in this instance is a single element, are called *cofactors*.

**805. Problem.** To develop a determinant.

Let it be required to develop

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

into a series of determinants of a lower order.

Let  $A_1$ ,  $A_2$ ,  $A_3$ , and  $A_4$  represent respectively the cofactors of  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$ . Then, it is readily seen that all the terms in the

development containing the factor  $a_1$  are formed from  $a_1$  and its cofactor  $A_1$ , and the sum of these terms is  $a_1 A_1$ . Similarly, the sum of all the terms containing  $a_2$  is  $a_2 A_2$ , the sum of all the terms containing  $a_3$  is  $a_3 A_3$ , and the sum of all the terms containing  $a_4$  is  $a_4 A_4$ . Now, in each term of  $a_2 A_2$  there occurs one more inversion than in each term of  $a_1 A_1$ , since a subscript 2 will precede a subscript 1; similarly, in each term of  $a_3 A_3$  there occur two more inversions of subscripts than in  $a_1 A_1$ , and in each term of  $a_4 A_4$  there occur three more inversions of subscripts than in  $a_1 A_1$ .

Therefore,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

Therefore,

**Rule.**—Multiply each element of the first row by its cofactor, making the products alternately plus and minus, and take the algebraic sum of the results.

**806. Scholtum.**—The successive application of this rule will eventually make the full development of any determinant depend upon the development of a determinant of the second order. Thus,

$$\begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} = b_2 \begin{vmatrix} c_3 & c_4 \\ d_3 & d_4 \end{vmatrix} - b_3 \begin{vmatrix} c_2 & c_4 \\ d_2 & d_4 \end{vmatrix} + b_4 \begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix} = b_2(c_3 d_4 - c_4 d_3) - b_3(c_2 d_4 - c_4 d_2) + b_4(c_2 d_3 - c_3 d_2)$$

**Illustrative Example.**—Find the value of  $\begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 4 \\ 4 & 3 & 2 \end{vmatrix}$

**Solution :**

$$\begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 4 \\ 4 & 3 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} - 3 \begin{vmatrix} 3 & 4 \\ 4 & 2 \end{vmatrix} + 4 \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} =$$

$$2(2 \times 2 - 4 \times 3) - 3(3 \times 2 - 4 \times 4) + 4(3 \times 3 - 2 \times 4) =$$

$$-16 + 30 + 4 = 18.$$

## EXERCISE 108.

Find the value of :

$$1. \begin{vmatrix} 3 & 1 & 5 \\ 4 & 2 & 7 \\ 1 & 6 & 4 \end{vmatrix}$$

$$2. \begin{vmatrix} 4 & 3 & 2 \\ 5 & 1 & 4 \\ 2 & 4 & 5 \end{vmatrix}$$

$$3. \begin{vmatrix} 3 & -1 & 4 \\ 2 & 1 & -3 \\ 4 & 2 & 7 \end{vmatrix}$$

$$4. \begin{vmatrix} a & 0 & b \\ -a & b & 0 \\ a & b & c \end{vmatrix}$$

$$5. \begin{vmatrix} m & 0 & n \\ m & p & 0 \\ 0 & p & n \end{vmatrix}$$

$$6. \begin{vmatrix} b+c & c & b \\ c & a+c & a \\ b & a & a+b \end{vmatrix}$$

$$7. \begin{vmatrix} 2 & 3 & 2 & 3 \\ 3 & 2 & 3 & 2 \\ 2 & 3 & 3 & 2 \\ 3 & 2 & 2 & 3 \end{vmatrix}$$

$$8. \begin{vmatrix} -1 & 2 & 3 & 4 \\ 2 & -1 & 3 & 4 \\ 3 & 2 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{vmatrix}$$

$$9. \begin{vmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 3 & 2 & 0 & 2 \\ 4 & 1 & 2 & 0 \end{vmatrix}$$

$$10. \begin{vmatrix} 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 4 \\ 2 & 3 & 1 & 2 \\ 1 & 4 & 3 & 2 \end{vmatrix}$$

$$11. \begin{vmatrix} 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 \end{vmatrix}$$

$$12. \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ a & b & c & d \\ i & k & l & m \end{vmatrix}$$

$$13. \begin{vmatrix} 3 & 4 & 1 & 2 \\ 6 & 8 & 2 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{vmatrix}$$

$$14. \begin{vmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \\ 4 & 8 & 0 & 1 \end{vmatrix}$$

$$15. \begin{vmatrix} a & b & c & d \\ a & b & c & d \\ \bar{n} & \bar{n} & \bar{n} & \bar{n} \\ a & c & b & d \\ c & a & d & b \end{vmatrix}$$

## Additional Properties of Determinants.

**807. Prin. 6.**—If every element of one row, or column, of a determinant is a binomial, the determinant can be expressed as the sum of two other determinants; one of which is derived from the original determinant by dropping the *second* terms of the binomials and the other by dropping the *first* terms.

**Demonstration.**—Each term of the development contains one and only one of the binomial elements as a factor. Therefore, each term of the development can be separated into two terms, one of which is the first term of the binomial factor times the remaining factors of

the term, and the other the second term of the binomial factor times the remaining factors of the term. The sum of the component parts that contain the first terms of the binomial elements will form a determinant which is independent of the second terms of the binomial elements, and the sum of the component parts that contain the second terms of the binomial elements will form a second determinant which is independent of the first terms of the binomial elements.

$$\text{Thus, } \begin{vmatrix} a & b+c & b \\ b & a+c & a \\ c & a+b & b \end{vmatrix} = \begin{vmatrix} a & b & b \\ b & a & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} a & c & b \\ b & c & a \\ c & b & b \end{vmatrix}$$

**808. Cor. 1.**—If every element in any row, or column, consists of  $m$  terms, the determinant can be expressed as the sum of  $m$  other determinants.

**809. Cor. 2.**—If the elements of  $r$  rows, or columns, consist of  $a, b, c, \dots m$  terms respectively, the determinant can be expressed as the sum of  $abc \dots m$  determinants.

**810. Schottum.**—These truths are of value in reducing a determinant when one or more of the derived determinants reduce to zero. Thus,

$$\begin{vmatrix} a & b+a+c & c \\ d & e+d+f & f \\ g & h+g+k & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} + \begin{vmatrix} a & a & c \\ d & d & f \\ g & g & k \end{vmatrix} + \begin{vmatrix} a & c & c \\ d & f & f \\ g & k & k \end{vmatrix} =$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} + 0 + 0 \text{ [801, P.]} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

**811. Prin. 7.**—If to all the elements of any row, or column, be added equimultiples of the corresponding elements of any other row, or column, the value of the determinant will remain unchanged.

**Demonstration.**—Consider a determinant of the third order. Thus,

$$\text{Prove } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + p a_3 & a_2 & a_3 \\ b_1 + p b_3 & b_2 & b_3 \\ c_1 + p c_3 & c_2 & c_3 \end{vmatrix}$$

**Demonstration.**—

$$\begin{vmatrix} a_1 + p a_2 & a_2 & a_3 \\ b_1 + p b_2 & b_2 & b_3 \\ c_1 + p c_2 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} p a_2 & a_2 & a_3 \\ p b_2 & b_2 & b_3 \\ p c_2 & c_2 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 0 \text{ [802, Cor. 2]} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The method of proof employed in this example is general, and is, therefore, applicable to a determinant of any order.

**812. Cor.**—*It may also be shown that*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + p a_2 + q a_3 & a_2 & a_3 \\ b_1 + p b_2 + q b_3 & b_2 & b_3 \\ c_1 + p c_2 + q c_3 & c_2 & c_3 \end{vmatrix}; \text{ etc.}$$

That is,

*To all the elements of any row, or column, may be added equimultiples of the corresponding terms of a second row, or column, and again equimultiples of the corresponding terms of a third row, or column, etc.*

**813. Scholium.**—*This principle is of practical value in the reduction of a determinant, if, by its application, two rows, or columns, can be made identical, or one of them a multiple of the other. Thus,*

$$\begin{vmatrix} 5 & 8 & 11 \\ 6 & 9 & 12 \\ 7 & 10 & 13 \end{vmatrix} = \begin{vmatrix} 5 + 8 + 11 & 8 & 11 \\ 6 + 9 + 12 & 9 & 12 \\ 7 + 10 + 13 & 10 & 13 \end{vmatrix} = \begin{vmatrix} 24 & 8 & 11 \\ 27 & 9 & 12 \\ 30 & 10 & 13 \end{vmatrix} = 0$$

[802, Cor. 2]. *It may also be used to simplify a complex determinant.*

**814. Prin. 8.**—*If the elements of one column of a determinant be multiplied by the cofactors of the corresponding elements of another column, the sum of the products, taken alternately plus and minus, is zero.*

**Demonstration.**—Take two determinants, alike in every respect, except that, in the second, the  $q$ th column is identical with the  $p$ th column. Now, in the first, the sum of the products of the elements

in the  $p$ th column and the cofactors of the corresponding elements of the  $q$ th column, when the products are taken alternately plus and minus, is  $a_p A_q - b_p B_q + \dots$ .

The value of the second determinant is

$$a_q A_q - b_q B_q + \dots [805, R.] = 0 [801, P.].$$

But it is evident that  $a_q, b_q, \dots$  are identical with  $a_p, b_p, \dots$ . Therefore,  $a_p A_q - b_p B_q + \dots = 0$ .

# EXERCISE 109.

Find the value of :

$$1. \begin{vmatrix} 3 & 2 & 3 \\ 5 & 3 & 4 \\ 7 & 4 & 5 \end{vmatrix} \quad 2. \begin{vmatrix} -4 & 2 & 0 \\ 6 & -2 & 2 \\ -9 & 3 & -3 \end{vmatrix} \quad 3. \begin{vmatrix} 12 & -5 & 2 \\ -3 & 4 & 5 \\ 7 & 0 & 7 \end{vmatrix}$$

$$4. \begin{vmatrix} 0 & -1 & -2 & 1 \\ 8 & 5 & 2 & 1 \\ 6 & 9 & 8 & 1 \\ -8 & 4 & 8 & 1 \end{vmatrix} \quad 5. \begin{vmatrix} y+z & x-z & x-y \\ y-z & z+x & y-x \\ z-y & z-x & x+y \end{vmatrix}$$

$$6. \begin{vmatrix} a+p & b+q & c+r \\ b+p & c+q & a+r \\ c+p & a+q & b+r \end{vmatrix} + \begin{vmatrix} a-p & b-q & c-r \\ b-p & c-q & a-r \\ c-p & a-q & b-r \end{vmatrix}$$

$$7. \text{ Prove } \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

$$8. \text{ Prove } \begin{vmatrix} a^2 - bc & c^2 - ab \\ c^2 - ab & b^2 - ac \end{vmatrix} = c(3abc - a^3 - b^3 - c^3)$$

## Multiplication of Determinants.

815. *Lemma.*—

$$\begin{vmatrix} a_1 & b_1 & c_1 & l & m & n \\ a_2 & b_2 & c_2 & p & q & r \\ a_3 & b_3 & c_3 & s & t & u \\ 0 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

**Demonstration.**—

$$\begin{vmatrix} a_1 & b_1 & c_1 & l & m & n \\ a_2 & b_2 & c_2 & p & q & r \\ a_3 & b_3 & c_3 & s & t & u \\ 0 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & p & q & r \\ b_3 & c_3 & s & t & u \\ 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & l & m & n \\ b_3 & c_3 & s & t & u \\ 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} \\
+ a_3 \begin{vmatrix} b_1 & c_1 & l & m & n \\ b_2 & c_2 & p & q & r \\ 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} = a_1 b_2 \begin{vmatrix} c_3 & s & t & u \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} - a_1 b_3 \begin{vmatrix} c_2 & p & q & r \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} \\
- a_2 b_1 \begin{vmatrix} c_3 & s & t & u \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} + a_2 b_3 \begin{vmatrix} c_1 & l & m & n \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} + a_3 b_1 \begin{vmatrix} c_2 & p & q & r \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} \\
- a_2 b_3 \begin{vmatrix} c_1 & l & m & n \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} = a_1 b_2 c_3 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} - a_1 b_3 c_2 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \\
- a_2 b_1 c_3 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} + a_2 b_3 c_1 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} + a_3 b_1 c_2 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \\
- a_3 b_2 c_1 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad [789, (A)].$$

**Scholium.**—It will be seen that the elements  $l, m, n, p, q, r, s, t, u$  of the first member do not appear in the second member. This is due to the fact that in the development of the first member all the terms containing these elements eventually vanish.

**816. Problem.** To find the product of two determinants of any order in terms of a determinant of the same order.

**Example.**—Find the product of :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

**Explanation :**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} \quad [815]$$

$$= \begin{vmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ a_1 x_1 + a_2 y_1 + a_3 z_1 & b_1 x_1 + b_2 y_1 + b_3 z_1 & c_1 x_1 + c_2 y_1 + c_3 z_1 & x_1 & y_1 & z_1 \\ a_1 x_2 + a_2 y_2 + a_3 z_2 & b_1 x_2 + b_2 y_2 + b_3 z_2 & c_1 x_2 + c_2 y_2 + c_3 z_2 & x_2 & y_2 & z_2 \\ a_1 x_3 + a_2 y_3 + a_3 z_3 & b_1 x_3 + b_2 y_3 + b_3 z_3 & c_1 x_3 + c_2 y_3 + c_3 z_3 & x_3 & y_3 & z_3 \end{vmatrix}$$

(This last form is derived from the preceding by adding to the first column  $a_1$  times the 4th column +  $a_2$  times the 5th column +  $a_3$  times the 6th column; to the second column,  $b_1$  times the 4th +  $b_2$  times the 5th +  $b_3$  times the 6th; and to the third column,  $c_1$  times the 4th +  $c_2$  times the 5th +  $c_3$  times the 6th [812])

$$= \begin{vmatrix} a_1 x_1 + a_2 y_1 + a_3 z_1 & b_1 x_1 + b_2 y_1 + b_3 z_1 & c_1 x_1 + c_2 y_1 + c_3 z_1 \\ a_1 x_2 + a_2 y_2 + a_3 z_2 & b_1 x_2 + b_2 y_2 + b_3 z_2 & c_1 x_2 + c_2 y_2 + c_3 z_2 \\ a_1 x_3 + a_2 y_3 + a_3 z_3 & b_1 x_3 + b_2 y_3 + b_3 z_3 & c_1 x_3 + c_2 y_3 + c_3 z_3 \end{vmatrix} \quad [805, R.].$$

Let the student observe how the elements of the first column of the product are derived from the elements of the multiplicand and multiplier; then, how the elements of the second and third columns are found. The laws which he will observe are general for determinants of any order.

**Example.**—Show that, according to the laws above observed,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} a_1 x_1 + a_2 y_1 & b_1 x_1 + b_2 y_1 \\ a_1 x_2 + a_2 y_2 & b_1 x_2 + b_2 y_2 \end{vmatrix}$$

#### EXERCISE 110.

Find the value of :

1.  $\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \times \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}$

2.  $\begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} \times \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$

3.  $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} \times \begin{vmatrix} 3 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{vmatrix}$

4.  $\begin{vmatrix} 2 & 0 & 3 \\ 3 & 1 & 0 \\ 2 & 0 & 3 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 3 & 2 \end{vmatrix}$

## Applications.

**817. I. Solution of Simultaneous Equations of the First Degree.**

$$1. \text{ Solve } a_1 x + b_1 y = r_1 \quad (\text{A})$$

$$a_2 x + b_2 y = r_2 \quad (\text{B})$$

**Solution:** Multiply (A) by  $A_1$  and (B) by  $-A_2$ , in which  $A_1$  and  $A_2$  are the cofactors of  $a_1$  and  $a_2$  in the determinant  $[a_1 \ b_2]$ , and take the sum,

$$(a_1 A_1 - a_2 A_2)x + (b_1 A_1 - b_2 A_2)y = r_1 A_1 - r_2 A_2 \quad (\text{C})$$

Now,  $b_1 A_1 - b_2 A_2 = 0$  [814, P.]. Therefore,

$$(a_1 A_1 - a_2 A_2)x = r_1 A_1 - r_2 A_2; \text{ whence,}$$

$$x = \frac{r_1 A_1 - r_2 A_2}{a_1 A_1 - a_2 A_2} = \frac{[r_1 \ b_2]}{[a_1 \ b_2]} \quad [805, 795].$$

Again, multiply (A) by  $B_1$  and (B) by  $-B_2$ , in which  $B_1$  and  $B_2$  are the cofactors of  $b_1$  and  $B_2$ , and take the sum,

$$(a_1 B_1 - a_2 B_2)x + (b_1 B_1 - b_2 B_2)y = r_1 B_1 - r_2 B_2 \quad (\text{D})$$

Now,  $a_1 B_1 - a_2 B_2 = 0$  [814, P.]. Therefore,

$$y = \frac{r_1 B_1 - r_2 B_2}{b_1 B_1 - b_2 B_2} = \frac{[a_1 \ r_2]}{[a_1 \ b_2]} \quad [805, 795].$$

$$2. \text{ Solve } a_1 x + b_1 y + c_1 z = r_1 \quad - (\text{A})$$

$$a_2 x + b_2 y + c_2 z = r_2 \quad (\text{B})$$

$$a_3 x + b_3 y + c_3 z = r_3 \quad (\text{C})$$

**Solution:** Multiply (A) by  $A_1$ , (B) by  $-A_2$ , (C) by  $A_3$ , in which  $A_1$ ,  $A_2$ , and  $A_3$  are respectively the cofactors of  $a_1$ ,  $a_2$ , and  $a_3$  in the determinant  $[a_1 \ b_2 \ c_3]$ , and add the resulting equations,

$$(a_1 A_1 - a_2 A_2 + a_3 A_3)x + (b_1 A_1 - b_2 A_2 + b_3 A_3)y + (c_1 A_1 - c_2 A_2 + c_3 A_3)z = r_1 A_1 - r_2 A_2 + r_3 A_3$$

Now, the coefficients of  $y$  and  $z$  vanish [814, P.];

$$\therefore x = \frac{r_1 A_1 - r_2 A_2 + r_3 A_3}{a_1 A_1 - a_2 A_2 + a_3 A_3} = \frac{[r_1 \ b_2 \ c_3]}{[a_1 \ b_2 \ c_3]} \quad [805, 795].$$

Then, by symmetry,

$$y = \frac{[a_1 \ r_2 \ c_3]}{[a_1 \ b_2 \ c_3]} \text{ and } z = \frac{[a_1 \ b_2 \ r_3]}{[a_1 \ b_2 \ c_3]}.$$

In a similar manner it may be shown that, if we take  $n$  equations of the first degree of the form of  $a_1 x + b_1 y + c_1 z + \dots = r_1$ , mul-

tively the first by  $A_1$ , the second by  $-A_2$ , the third by  $A_3$ , ..., in which  $A_1, A_2, A_3, \dots$ , are the cofactors of  $a_1, a_2, a_3, \dots$ , and take the sum, the coefficients of all the unknown quantities, except  $x$ , will vanish, and we shall have

$$x = \frac{[r_1 \ b_2 \ c_3 \ \dots]}{[a_1 \ b_2 \ c_3 \ \dots]}; \text{ and, by symmetry,}$$

$$y = \frac{[a_1 \ r_2 \ c_3 \ \dots]}{[a_1 \ b_2 \ c_3 \ \dots]}, \text{ etc. Therefore,}$$

**Principle.**—Any unknown quantity in a complete system of simultaneous equations of the first degree equals a fraction whose denominator is a determinant formed from the coefficients of the terms of the equations taken in order, and whose numerator is formed from the denominator by replacing the coefficients of the unknown quantity by the corresponding right members of the equations.

## EXERCISE 111.

Solve :

1.  $3x + 2y = 16$

$2x - 3y = 2$

2.  $5x - 3y = 6$

$2x + 5y = 21$

3.  $ax + by = c$

$mx + ny = d$

4.  $(a+b)x - (c+d)y = m$

$(a-b)x + (c-d)y = n$

5.  $2x + 3y - 2z = 5$

$3x - 2y + 4z = 16$

$4x - 3y - z = -5$

6.  $x - y + z = 6$

$3x + 5y - 3z = 14$

$2x + 4y + 3z = 20$

7.  $ax + by + cz = d$

$cx + by + az = e$

$bx + cy + az = h$

8.  $(a+b)x + by + az = m$

$ax + (a+b)y + bz = n$

$bx + ay + (a+b)z = r$

9.  $x + y + z + u = 14$

$x - y + z - u = -2$

$x + y - z - u = -4$

$x - y - z + u = 0$

10.  $ax + by + cz = m$

$bx + cy + au = n$

$cx + az + bu = p$

$ay + bz + cu = q$

11.  $2x - 3y + 2z + u = -12$

$3x + 2y - 3z + 2u = 12$

$x - 3y + 4z + 3u = -24$

$2x + 2y - 3z - 4u = 28$

12.  $ax + by + cz + du = p$   
 $ax - by + cz - du = q$   
 $ax + by - cz - du = r$   
 $ax - by - cz + du = s$
13.  $2x + 3y - 4z + 2u + 3v = 19$   
 $3x - 2y + 2z - 3u + 4v = 13$   
 $2x - 4y + 3z - 2u + 2v = 5$   
 $x + y - 3z + 2u + v = 7$   
 $x + 2y + 3z - 4u + 5v = 23$
- 

**818. II.** *To determine under what condition  $(n + 1)$  equations of  $n$  unknown quantities may be simultaneously true.*

Assume the equations

$$a_1x + b_1y + c_1 = 0, \quad (\text{A})$$

$$a_2x + b_2y + c_2 = 0, \quad (\text{B})$$

and  $a_3x + b_3y + c_3 = 0 \quad (\text{C})$

to be simultaneously true.

Multiply (A) by  $C_1$ , (B) by  $-C_2$ , and (C) by  $C_3$ , in which  $C_1$ ,  $C_2$ , and  $C_3$  are the cofactors of  $c_1$ ,  $c_2$ , and  $c_3$  in the determinant  $[a_1 \ b_1 \ c_1]$ , and add the results. Then,

$$(a_1 C_1 - a_2 C_2 + a_3 C_3)x + (b_1 C_1 - b_2 C_2 + b_3 C_3)y + (c_1 C_1 - c_2 C_2 + c_3 C_3) = 0 \quad [\text{Ax. 2}].$$

$$\text{But, } a_1 C_1 - a_2 C_2 + a_3 C_3 = b_1 C_1 - b_2 C_2 + b_3 C_3 =$$

$$c_1 C_1 - c_2 C_2 + c_3 C_3 = [a_1 \ b_1 \ c_1] \quad [814, \text{P.}; 805, \text{R.}].$$

$\therefore [a_1 \ b_1 \ c_1] = 0$  is the condition under which (A), (B), and (C) are simultaneously true.

**Note.**—The equation  $[a_1 \ b_1 \ c_1] = 0$  is called the *eliminant* of the group.

In a similar manner it may be shown that  $n + 1$  equations of  $n$  unknown quantities, of the form of

$$a_1x + b_1y + \dots + r_1 = 0,$$

are simultaneously true, when

$$[a_1 \ b_1 \ c_1 \ \dots \ r_{n+1}] = 0.$$

**Note.**—Equations that are simultaneously true are said to *consist*, that is, they are *consistent*.

## EXERCISE 112.

Test the consistency of :

$$\begin{array}{ll} 1. \quad 2x + 3y - 13 = 0 & 2. \quad 3x + 2y - 17 = 0 \\ \quad 3x + 2y - 12 = 0 & \quad 5x - 3y - 3 = 0 \\ \quad x + 3y - 11 = 0 & \quad 2x - 5y + 15 = 0 \end{array}$$

819. III. To eliminate  $x$  from any two rational integral equations in  $x$ .

Illustrations.—1. Eliminate  $x$  from the equations

$$ax^2 + bx + c = 0 \quad (A)$$

$$mx^2 + nx + r = 0 \quad (B)$$

Solution : It is evident that

$$ax^2 + bx + c = 0$$

$$ax^2 + bx + c = 0$$

$$mx^2 + nx + r = 0$$

and

$$mx^2 + nx + r = 0$$

are simultaneously true. Therefore,

$$\begin{vmatrix} a & b & c & 0 \\ 0 & a & b & c \\ m & n & r & 0 \\ 0 & m & n & r \end{vmatrix} = 0$$

2. Eliminate  $x$  from the equations

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$mx^3 + nx^2 + px + q = 0$$

Solution : It is evident that

$$ax^5 + bx^4 + cx^3 + dx^2 + ex = 0$$

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$mx^5 + nx^4 + px^3 + qx^2 = 0$$

$$mx^4 + nx^3 + px^2 + qx = 0$$

$$mx^3 + nx^2 + px + q = 0$$

are simultaneously true. Therefore,

$$\begin{vmatrix} a & b & c & d & e & 0 \\ 0 & a & b & c & d & e \\ m & n & p & q & 0 & 0 \\ 0 & m & n & p & q & 0 \\ 0 & 0 & m & n & p & q \end{vmatrix} = 0$$

This method is known as Sylvester's Method of Elimination.

## Probabilities.

## Definitions and Fundamental Principles.

**820.** When the number of ways in which an event may occur is greater than the number of ways in which it may fail, and the ways are equally likely to happen, we say :

1. *The event is probable.*
2. *The event is likely to happen.*
3. *The chance is in favor of the event.*
4. *The odds are in favor of the event.*

**821.** When the number of ways in which an event may fail is greater than the number of ways in which it may occur, and the ways are equally likely to happen, we say :

1. *The event is improbable.*
2. *The event is not likely to happen.*
3. *The chance is against the event.*
4. *The odds are against the event.*

**822.** When the number of ways in which an event may occur is equal to the number of ways in which it may fail, and the ways are equally likely to happen, we say :

1. *The occurrence and failure of the event are equally probable.*
2. *The event is as likely to happen as to fail.*
3. *There is an even chance for and against the event.*
4. *The odds are even for and against the event.*

**823.** If an event can occur in  $a$  ways and fail in  $b$  ways, and the ways are equally likely to happen, we need more definite language to express the exact probability or chance of the event. Thus, we say :

1. *The odds are as  $a$  to  $b$  in favor of the event.*
2. *The odds are as  $b$  to  $a$  against the event.*

**824.** If we let  $k$  represent the probability of any particular way happening,  $a$  the number of ways favorable to the event, and  $b$  the number of unfavorable ways, then will  $ak$  represent the probability of the event happening, and  $bk$  the probability of its failing, and  $ak + bk$ , or  $(a + b)k$ , *certainty*, which is taken as the unit of measure,

then  $(a + b)k = 1$ ; whence,  $k = \frac{1}{a + b}$ ;

and  $ak = \frac{a}{a + b}$ , the probability or chance of the event, and  $bk = \frac{b}{a + b}$ , the probability or chance against the event. Therefore,

*Prin. 1.*—The probability or chance of an event happening equals the number of favorable ways divided by the whole number of ways.

*Prin. 2.*—The probability or chance of an event failing equals the number of unfavorable ways divided by the whole number of ways.

**825.** Since an event is certain to happen or fail, and certainty is expressed by unity, it follows that,

*Prin. 3.*—The probability of an event happening equals unity minus the probability that it will fail; and the probability that it will fail equals unity minus the probability that it will happen.

**Illustration.**—If there are 3 black and 2 white balls in a bag containing only 5 balls, what is the chance,

1. That a black ball will be drawn on the first trial?

2. That a black ball will not be drawn on the first trial?

**Solution:** 1. There are 3 favorable ways out of 5 to draw a black ball; therefore, the chance is  $\frac{3}{5}$  (Prin. 1).

2. There are 2 unfavorable ways out of 5 to draw a black ball, namely, the two favorable ways for drawing a white ball; therefore, the chance of failing to draw a black ball is  $\frac{2}{5}$ . Or,

That a black ball will be drawn or not drawn on the first trial is *certainly*. The chance for drawing a black ball is  $\frac{3}{5}$ ; therefore, the chance of failure is  $1 - \frac{3}{5} = \frac{2}{5}$ .

**826. Exclusive Events.**—Two or more events are mutually exclusive when the happening of one of them precludes the possibility of any other one happening. Thus, if a coin be thrown up, it may fall either head or tail. If it fall head, or is supposed to fall head, it can not fall tail, or be supposed to fall tail, in the same throw. Falling head and falling tail are, therefore, mutually exclusive events.

**827.** In a bag are  $d$  balls;  $a$  of them are white,  $b$  blue,  $c$  red, and the remaining ones yellow. What is the chance of drawing, on the first trial,

1. Either a red or a white ball?
2. A red, a white, or a blue ball?

**Solution:** 1. The chance of drawing a red ball is  $\frac{c}{d}$ , and the chance of drawing a white ball is  $\frac{a}{d}$ ; and the chance of drawing either a red or a white ball is  $\frac{a+c}{d} = \frac{a}{d} + \frac{c}{d}$ .

2. The chance of drawing a red ball is  $\frac{c}{d}$ ; of drawing a white ball,  $\frac{a}{d}$ ; of drawing a blue ball,  $\frac{b}{d}$ ; and of drawing a red, a white, or a blue ball,  $\frac{c+a+b}{d}$ ; which equals  $\frac{c}{d} + \frac{a}{d} + \frac{b}{d}$ . Therefore,

**Prin. 4.**—*The chance that one of several mutually exclusive events will happen equals the sum of their separate chances of happening.*

#### EXERCISE 118.

1. What is the chance of throwing 4 with a single die?

**Suggestion.**—A die has six faces, which are equally liable to turn up, but only one of these contains four dots. Therefore, the chance is  $\frac{1}{6}$ .

2. What is the chance of throwing an even number with a single die?

**Suggestion.**—Three of the faces have an even number of dots; therefore, the chance is  $\frac{3}{6}$ , or  $\frac{1}{2}$ .

3. If the odds be 4 to 3 in favor of an event, what are the respective chances of the success and failure of the event?

**Suggestion.**—There are 4 points out of 7 favorable and 3 out of 7 unfavorable to the happening of the event; therefore, the respective chances of success and failure are  $\frac{4}{7}$  and  $\frac{3}{7}$ .

4. If 4 coppers are tossed, what are the odds against exactly 2 turning up head?

**Suggestion.**—Each coin may fall in two ways; hence, the four coins may fall in  $2^4 = 16$  ways [550, Cor.]. The two coins that may turn up head can be selected from the four coins in  $\frac{4 \times 3}{2}$ , or 6 ways. Therefore, the chance of success is  $\frac{6}{16}$ , or  $\frac{3}{8}$ , and the chance of failure is  $1 - \frac{3}{8} = \frac{5}{8}$ . Therefore, the odds are as 5 to 3 against the event.

5. In a bag are 7 white and 5 red balls; if two are drawn, find the chance that 1 is red and 1 white.

**Solution:** Two balls can be selected from 12 balls in  $\frac{12 \times 11}{2} = 66$  ways. One white ball can be selected from 7 white balls in 7 ways, and 1 red ball from 5 red balls in 5 ways. Hence, 1 white ball and 1 red ball can be selected from 7 white and 5 red balls in  $7 \times 5$ , or 35 ways. Therefore, 35 out of 66 ways are favorable to drawing 1 white and 1 red ball. Therefore, the chance is  $\frac{35}{66}$ .

6. Twenty persons take their seats at a round table. What are the odds against two persons thought of sitting together?

**Solution:** Let the two persons be A and B. Besides the place where A may sit, there are 19 places, two of which are adjacent to him, and the remaining 17 not adjacent. Any of these B may select. Therefore, the odds are as 17 to 2 against A and B sitting together.

**828. Expectation.**—The value of any probability of prize or property depending upon the occurrence of some uncertain event is called an *Expectation*.

7. A person holds  $a$  tickets in a lottery in which the whole number of tickets issued is  $n$ . There is only one prize offered, and this is worth  $\$p$ . What is the person's expectation?

**Solution:** It is evident that the  $n$  tickets are worth  $\$p$ , and that the tickets are of equal value before the drawing; therefore, the  $a$  tickets are worth  $\frac{a}{n}$  of  $\$p$ , which is  $\$p \times \frac{a}{n}$ . Therefore,

**829. Prin. 5.**—*The expectation of an event equals the product of the sum to be realized and the chance of the event.*

8. A person is allowed to draw two bank-notes from a bag containing 8 ten-dollar bills and 20 two-dollar bills. What is his expectation?

**Solution:** The two notes can be drawn from 28 notes in  $\frac{28 \times 27}{2}$   
 $= 378$  ways. Two ten-dollar notes can be drawn from 8 ten-dollar notes in  $\frac{8 \times 7}{2} = 28$  ways. Two two-dollar notes can be drawn from 20 two-dollar notes in  $\frac{20 \times 19}{2} = 190$  ways.

One ten-dollar note and one two-dollar note can be drawn from 8 ten-dollar notes and 20 two-dollar notes in  $8 \times 20 = 160$  ways. Therefore,

The chance of drawing  $\$20$  is  $\frac{28}{378}$ , and the expectation is  $\$1.48\frac{4}{9}$ .

The chance of drawing  $\$4$  is  $\frac{190}{378}$ , and the expectation is  $\$2.01\frac{11}{18}$ .

The chance of drawing  $\$12$  is  $\frac{160}{378}$ , and the expectation is  $\$5.07\frac{5}{9}$ .

$\therefore$  The entire expectation is  $\$1.48\frac{4}{9} + \$2.01\frac{11}{18} + \$5.07\frac{5}{9} = \$8.57\frac{1}{7}$ .

9. A bag contains a £5 note, a £10 note, and six pieces of blank paper of the same size and texture as a bank-note. Show that the expectation of a man who is allowed to draw out one piece of paper is £1 17s. 6d.

**830. Independent Events.**—Two or more events are independent of each other when the happening of one of them does not affect the probability of any other one's happening.

**10.** There are  $b$  balls in one bag,  $a$  of which are white;  $d$  in another,  $c$  of which are white; and  $f$  in another,  $e$  of which are white. Show that the chance of drawing one white ball from each bag in a single trial is  $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$ .

**Solution:** One ball can be drawn from each bag in  $b \times d \times f$  ways [550]. One white ball can be drawn from each bag in  $a \times c \times e$  ways [550]. Therefore, the chance of drawing a white ball from each bag is  $\frac{a \times c \times e}{b \times d \times f}$  [824, P. 1]  $= \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$ . Therefore,

**831. Prin. 6.**—*The chance of two or more independent events happening simultaneously is the product of their several chances of happening.*

**832. Cor. 1.**—*The chance of two or more independent events failing simultaneously is the product of their several chances of failing.*

**833. Cor. 2.**—*The chance of one of two independent events failing and the other happening is the product of the chance that one fails and the chance that the other happens. .*

**11.** A can solve 3 problems out of 4, B 5 out of 6, and C 7 out of 8. What is the chance that a certain problem will be solved, if all try?

**Solution:** Unless all fail, the problem will be solved. The chance that A will fail is  $\frac{1}{4}$ , that B will fail  $\frac{1}{6}$ , that C will fail  $\frac{1}{8}$ , that all will fail  $\frac{1}{4} \times \frac{1}{6} \times \frac{1}{8} = \frac{1}{192}$ . Therefore, the chance of success is  $\frac{191}{192}$ .

**834. Dependent Events.**—In a series of events, any assumed event is said to be *dependent* upon a preceding

event, if the happening of the preceding event changes the probability of the happening of the assumed event.

**12.** Find the chance of drawing 3 white balls in succession from a bag containing 5 white and 3 red balls.

**Solution:** The chance of drawing a white ball on the first trial is  $\frac{5}{8}$ . Having drawn a white ball, there remain in the bag 7 balls, 4 of which are white. The chance of drawing a white ball on the second trial is therefore  $\frac{4}{7}$ . Similarly, the chance of drawing a white ball on the third trial is  $\frac{3}{6}$ . Therefore, the chance of drawing three white balls in succession is  $\frac{5}{8} \times \frac{4}{7} \times \frac{3}{6}$  [831] =  $\frac{5}{28}$ . Therefore,

**835. Prin. 7.**—*The chance that a series of events should happen is the continued product of the chance that the first should happen, the chance that the second should then happen, the chance that the third should follow, and so on.*

**13.** In one of two bags are 3 red and 4 white balls, and in the other 5 red and 3 white balls, and a ball is to be drawn from one or other of the bags. Find the chance that the ball drawn will be white.

**Solution:** The chance that the first bag will be chosen is  $\frac{1}{2}$ . Then, the chance of drawing a white ball from the first bag is  $\frac{4}{7}$ ; hence, the real chance of drawing a white ball from the first bag is  $\frac{1}{2}$  of  $\frac{4}{7} = \frac{2}{7}$ . Similarly, the chance of drawing a white ball from the second bag is  $\frac{1}{2}$  of  $\frac{3}{8} = \frac{3}{16}$ . These events are mutually exclusive; therefore, the chance required is  $\frac{2}{7} + \frac{3}{16} = \frac{53}{112}$ .

**836. Inverse Probability.**—When an event is known to have happened from one of two or more known causes, the determination of the chance that it has happened from

any particular one of these causes is a problem of *inverse probability*.

14. It is known that a black ball has been drawn from one of two bags. The first of these bags contained  $m$  balls,  $a$  of which were black, and the second  $n$  balls,  $b$  of which were black. What is the chance that the ball was drawn from the first bag?

**Solution:** Suppose that  $2N$  drawings were made. The chance is that  $N$  were made from each bag. In the  $N$  drawings from the first bag the chance is that  $\frac{a}{m} \times N$  were black balls. In the drawings from the second bag the chance is that  $\frac{b}{n} \times N$  were black balls. Therefore, in  $2N$  drawings, the chance is that  $\left(\frac{a}{m} + \frac{b}{n}\right)N$  were black balls. Therefore, the chance that a black ball was drawn from the first bag is  $\left(\frac{a}{m} \times N\right) \div \left(\frac{a}{m} + \frac{b}{n}\right)N = \frac{an}{an + bm}$ .

**837. Theorem.**—*If an event is believed to have been produced by some one of the causes  $P_1, P_2, P_3, \dots, P_n$ , which are mutually exclusive, and  $p_1, p_2, p_3, \dots, p_n$  represent the respective probabilities of these causes when no other causes exist, then the probability that  $P_r$  produced the event is*

$$\frac{P_r p_r}{P_1 p_1 + P_2 p_2 + \dots + P_n p_n}.$$

**Demonstration.**—Let  $N$  be the number of trials made in producing the event. The first cause operated  $N \times P_1$  times; therefore, on the supposition that no other causes operated than those named, the probability that the event was produced by the first cause is  $N \times P_1 \times p_1$ . Under similar restrictions, the probability that the event was produced by the second cause is  $N \times P_2 \times p_2$ ; by the third cause,  $N \times P_3 \times p_3$ ; by the  $r$ th cause,  $N \times P_r \times p_r$ ; by any one of the causes,  $N(P_1 p_1 + P_2 p_2 + P_3 p_3 + \dots + P_n p_n)$ . Therefore, the real chance of its having been caused by the  $r$ th cause, or  $P_r$ , is

$$\frac{N \times P_r \times p_r}{N(P_1 p_1 + P_2 p_2 + \dots + P_n p_n)} = \frac{P_r \times p_r}{P_1 p_1 + P_2 p_2 + \dots + P_n p_n}.$$

15. Four bags were known to contain 3 red and 4 white, 4 red and 3 white, 5 red and 1 white, and 4 red and 4

white balls respectively. A white ball was drawn at random from one of the bags. Find the chance that it was drawn from the second bag.

**Solution:**  $P_1 = P_2 = P_3 = P_4 = \frac{1}{4}$ ,  $p_1 = \frac{4}{7}$ ,  $p_2 = \frac{3}{7}$ ,  $p_3 = \frac{1}{6}$ , and  $p_4 = \frac{1}{2}$ . Therefore, the required probability is

$$\frac{\frac{1}{4} \times \frac{3}{7}}{\frac{1}{4} \left( \frac{4}{7} + \frac{3}{7} + \frac{1}{6} + \frac{1}{2} \right)} = \frac{\frac{3}{7}}{\frac{5}{3}} = \frac{9}{35}.$$

**838. Probability of Testimony.**—The following examples illustrate how to deal with questions relating to the credibility of testimony:

**16.** A speaks the truth  $a$  times in  $m$ , B  $b$  times in  $n$ , and C  $c$  times in  $r$ . What is the chance that a statement is true which all affirm? Which A and B affirm and C denies?

**Solution:** 1. The statement is either true or false. If true, all have spoken the truth; the probability of which is  $\frac{a}{m} \times \frac{b}{n} \times \frac{c}{r} = \frac{abc}{mnr}$ . If false, all have lied; the probability of which is

$$\left(1 - \frac{a}{m}\right) \left(1 - \frac{b}{n}\right) \left(1 - \frac{c}{r}\right) = \frac{(m-a)(n-b)(r-c)}{mnr}.$$

Hence, the probability of the truth of the statement is,

$$\frac{abc}{mnr} + \left\{ \frac{abc}{mnr} + \frac{(m-a)(n-b)(r-c)}{mnr} \right\} = \frac{abc}{abc + (m-a)(n-b)(r-c)}.$$

2. If the statement is true, A and B have told the truth and C has lied; the probability of which is  $\frac{a}{m} \times \frac{b}{n} \times \left(1 - \frac{c}{r}\right) = \frac{ab(r-c)}{mnr}$ . If the statement is false, A and B have lied and C has told the truth; the probability of which is  $\left(1 - \frac{a}{m}\right) \left(1 - \frac{b}{n}\right) \left(\frac{c}{r}\right) = \frac{(m-a)(n-b)c}{mnr}$ .

Hence, the probability of the truth of the statement is,

$$\frac{ab(r-c)}{mnr} + \left\{ \frac{ab(r-c)}{mnr} + \frac{(m-a)(n-b)c}{mnr} \right\} = \frac{ab(r-c)}{ab(r-c) + (m-a)(n-b)c}$$

**17.** A, B, and C tell the truth to the best of their knowledge and belief. A observes correctly 4 times out of 5, B 3 times out of 5, and C 5 times out of 7. What is the probability that a phenomenon occurred (which was

just as likely to fail as to occur), provided all had equal opportunity of observing, and all report its occurrence? What if A and B report its occurrence and C its failure?

**Solution:** 1. The phenomenon either occurred or failed. If it occurred, A, B, and C observed correctly; the probability of which is  $\frac{4}{5} \times \frac{3}{5} \times \frac{5}{7}$ . The inherent probability that it would occur is  $\frac{1}{2}$ . Hence, the probability that the assumption that it occurred is correct is  $\frac{1}{2} \times \frac{4}{5} \times \frac{3}{5} \times \frac{5}{7} = \frac{6}{35}$ .

If it did not occur, all observed falsely; the probability of which is  $\frac{1}{5} \times \frac{2}{5} \times \frac{2}{7}$ ; and the probability of the correctness of the assumption that the phenomenon failed is  $\frac{1}{2} \times \frac{1}{5} \times \frac{2}{5} \times \frac{2}{7} = \frac{2}{175}$ . Hence, the chance that the phenomenon occurred is  $\frac{6}{35} + \left( \frac{6}{35} + \frac{2}{175} \right) = \frac{15}{16}$ .

2. The probability of the correctness of the assumption that the phenomenon occurred is  $\frac{1}{2} \times \frac{4}{5} \times \frac{3}{5} \times \frac{2}{7} = \frac{12}{175}$ .

The probability of the correctness of the assumption that the phenomenon failed is  $\frac{1}{2} \times \frac{1}{5} \times \frac{2}{5} \times \frac{5}{7} = \frac{1}{35}$ .

Hence, the chance of the event is  $\frac{12}{175} + \left( \frac{12}{175} + \frac{1}{35} \right) = \frac{12}{17}$ .

**Note.**—For a fuller treatment of Choice and Chance than space will permit to give in this book, see Whitworth's "Choice and Chance."

#### EXERCISE 114.

1. If A's chance of winning a race is  $\frac{1}{6}$  and B's chance  $\frac{1}{8}$ , show that the chance that both will fail is  $\frac{17}{24}$ .

2. If the odds be  $m$  to  $n$  in favor of an event, show that the chance of the event is  $\frac{m}{m+n}$ , and the chance against the event is  $\frac{n}{m+n}$ .

3. If the letters  $e, t, s, n$  be arranged in a row at random, show that the chance of having an English word is  $\frac{1}{6}$ .

4. Show that the chance that the year  $1900 + x$ , in which  $x < 100$ , is a leap-year, is  $\frac{8}{33}$ .

5. A draws 3 balls from a bag containing 3 white and 6 black balls; B draws 1 ball from another bag containing 1 white and 2 black balls. Show that A's chance of drawing a white ball is to B's chance as 16 to 7.

6. Show that when two dice are thrown the chance that the throw will amount to more than 8 is  $\frac{5}{18}$ .

7. Show that the chance of throwing exactly 11 in one throw with two dice is  $\frac{1}{18}$ .

8. One purse contains 5 sovereigns and 4 shillings; another contains 5 sovereigns and 3 shillings. Show that the chance of drawing a sovereign is  $\frac{85}{144}$ , if a purse is selected at random and a coin drawn from it at random. Show that the expectation of the privilege is  $12s. 2\frac{7}{12}d$ .

9. There are three independent events whose several chances are  $\frac{2}{3}$ ,  $\frac{3}{5}$ , and  $\frac{1}{2}$ . Show that the chance that one of them will happen and only one is  $\frac{3}{10}$ .

10. If two letters are taken at random out of *esteemed*, show that the odds against both being *e* are the same as the odds in favor of one at least being *e*.

11. A letter is taken at random out of each of the words *choice* and *chance*. Show that the chance that they are the same letter is  $\frac{1}{6}$ .

12. A bag contains 6 black and 1 red ball. Show that the expectation of a person who is to receive a shilling for every ball he draws out before drawing the red one is 3 shillings.

13. Two numbers are chosen at random. Show that the chance is  $\frac{1}{2}$  that their sum is even.

14. An archer hits his target on an average 3 times out of 4. Show that the chance that he will hit it exactly 3 times in 4 successive trials is  $\frac{27}{64}$ .

15. A's reputation for telling the truth is  $\frac{3}{7}$ , B's  $\frac{4}{9}$ , and C's  $\frac{7}{8}$ . 1st, Show that the probability of the truth of a statement which all deny is  $\frac{5}{26}$ ; 2d, Show that the probability of the truth of the statement is  $\frac{35}{38}$ , if A and B deny it and C affirms it.

16. Show that with two dice the chance of throwing more than 7 is equal to the chance of throwing less than 7.

17. Two persons throw a die alternately, with the understanding that the first who throws 6 is to receive 11 cents. Show that the expectation of the first is to that of the second as 6 to 5.

18. A's chance of winning a single game against B is  $\frac{3}{5}$ . Show that his chance of winning at least 2 games out of 3 is  $\frac{81}{125}$ .

19. A party of  $n$  persons take their seats at random at a round table. Show that it is  $n - 3$  to 2 against two specified persons sitting together.

20. Show that the chance that a person with 2 dice will throw double aces exactly 3 times in 5 trials is  $\left(\frac{1}{36}\right)^3 \times \left(\frac{35}{36}\right)^2 \times 10$ .

21. There are 10 tickets, five of which are numbered 1, 2, 3, 4, 5, and the rest are blank. Show that the probability of drawing a total of *ten* in three trials, one ticket being drawn each time and replaced, is  $\frac{33}{1000}$ .

# SUPPLEMENT.

## CONTINUED FRACTIONS.

### 1. Definitions.

**839.** An expression in the form of

$$\frac{a}{b + \frac{c}{d + \frac{e}{f + \text{etc.}}}}$$

is a *Continued Fraction*.

**840.** The discussion in this section will be limited to continued fractions in the form of

$$\frac{1}{a + \frac{1}{b + \frac{1}{c + \text{etc.}}}}$$

and  $\frac{1}{a}$ ,  $\frac{1}{b}$ ,  $\frac{1}{c}$ , etc., will be called *Partial Fractions*.

**841.** A continued fraction may be written in a more convenient form, as follows :

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \dots + \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \dots$$

**842.** When the number of partial fractions in a continued fraction is finite, it is a *terminating* continued fraction ; when infinite, an *interminate* continued fraction.

**843.** If at some stage in an interminate continued fraction one or more partial fractions begin to repeat in the same order, it is called a *periodic* continued fraction.

**844.** A periodic continued fraction is *pure* when it contains no other than repeating partial fractions, and *mixed* when it contains one or more partial fractions before the repeating ones.

Thus,  $\frac{1}{a} + \frac{1}{b} + \frac{1}{a} + \frac{1}{b} + \dots$ , or  $\frac{1}{a} + \frac{1}{b}$ ,  
is a pure periodic fraction ;  
and  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{b} + \frac{1}{c} + \dots$ , or  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ ,  
is a mixed periodic fraction.

**845.** The fraction resulting from stopping at any stage is called a *convergent*.

## 2. The Formative Law of Successive Convergents.

**846.** In the continued fraction

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \dots + \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} + \dots,$$

$$\frac{1}{a} = \text{the first convergent.}$$

$$\frac{1}{a} + \frac{1}{b} = \frac{b}{ab+1} = \text{the second convergent.}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc+1}{(ab+1)c+a} = \text{the third convergent.}$$

It will be seen that

1. *The numerator of the third convergent is the numerator of the second convergent multiplied by the denominator of the third partial fraction, plus the numerator of the first convergent ; and*

2. *The denominator of the third convergent is the denominator of the second convergent multiplied by the denominator of the third partial fraction, plus the denominator of the first convergent.*

Will these laws hold true in the formation of any convergent from the two preceding convergents?

Let  $\frac{P}{P_1}$ ,  $\frac{Q}{Q_1}$ ,  $\frac{R}{R_1}$ , and  $\frac{S}{S_1}$  be respectively the  $(n-2)$ th,  $(n-1)$ th,  $n$ th, and  $(n+1)$ th convergents; and  $p$ ,  $q$ ,  $r$ , and  $s$  the denominators of the  $(n-2)$ th,  $(n-1)$ th,  $n$ th, and  $(n+1)$ th partial fractions.

Suppose the laws to hold true in the formation of the convergent  $\frac{R}{R_1}$ , then will  $\frac{R}{R_1} = \frac{rQ + P}{rQ_1 + P_1}$ . (A)

Now, from the nature of the continued fraction,  $\frac{S}{S_1}$  may be formed by putting  $r + \frac{1}{s}$  for  $r$  in  $\frac{R}{R_1}$ . Therefore,

$$\begin{aligned}\frac{S}{S_1} &= \frac{\left(r + \frac{1}{s}\right)Q + P}{\left(r + \frac{1}{s}\right)Q_1 + P_1} = \frac{(sr + 1)Q + sP}{(sr + 1)Q_1 + sP_1} \\ &= \frac{s(rQ + P) + Q}{s(rQ_1 + P_1) + Q_1} = \frac{sR + Q}{sR_1 + Q_1}.\end{aligned}$$

Therefore, if the laws are applicable in the formation of the  $n$ th convergent, they are also applicable in the formation of the  $(n+1)$ th convergent. But we have seen that they do apply in the formation of the third convergent, and, hence, apply in the formation of the fourth convergent, and so on. Therefore, in general,

1. The numerator of the  $n$ th convergent equals the numerator of the  $(n-1)$ th convergent multiplied by the denominator of the  $n$ th partial fraction, plus the numerator of the  $(n-2)$ th convergent; and

2. The denominator of the  $n$ th convergent equals the denominator of the  $(n-1)$ th convergent multiplied by the denominator of the  $n$ th partial fraction, plus the denominator of the  $(n-2)$ th convergent.

**Example.**—Find the first 8 convergents of the continued fraction  $\frac{1}{2} + \frac{1}{3} + \frac{1}{1} + \frac{1}{4} + \frac{1}{5} + \frac{1}{2} + \frac{1}{4} + \frac{1}{3}$ .

**Solution :**

$$\frac{R}{R_1} = \frac{rQ + P}{rQ_1 + P_1} = \frac{1}{2}, \frac{3}{7}, \frac{4}{9}, \frac{19}{43}, \frac{99}{224}, \frac{217}{491}, \frac{967}{2188}, \frac{3118}{7055}.$$

### Properties of Convergents.

**847.** Take the continued fraction

$$y = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \dots$$

1.  $a < a + \frac{1}{b} + \frac{1}{c} + \dots, \therefore \frac{1}{a} > y$
2.  $b < b + \frac{1}{c} + \frac{1}{d} + \dots, \therefore a + \frac{1}{b} > a + \frac{1}{b} + \frac{1}{c} + \dots;$   
whence,  $\frac{1}{a} + \frac{1}{b} < y$
3.  $c < c + \frac{1}{d} + \dots, \therefore b + \frac{1}{c} > b + \frac{1}{c} + \frac{1}{d} + \dots;$   
whence,  $a + \frac{1}{b} + \frac{1}{c} < a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \dots;$   
and  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > y$ , etc. Therefore,

**Prin. 1.**—*The successive convergents are alternately greater and less than the continued fraction (the odd orders being too great and the even orders too small).*

**848.** The difference between the first two convergents  $= \frac{1}{a} - \frac{b}{ab+1} = \frac{1}{a(ab+1)} =$  *unity divided by the product of their denominators.* Is this a general law?

Let  $\frac{P}{P_1}$ ,  $\frac{Q}{Q_1}$ , and  $\frac{R}{R_1}$  be the  $(n-1)$ th,  $n$ th, and  $(n+1)$ th convergents, and  $p$ ,  $q$ , and  $r$  the denominators of the

$(n-1)$ th,  $n$ th, and  $(n+1)$ th partial fractions, and let  $\sim$  denote *difference between*.

$$\begin{aligned} \text{Assume } \frac{P}{P_1} \sim \frac{Q}{Q_1} &= \frac{P Q_1 \sim Q P_1}{P_1 Q_1} = \frac{1}{P_1 Q_1}; \text{ then will} \\ \frac{R}{R_1} \sim \frac{Q}{Q_1} &= \frac{R Q_1 \sim Q R_1}{Q_1 R_1} = \frac{Q_1 (r Q + P) \sim Q (r Q_1 + P_1)}{Q_1 R_1} \\ &= \frac{P Q_1 \sim Q P_1}{Q_1 R_1} = \frac{1}{Q_1 R_1}. \end{aligned} \quad (\text{A})$$

Therefore, if the law holds good for the difference between the  $(n-1)$ th and  $n$ th convergents, it will also for the difference between the  $(n+1)$ th and the  $n$ th convergents. But we have seen that it does hold good for the difference between the first and second convergents, and, hence, it will for the difference between the next higher pair, and so on. Therefore,

**Prin. 2.**—*The difference between any two consecutive convergents equals unity divided by the product of their denominators.*

**849.** Since  $P Q_1 \sim Q P_1 = 1$  [848, A],  $P$  and  $P_1$  can not have a common factor, neither can  $Q$  and  $Q_1$ .

Therefore,

**Prin. 3.**—*Every convergent is in its lowest terms.*

**850.** If we let  $\frac{U}{U_1}$  represent the true value of the continued fraction; then will

$$\begin{aligned} \frac{P}{P_1} \sim \frac{U}{U_1} &< \frac{P}{P_1} \sim \frac{Q}{Q_1} \text{ [P. 1]}; \text{ also, } \frac{U}{U_1} \sim \frac{Q}{Q_1} < \frac{P}{P_1} \sim \frac{Q}{Q_1} \\ \therefore \frac{P}{P_1} \sim \frac{U}{U_1} &< \frac{1}{P_1 Q_1} \text{ [P. 2]}; \text{ also, } \frac{U}{U_1} \sim \frac{Q}{Q_1} < \frac{1}{P_1 Q_1}. \end{aligned}$$

Hence, if either  $\frac{P}{P_1}$  or  $\frac{Q}{Q_1}$  be used for  $\frac{U}{U_1}$ , the error will be less than  $\frac{1}{P_1 Q_1}$ , or less than  $\frac{1}{Q_1^2}$ .

**851.** Let  $\frac{P}{P_1}$ ,  $\frac{Q}{Q_1}$ , and  $\frac{R}{R_1}$  be three consecutive convergents whose terminal partial fractions are  $\frac{1}{p}$ ,  $\frac{1}{q}$ , and  $\frac{1}{r}$ ; and  $\frac{U}{U_1}$ , the true value of the continued fraction. Then,  $\frac{U}{U_1}$  differs from  $\frac{R}{R_1}$  only in the use of  $r + \frac{1}{s +}$  etc. for  $r$ . Put  $r + \frac{1}{s +}$  etc. =  $x$ .

$$\text{Now, } \frac{R}{R_1} = \frac{r Q + P}{r Q_1 + P_1} \quad \therefore \frac{U}{U_1} = \frac{x Q + P}{x Q_1 + P_1}$$

$$\therefore \frac{U}{U_1} \sim \frac{Q}{Q_1} = \frac{x Q + P}{x Q_1 + P_1} \sim \frac{Q}{Q_1} = \frac{P Q_1 \sim P_1 Q}{Q_1 (x Q_1 + P_1)} \\ = \frac{1}{Q_1 (x Q_1 + P_1)}$$

$$\text{and, } \frac{P}{P_1} \sim \frac{U}{U_1} = \frac{P}{P_1} \sim \frac{x Q + P}{x Q_1 + P_1} = \frac{x (P Q_1 - P_1 Q)}{P_1 (x Q_1 + P_1)} \\ = \frac{x}{P_1 (x Q_1 + P_1)}$$

Now,  $x > 1$ , and  $P_1 < Q_1$ ,

$$\therefore \frac{1}{Q_1 (x Q_1 + P_1)} < \frac{x}{P_1 (x Q_1 + P_1)}; \text{ or,}$$

$\frac{Q}{Q_1}$  is nearer  $\frac{U}{U_1}$  than is  $\frac{P}{P_1}$ . Therefore,

**Prin. 4.**—The higher the order of a convergent the nearer does it approach to the true value of the continued fraction.

**852. Cor.**—A continued fraction is the limit of its convergents; or, if  $y$  be a continued fraction and  $x$  its variable convergent,  $y = \lim. x$ .

**853.** The higher the order, the greater will be the denominator of a convergent [846, 2] and the nearer will the value of the convergent be to the value of the continued fraction [851, Prin. 4]. But may there not be some other

fraction, not a convergent, with smaller denominator, that is a nearer approximation to a continued fraction than a given convergent?

Suppose  $\frac{M}{M_1}$  not a convergent, and nearer to  $\frac{U}{U_1}$  than  $\frac{Q}{Q_1}$ , and  $M_1 < Q_1$ ; then  $\frac{U}{U_1} \sim \frac{M}{M_1} < \frac{Q}{Q_1} \sim \frac{U}{U_1}$ ;

$$\therefore \frac{M}{M_1} \sim \frac{P}{P_1} < \frac{Q}{Q_1} \sim \frac{P}{P_1}; \text{ or, } \frac{MP_1 \sim M_1P}{M_1P_1} < \frac{1}{Q_1P_1}.$$

But  $M_1P_1 < Q_1P_1$ , since  $M_1 < Q_1$ .

$\therefore MP_1 \sim M_1P < 1$ ; which is impossible, since  $M$ ,  $M_1$ ,  $P$ , and  $P_1$  are integral. Therefore,

**Prin. 5.**—*Any convergent is nearer the true value of a continued fraction than any fraction with smaller denominator.*

#### Problems.

**854. 1.** To reduce a common fraction to a terminating continued fraction.

Since an improper fraction is equivalent to an integer and a proper fraction, it will be necessary only to investigate a method for extending a proper fraction.

Let  $\frac{b}{a}$  = a proper fraction in its lowest terms.

Divide both terms by  $b$ , and put for the improper fraction  $\frac{a}{b}$ , the mixed number  $p + \frac{c}{b}$ ; then,

$$\frac{b}{a} = \frac{1}{p + \frac{c}{b}}$$

Divide both terms of  $\frac{c}{b}$  by  $c$ , and put  $\frac{b}{c} = q + \frac{d}{c}$ ;

then,

$$\frac{b}{a} = \frac{1}{p + \frac{1}{q + \frac{d}{c}}}$$

Again, divide both terms of  $\frac{d}{c}$  by  $d$ , and put  $\frac{c}{d} = r + \frac{e}{d}$ ;  
 then,

$$\frac{b}{a} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{e}{d}$$

It will now be seen that the denominators of the successive partial fractions have been obtained as follows :

$$\begin{array}{r} b) \ a \ (p \\ \underline{bp} \\ c) \ b \ (q \\ \underline{cq} \\ d) \ c \ (r \\ \underline{rd} \\ e \text{ etc.} \end{array}$$

Since  $a$  and  $b$  are integral, they have a highest common divisor, and the division will eventually terminate.

Therefore, the continued fraction will be a *terminating one*.

**Rule.**—To reduce a proper fraction to a terminating continued fraction, find the highest common divisor of its terms by successive division, and use the quotients in regular order for the denominators of the partial fractions.

**855. 2. To reduce a quadratic surd to a continued fraction.**

**Illustrations.**—1. Reduce  $\sqrt{26}$  to a continued fraction.

**Solution:**  $\sqrt{26} = 5 + \frac{1}{x}$

$$\begin{aligned} \therefore x &= \frac{1}{\sqrt{26} - 5} = \sqrt{26} + 5 = 10 + \frac{1}{x} \\ &= 10 + \frac{1}{10 + \frac{1}{x}} = 10 + \frac{1}{10 + \frac{1}{10 + \frac{1}{x}}} \end{aligned}$$

$$\therefore \sqrt{26} = 5 + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \dots = 5 + \frac{1}{10}$$

2. Reduce  $\sqrt{19}$  to a continued fraction.

**Solution:**  $\sqrt{19} = 4 + \frac{1}{x}$

$$\therefore x = \frac{1}{\sqrt{19} - 4} = \frac{\sqrt{19} + 4}{3} = 2 + \frac{1}{x_1}$$

$$\therefore x_1 = \frac{3}{\sqrt{19} - 2} = \frac{\sqrt{19} + 2}{5} = 1 + \frac{1}{x_2}$$

$$\therefore x_2 = \frac{5}{\sqrt{19} - 3} = \frac{\sqrt{19} + 3}{2} = 3 + \frac{1}{x_3}$$

$$\therefore \sqrt{19} = 4 + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \dots$$

**Scholium.**—A quadratic surd may always be reduced to a periodic continued fraction if the expansion is carried sufficiently far.

**856. 3.** To reduce a periodic continued fraction to a simple fraction.

The periodic continued fraction

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + x = x$$

$$\therefore \frac{qr + qx + 1}{pqr + pqx + p + r + x} = x$$

whence,  $(pq + 1)x^2 + (pqr - q + p + r)x = qr + 1$

The value of  $x$  found from this equation is the value of the continued fraction.

**857. 4.** To approximate the ratio of two numbers.

**Example.**—When the diameter of a circle is 1, the circumference is  $3.1415926 +$ . Approximate the ratio of the diameter to the circumference.

**Solution:**

$$1 : 3.1415926 = \frac{10000000}{31415926} = \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \dots \text{ [Prob. 1].}$$

The successive convergents, which are also the successive approximations of the ratio, are:  $\frac{1}{3}$ ,  $\frac{7}{22}$ ,  $\frac{106}{333}$ ,  $\frac{113}{355}$ , etc.

EXERCISE 113.

Reduce to continued fractions :

1.  $\frac{125}{317}$

2.  $\frac{140}{213}$

3.  $\frac{100}{999}$

4.  $\frac{106}{729}$

5.  $\sqrt{10}$

6.  $\sqrt{12}$

7.  $\sqrt{30}$

8.  $\sqrt{57}$

9.  $\cdot 3183$

10.  $3\cdot 1416$

11.  $67^{\circ} 20' 30''$

Find the successive convergents of :

12.  $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

13.  $\frac{1}{3} + \frac{1}{1} + \frac{1}{9} + \frac{1}{1}$

14.  $\frac{i}{2} + \frac{i}{3}$

15.  $\frac{1}{1} + \frac{i}{2} + \frac{i}{3}$

16.  $\frac{1}{2} + \frac{i}{3}$

Find the true value of :

17.  $\frac{i}{4}$

18.  $\frac{i}{2} + \frac{i}{4}$

19.  $\frac{1}{2} + \frac{i}{4}$

20.  $\frac{i}{1} + \frac{1}{2} + \frac{i}{1}$

21. Find a series of common fractions converging to  $1 : \sqrt{3}$ .

22. Express approximately the ratio of a liquid quart ( $57\cdot 75$  cu. in.) to a dry quart ( $67\cdot 2$  cu. in.).

23. The square root of 600 is  $24\cdot 494897$ , and the cube root of 600 is  $8\cdot 434327$ . Find a series of four common fractions approximating nearer and nearer to the ratio of the latter to the former.

24. The imperial bushel of Great Britain contains  $2218\cdot 192$  cu. in., and the Winchester bushel  $2150\cdot 42$  cu. in. Find the nearest approximation, that can be expressed by a common fraction whose denominator is less than 100, of the ratio of the latter to the former.

25. Two scales of equal length having their zero points coinciding also have the 27th gradation of the one to coincide with the 85th gradation of the other. Show that the 7th and 22d more nearly coincide than any other two gradations.

## THEORY OF NUMBERS.

## Systems of Notation.

## 1. Definitions.

**858.** *Notation* is the art of expressing numbers by means of characters.

**859.** A *system of notation* is a method of expressing numbers in a series of powers of some fixed number.

**860.** The order of progression on which any system of notation is founded is called the *scale* of the system, and the fixed number on which the scale is based is called the *radix*.

**861.** Any integral number, except unity, may be taken as the *radix*. When the radix is two, the scale and system are called *binary*; when three, *ternary*; when four, *quaternary*; when five, *quinary*; when six, *senary*; when seven, *septenary*; when eight, *octary*; when nine, *nonary*; when ten, *denary* or *decimal*; when eleven, *undenary*; when twelve, *duodenary*; etc.

**862.** In the decimal or denary system,

$$56342 = 5 \times 10,000 + 6 \times 1000 + 3 \times 100 + 4 \times 10 + 2 = 5 \times 10^4 + 6 \times 10^3 + 3 \times 10^2 + 4 \times 10 + 2;$$

or, in inverse order,

$$2 + 4 \times 10 + 3 \times 10^2 + 6 \times 10^3 + 5 \times 10^4.$$

In the octary system,

$$34725 = 3 \times 8^4 + 4 \times 8^3 + 7 \times 8^2 + 2 \times 8 + 5;$$

or, in inverse order,

$$5 + 2 \times 8 + 7 \times 8^2 + 4 \times 8^3 + 3 \times 8^4.$$

∴ In general, if  $r$  be taken as the radix, and

$$a_0, a_1, a_2, a_3 \dots a_{n-1}$$

as the  $n$  digits of a number, reckoning in order from right to left, the number is represented by

$$a_{n-1}r^{n-1} + a_{n-2}r^{n-2} + a_{n-3}r^{n-3} + \dots + a_2r^2 + a_1r + a_0$$

**863. Theorem.**—*Any integral number may be expressed in the form of*

$$ar^n + br^{n-1} + cr^{n-2} + \dots + pr^2 + qr + s,$$

*in which the coefficients are each less than  $r$ .*

**Demonstration:** Let  $N$  equal the number of units in any number, and  $r^n$  the highest power of the radix less than  $N$ .

Divide  $N$  by  $r^n$ , and let the quotient be  $a$  and the remainder  $N'$ . Then  $N = ar^n + N'$ .

Now,  $a$  is less than  $r$ , else  $r^n$  would not be the highest power of  $r$  less than  $N$ ; and  $N'$  is less than  $r^n$ .

Divide  $N'$  by  $r^{n-1}$ , and let the quotient be  $b$  and the remainder  $N''$ . Then  $N' = br^{n-1} + N''$ , in which  $b < r$  and  $N'' < r^{n-1}$ .

In like manner, divide  $N''$  by  $r^{n-2}$ , and let the quotient be  $c$ , and the remainder  $N'''$ . Then  $N'' = cr^{n-2} + N'''$ , in which  $c < r$  and  $N''' < r^{n-2}$ .

If this process be continued, a remainder,  $s$ , will eventually be reached less than  $r$ . Therefore,  $N = ar^n + br^{n-1} + cr^{n-2} + \dots + pr^2 + qr + s$ , in which the coefficients are each less than  $r$ .

**864. Cor.**—*In any system of notation, the number of digits including 0 is equal to the radix.*

**865. Problem.** To express a given number in any proposed scale.

**Solution:** Let  $N$  be the number and  $r$  the radix of the proposed scale.

Suppose  $N = ar^n + br^{n-1} + cr^{n-2} + \dots + pr^2 + qr + s$ , it is required to find the values of  $a, b, c, \dots, p, q, s$ .

$$\frac{N}{r} = ar^{n-1} + br^{n-2} + cr^{n-3} + \dots + pr + q + \frac{s}{r}.$$

Therefore, the remainder, after dividing  $N$  by  $r$ , is the last digit.

Suppose  $N' = ar^{n-1} + br^{n-2} + cr^{n-3} + \dots + pr + q$ .

$$\frac{N'}{r} = ar^{n-2} + br^{n-3} + cr^{n-4} + \dots + p + \frac{q}{r}.$$

Therefore, the remainder, after dividing  $N'$  by  $r$ , is the next digit.

Suppose  $N'' = ar^{n-3} + br^{n-4} + cr^{n-5} + \dots + p$ .

$$\frac{N''}{r} = ar^{n-4} + br^{n-5} + cr^{n-6} + \dots + \frac{p}{r}.$$

Therefore, the remainder, after dividing  $N''$  by  $r$ , is the next digit.  
Etc.,                      etc.,                      etc.

Therefore,

**Rule.**—Divide the number by the radix, then the quotient by the radix, and so on until the quotient becomes less than the radix; the successive remainders will be the digits of the number, beginning with the units.

**Illustrative Examples.**—1. Express 35432 (denary scale) in the senary scale; also, 35432 (senary scale) in the octary scale.

$$\begin{array}{r} (1) \quad 6 \overline{) 35432} \\ \quad 6 \overline{) 5905} - 2 \\ \quad \quad 6 \overline{) 984} - 1 \\ \quad \quad \quad 6 \overline{) 164} - 0 \\ \quad \quad \quad \quad 6 \overline{) 27} - 2 \\ \quad \quad \quad \quad \quad 4 - 3 \end{array}$$

$$\begin{array}{r} (2) \quad 8 \overline{) 35432} \\ \quad 8 \overline{) 2545} - 4 \\ \quad \quad 8 \overline{) 212} - 1 \\ \quad \quad \quad 8 \overline{) 14} - 0 \\ \quad \quad \quad \quad 1 - 2 \end{array}$$

$$\therefore 35432_{r=3} = 12014_{r=8}$$

$$\therefore 35432_{r=10} = 432012_{r=6}$$

**Explanation of (2):**

$$35 + 8 = (3 \times 6 + 5) + 8 = 23 + 8 = 2, \text{ and } 7 \text{ over;}$$

$$74 + 8 = (7 \times 6 + 4) + 8 = 46 + 8 = 5, \text{ and } 6 \text{ over;}$$

$$63 + 8 = (6 \times 6 + 3) + 8 = 39 + 8 = 4, \text{ and } 7 \text{ over;}$$

$$72 + 8 = (7 \times 6 + 2) + 8 = 44 + 8 = 5, \text{ and } 4 \text{ over.}$$

2. Express 35439 (denary scale) in duodenary scale; also, 34439 (nonary scale) in undenary scale.

**Note.**—The undenary scale needs a character to represent *ten*, and the duodenary scale two characters to represent *ten* and *eleven*. We will represent ten by *t* and eleven by *e*.

$$\begin{array}{r} (1) \quad 12 \overline{) 34439} \\ \quad 12 \overline{) 2869} - e \\ \quad \quad 12 \overline{) 239} - 1 \\ \quad \quad \quad 12 \overline{) 19} - e \\ \quad \quad \quad \quad 1 - 7 \end{array}$$

$$\begin{array}{r} (2) \quad 11 \overline{) 35439} \\ \quad 11 \overline{) 2852} - 5 \\ \quad \quad 11 \overline{) 236} - 8 \\ \quad \quad \quad 11 \overline{) 18} - 8 \\ \quad \quad \quad \quad 1 - 6 \end{array}$$

$$\therefore 34439_{r=10} = 17e1e_{r=12}$$

$$\therefore 35439_{r=9} = 16885_{r=11}$$

## EXERCISE 116.

1. Find the sum in senary scale of  $4532_{r=6}$ ,  
 $3452_{r=6}$ ,  $5423_{r=6}$ , and  $3251_{r=6}$
  2. Find the difference (octary scale) of  $3574_{r=8}$  and  
 $2756_{r=8}$
  3. Multiply  $36425_{r=7}$  by 8 ; also  $25436_{r=8}$  by 10
  4. Divide  $47654_{r=11}$  by 9 ; also  $2e58t3_{r=12}$  by 11
  5. Express  $43250_{r=5}$  in the denary scale.
  6. Express  $38472_{r=9}$  in the septenary scale.
  7. Express  $35243_{r=6}$  in the duodenary scale.
  8. Express  $8et950_{r=12}$  in the quaternary scale.
  9. Find the sum (denary scale) of  $3472_{r=8}$  and  $5842_{r=10}$
  10. Find the difference (nonary scale) of  $5t34_{r=11}$  and  
 $6432_{r=7}$
  11. What is the radix of the scale in which  $476_{r=10}$   
 $= 2112$  ?
- Suggestion.**—Let  $r$  = the radix ; then will  $2r^3 + r^2 + r + 2 = 476$ .
12. In what scale is 3 times  $134 = 450$  ?
  13. In what scale is  $135_{r=6} = 43$  ?
  14. What is the H. C. D. of  $36_{r=8}$ ,  $48_{r=8}$ , and  $60_{r=8}$  ?
  15. Multiply  $28_{r=9}$  by  $45_{r=9}$  ; also square  $25_{r=6}$
  16. In what scale is 1552 the square of 34 ?
  17. Show that 35, 44, and 53 are in arithmetical progression in any scale of notation.
  18. Show that 1331 is a perfect cube in any system of notation.
  19. Show that 14641 is a perfect fourth power in any system of notation.
  20. Show that 11, 220, and 4400 are in geometrical progression in any system of notation.

### Divisibility of Numbers and their Digits.

**866. Theorem I.**—*If a number,  $N$ , be divided by any factor of  $r$ ,  $r^2$ ,  $r^3$ , etc., respectively ( $r$  being the radix), it will leave the same remainder as when the number expressed by the last term, the last two terms, the last three terms, etc., is divided by the same factor.*

**Demonstration:** Suppose  $x$  a factor of  $r$ ,  $y$  a factor of  $r^2$ , and  $z$  a factor of  $r^3$ , etc.

$$N = ar^{n-1} + br^{n-2} + \dots + pr^2 + qr + s.$$

Now,  $x$  is certainly a factor of every term of  $N$ , except  $s$ ;  $y$ , a factor of every term, except  $qr + s$ ; and  $z$ , a factor of every term, except  $pr^2 + qr + s$ , etc. Therefore,

1.  $\frac{N}{x} = \text{an integer} + \frac{s}{x}.$
2.  $\frac{N}{y} = \text{an integer} + \frac{qr + s}{y}.$
3.  $\frac{N}{z} = \text{an integer} + \frac{pr^2 + qr + s}{z};$  which was to be proved.

**867. Cor.**—*In the decimal system of notation,*

1. *A number is divisible by any factor of 10, if the units' digit is divisible by that factor.*

2. *A number is divisible by any factor of 100, if the number expressed by the last two figures is divisible by that factor.*

3. *A number is divisible by any factor of 1000, if the number expressed by the last three figures is divisible by that factor.*

**868. Theorem II.**—*The difference between a number and the sum of its digits is divisible by the radix less one.*

**Demonstration:**

Let  $N = ar^{n-1} + br^{n-2} + \dots + pr^2 + qr + s = \text{any number};$   
then,  $a + b + \dots + p + q + s = \text{the sum of the digits}.$

Now,  $a(r^{n-1} - 1) + b(r^{n-2} - 1) + \dots + p(r^2 - 1) + q(r - 1) = \text{the difference between the number and the sum of its digits, and every term is divisible by } r - 1.$

**869. Cor.**—*In the decimal system of notation, The difference between a number and the sum of its digits is divisible by 9 or 3.*

**870. Theorem III.**—*A number,  $N$ , divided by  $r - 1$ , leaves the same remainder as the sum of its digits divided by  $r - 1$ ,  $r$  being the radix.*

**Demonstration.**—Put  $s$  for the sum of the digits;  $q$  and  $q'$  for the quotients; and  $c$  and  $c'$  for the remainders.

$$1. \quad N = q(r - 1) + c$$

$$2. \quad s = q'(r - 1) + c'$$

$$\therefore N - s = (q - q')(r - 1) + (c - c').$$

Now,  $N - s$  is divisible by  $r - 1$  [T. II], and  $(q - q')(r - 1)$  is evidently divisible by  $r - 1$ ; therefore,  $c - c'$  is divisible by  $r - 1$ . But  $c$  and  $c'$  are each less than  $r - 1$ ; hence,  $c - c' = 0$ , or  $c = c'$ .

**871. Cor.**—*In the decimal system, A number is divisible by 9, if the sum of its digits is divisible by 9.*

**872. Theorem IV.**—*If from a number,  $N$ , we subtract the digits of the even powers of  $r$ , and add those of the odd powers, the result will be divisible by  $r + 1$ .*

**Demonstration.**—Let  $N = ar^4 + br^3 + cr^2 + dr + e$ .

$$\text{Add} \quad -a \quad +b \quad -c \quad +d \quad -e,$$

then,  $a(r^4 - 1) + b(r^3 + 1) + c(r^2 - 1) + d(r + 1)$ , the result, is divisible by  $r + 1$ , since every term is divisible by  $r + 1$ .

**873. Theorem V.**—*If a number,  $N$ , be divided by  $r + 1$ , the remainder will be the same as when the difference between the sums of the digits of the even and odd powers of  $r$  is divided by  $r + 1$ .*

**Demonstration.**—Put  $d$  for the difference between the sums of the digits of the even and odd powers of  $r$ ;  $q$  and  $q'$  for the quotients; and  $c$  and  $c'$  for the remainders; then will

$$N = q(r + 1) + c,$$

$$\text{and} \quad d = q'(r + 1) + c'.$$

$$\therefore N - d = (q - q')(r + 1) + c - c'.$$

Now,  $N - d$  is divisible by  $r + 1$  [T. IV], and  $(q - q')(r + 1)$  is evidently divisible by  $r + 1$ . Therefore,  $c - c'$  is divisible by  $r + 1$ . But  $c$  and  $c'$  are each less than  $r + 1$ ; hence,  $c - c' = 0$ , or  $c = c'$ .

**874. Cor.**—*In the decimal system of notation,*

*A number is divisible by 11, if the difference between the sums of the digits in the even and odd places is divisible by 11.*

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### Even and Odd Numbers.

**875.** An *even number* is a number that is exactly divisible by 2.

**876.** An *odd number* is a number that is not exactly divisible by 2.

**877.** If we let  $x$  represent any integral number including zero, and regard zero as an even number, it becomes evident that the general formula for an even number is  $2x$ , and for an odd number  $2x + 1$ .

**878. Theorem I.**—*The sum of any number of even numbers is even.*

**Demonstration.**—Let  $2x_1, 2x_2, 2x_3, \dots, 2x_n$  represent  $n$  even numbers; then will their sum be

$2x_1 + 2x_2 + 2x_3 + \dots + 2x_n = 2(x_1 + x_2 + x_3 + \dots + x_n)$   
an even number.

**879. Theorem II.**—*The sum of an even number of odd numbers is even.*

**Demonstration.**—Let  $2x_1 + 1, 2x_2 + 1, 2x_3 + 1, \dots, 2x_{2n} + 1$  represent  $2n$  odd numbers; then will their sum be

$$\begin{aligned} &(2x_1 + 1) + (2x_2 + 1) + (2x_3 + 1) + \dots + (2x_{2n} + 1) = \\ &2x_1 + 2x_2 + 2x_3 + \dots + 2x_{2n} + 2n = \\ &2(x_1 + x_2 + x_3 + \dots + x_{2n} + n), \text{ an even number.} \end{aligned}$$

**880. Theorem III.**—*The sum of an odd number of odd numbers is odd.*

**Demonstration.**—Let  $2x_1 + 1, 2x_2 + 1, 2x_3 + 1, \dots, 2x_{2n+1} + 1$  represent  $2n + 1$  odd numbers; then will their sum be

$$\begin{aligned} &(2x_1 + 1) + (2x_2 + 1) + (2x_3 + 1) + \dots + (2x_{2n+1} + 1) = \\ &2x_1 + 2x_2 + 2x_3 + \dots + 2x_{2n+1} + 2n + 1 = \\ &2(x_1 + x_2 + x_3 + \dots + x_{2n+1} + n) + 1, \text{ an odd number.} \end{aligned}$$

**881. Theorem IV.**—*The sum of an equal even number of even and odd numbers is even.*

**Demonstration.**—Let  $(2x_1 + 1) + (2x_2 + 1) + \dots + (2x_n + 1) =$   
the sum of  $2n$  odd numbers;

and  $2x'_1 + 2x'_2 + \dots + 2x'_n =$   
the sum of  $2n$  even numbers; then will their sum be

$\{2(x_1 + x'_1) + 1\} + \{2(x_2 + x'_2) + 1\} + \dots + \{2(x_n + x'_n) + 1\},$   
which is even [T. II].

**882. Theorem V.**—*The sum of an equal odd number of even and odd numbers is odd.*

**Demonstration.**—Let  $(2x_1 + 1) + (2x_2 + 1) + \dots + (2x_{n+1} + 1) =$   
the sum of  $2n + 1$  odd numbers;

and  $2x'_1 + 2x'_2 + \dots + 2x'_{n+1} =$   
the sum of  $2n + 1$  even numbers; then will their sum be

$\{2(x_1 + x'_1) + 1\} + \{2(x_2 + x'_2) + 1\} + \dots + \{2(x_{n+1} + x'_{n+1}) + 1\},$   
which is odd [T. III].

**883. Theorem VI.**—*The difference between two numbers, if both are odd or both even, is even.*

**Demonstration.**—1. Let  $2x$  and  $2x'$  be two even numbers.

Their difference is  $2x - 2x' = 2(x - x')$ , which is even.

2. Let  $2x + 1$  and  $2x' + 1$  be two odd numbers.

Their difference is  $(2x + 1) - (2x' + 1) = 2x - 2x' = 2(x - x')$ ,  
which is even.

**884. Theorem VII.**—*The difference between an odd and an even number is odd.*

**Demonstration.**—Let  $2x + 1$  be any odd number, and  $2x'$  any even number.

Their difference is  $(2x + 1) - 2x' = 2(x - x') + 1$ , which is odd.

**885. Theorem VIII.**—*The product of any number of even numbers is even.*

**Demonstration.**—Let  $2x_1, 2x_2, 2x_3, \dots, 2x_n$  be  $n$  even numbers.

Their product is  $2(2^{n-1}x_1, x_2, x_3, \dots, x_n)$ , which is even.

**Cor.**—*Any power of an even number is even.*

**886. Theorem IX.**—*The product of any number of odd numbers is odd.*

**Demonstration.**—Let  $2x_1 + 1, 2x_2 + 1, \dots, 2x_n + 1$  be  $n$  odd numbers. It is evident, from the nature of multiplication, that the product of these numbers will contain the factor 2 in every term, except the last, which will be 1. That is, the product will have the form of  $2x' + 1$ , which is odd.

**887. Cor.**—*Any power of an odd number is odd.*

**888. Theorem X.**—*The product of any number of odd and even numbers is even.*

**Demonstration.**—The product of the odd numbers is odd [T. IX], and may be represented by  $2x + 1$ .

The product of the even numbers is even [T. VIII], and may be represented by  $2x'$ .

$\therefore$  The entire product is  $2x'(2x + 1) = 2(x'x + x')$ , which is even.

**Example.**—It is required to divide one dollar among 15 boys, giving to each boy an odd number of cents. Is this question possible?

## Prime, Composite, Square, and Cubic Numbers.

### 1. Definitions.

**889. A Prime Number** is a number that can not be produced by multiplying together factors other than itself and unity.

*A prime number is divisible only by itself and unity.*

**890. A Composite Number** is a number that may be produced by multiplying together other factors than itself and unity.

*A composite number is divisible by other factors than itself and unity.*

**891. A Square Number** is one that may be resolved into two equal factors.

**892. A Cubic Number** is one that may be resolved into three equal factors.

**893.** Two or more numbers are *prime to each other* when they have no common factor, except unity.

2. Primes.

**894. Theorem I.**—*The number of primes is unlimited.*

For, let  $n$  be the number of primes, and, if  $n$  is not unlimited, let  $p$  be the greatest prime number. Then will  $2 \times 3 \times 5 \times 7 \times 11 \times \dots \times p$  be divisible by all primes not greater than  $p$ ; and  $(2 \times 3 \times 5 \times 7 \times 11 \times \dots \times p) + 1$  not be divisible by any prime not greater than  $p$ . Therefore,  $(2 \times 3 \times 5 \times 7 \times 11 \times \dots \times p) + 1$  is itself a prime greater than  $p$ , or is divisible by a prime greater than  $p$ . In either case,  $p$  is not the greatest prime. Therefore,  $n$  is unlimited.

**895. Theorem II.**—*Every prime number, except 2 and 3, belongs to the form  $6n \pm 1$ .*

For, every number evidently belongs to one of the forms  $6n$ ,  $6n + 1$ ,  $6n + 2$ ,  $6n + 3$ ,  $6n + 4$ , or  $6n + 5$ , in which  $n$  may be any integer including 0. Now,  $6n$ ,  $6n + 2$ , and  $6n + 4$ , are each divisible by 2, and  $6n + 3$  by 3; hence, these forms are composite, except when  $n = 0$  in  $6n + 2$  and  $6n + 3$ , in which case we have the primes 2 and 3.

The only forms remaining to contain primes are  $6n + 1$  and  $6n + 5$ . But  $6n + 5 = (6n + 6) - 1 = 6(n + 1) - 1 = 6n' - 1$ . Therefore, the general form  $6n \pm 1$  contains all primes, except 2 and 3.

**Schollum.**—*It must not be inferred from this proposition that all numbers expressed by  $6n \pm 1$  are prime. Thus, when  $n = 4$ ,  $6n + 1 = 25$ ; and when  $n = 11$ ,  $6n - 1 = 65$ .*

*Cor.—Every prime above 3, increased or diminished by unity, is divisible by 6.*

**896. Theorem III.**—No rational formula can represent primes only.

For, if possible, let  $a + bx + cx^2 + dx^3 + \dots$  be prime for all values of  $x$ .

When  $x = m$ , let  $a + bx + cx^2 + dx^3 + \dots = p$ ;  
then,  $p = a + bm + cm^2 + dm^3 + \dots$

When  $x = m + np$ , let  $a + bx + cx^2 + dx^3 + \dots = q$ ;  
then,  $q = a + b(m + np) + c(m + np)^2 +$   
 $d(m + np)^3 + \dots$   
 $= a + bm + cm^2 + dm^3 + \dots + rp$   
 $= p + rp = p(1 + r)$ , a composite number.

**897. Scholium.**—The form  $n^2 + n + 41$  is prime for all values of  $n$  from 0 to 39 inclusive, and the form  $2n^2 + 29$  for all values of  $n$  from 0 to 28 inclusive. These forms have been discovered by trial, and are not demonstrable.

**898. Theorem IV.**—If a number is not divisible by a factor equal to or less than its square root, it is a prime.

For, let  $N = x \times y$  be any number not prime. Then, if  $x = y$ ,  $N = y^2$ , and  $y = \sqrt{N}$ . But, if  $x > \sqrt{N}$ , then  $y < \sqrt{N}$ , since  $x \times y = N$ . But  $N$  is divisible by  $y$ . Therefore, if  $N$  is not prime, it is divisible by a factor equal to or less than  $\sqrt{N}$ . Hence, too, if a number is not divisible by a factor equal to or less than its square root, it is prime.

### 3. Composites.

**899. Theorem I.**—If a number is a factor of the product of two numbers and is not a factor of one of them, it is a factor of the other.

Thus, let  $x$  be a factor of  $ab$ , and not a factor of  $a$ ; then will it be a factor of  $b$ .

For,  $\frac{a}{x}$  may be reduced to a terminating continued fraction [854]. Let  $\frac{p}{q}$  be the convergent next in value to  $\frac{a}{x}$ . Then,  $\frac{p}{q} \sim \frac{a}{x} = \frac{1}{qx}$  [848, P.]; whence,

$$px \sim aq = 1; \text{ and } bpx \sim abq = b.$$

Now,  $bpx$  and  $abq$  are each divisible by  $x$ ; therefore, their difference,  $b$ , is divisible by  $x$ .

**900. Cor.**—*If a number is prime to each of two or more other numbers, it is prime to their product.*

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**901. Theorem II.**—*Every composite number may be resolved into one set of prime factors and into only one set.*

1. Any composite number ( $N$ ) is the product of two or more factors each less than  $N$ , which are all composite, all prime, or some composite and some prime. As many of these as are composite are again resolvable into other factors less than themselves, and so on, until no factor is further resolvable into factors less than itself and greater than unity, at which stage all the factors are prime.

2. Let one set of prime factors of  $N$  be  $a, b, c, \dots$ , and, if possible, let another set be  $p, q, r, \dots$ ; then will  $a \times b \times c \times \dots = p \times q \times r \times \dots$ .

Now, suppose  $a$  different from  $q, r, \dots$ , then it is not contained in  $q \times r \times \dots$  [900]; it must, therefore, be contained in  $p$ , but this can only be when  $a = p$ , since  $p$  is a prime. But, if  $a = p$ ,  $b \times c \times \dots = q \times r \times \dots$ ; from which it follows as before that  $b$  is identical with one of the factors in  $q \times r \times \dots$ ; etc.

**902. Theorem III.**—*The product of any  $r$  consecutive numbers is divisible by  $\lfloor r \rfloor$ .*

For,  $\frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r \rfloor}$  is the product of  $r$  consecutive numbers divided by  $\lfloor r \rfloor$ , and it is also the number of combinations of  $n$  things taken  $r$  together, which is evidently a whole number.

**903. Cor. 1.**—The coefficient of the  $(r+1)$ th term of the binomial theorem is  $\frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r \rfloor}$  [595]; therefore,

*The coefficient of every term of the binomial theorem is integral when  $n$  is a positive integer.*

**904. Cor. 2.**—If we represent  $\frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r \rfloor}$  by  $q$ , it follows that,

*All factors of the numerator that are prime and are greater than  $r$  are divisors of  $q$ .*

**905. Theorem IV.**—*Fermat's Theorem. If  $p$  be any prime number, and  $a$  be a number prime to  $p$ , then  $a^{p-1} - 1$  will be divisible by  $p$ .*

**Demonstration:**  $a^p = [1 + (a-1)]^p$

$$= 1 + p(a-1) + \frac{p(p-1)}{\lfloor 2 \rfloor} (a-1)^2 + \dots + (a-1)^p \quad (\text{A})$$

$$\begin{aligned} \therefore a^p - (a-1)^p - 1 &= p(a-1) + \frac{p(p-1)}{\lfloor 2 \rfloor} + \text{etc.} \\ &= \text{a multiple of } p \text{ [901. 140, P.].} \end{aligned} \quad (\text{B})$$

Let  $a=2$ , then

$$a^p - (a-1)^p - 1 = 2^p - 2 = \text{a multiple of } p.$$

Let  $a=3$ , then

$$\begin{aligned} a^p - (a-1)^p - 1 &= 3^p - 2^p - 1 = (3^p - 3) - (2^p - 2) \\ &= \text{a multiple of } p. \end{aligned}$$

$$\therefore 3^p - 3 \text{ is a multiple of } p \text{ [157, P.].}$$

By continuing this process, it may be shown by induction that  $a^p - a$  is a multiple of  $p$ .

But  $a^p - a = a(a^{p-1} - 1)$  and  $a$  is prime to  $p$ ; therefore,  $a^{p-1} - 1$  is divisible by  $p$ .

### Perfect Squares.

**906. Theorem I.**—Every square number is of the form  $3m$  or  $3m + 1$ .

For, every number is of the form of  $3x$  or  $3x \pm 1$ .

Now,  $(3x)^2 = 9x^2 = 3(3x^2) = 3m$ ; and

$$(3x \pm 1)^2 = (9x^2 \pm 6x + 1) = 3(3x \pm 2) + 1 = 3m + 1.$$

**907. Theorem II.**—Every square number is of the form  $4m$  or  $4m + 1$ .

For, every number is of the form of  $4x$ ,  $4x + 1$ ,  $4x + 2$ , or  $4x + 3$ .

$$\text{Now, } (4x)^2 = 16x^2 = 4(4x^2) = 4m;$$

$$(4x + 1)^2 = 16x^2 + 8x + 1 = 4(4x^2 + 2x) + 1 \\ = 4m + 1;$$

$$(4x + 2)^2 = 16x^2 + 16x + 4 = 4(4x^2 + 4x + 1) \\ = 4m;$$

$$\text{and } (4x + 3)^2 = 16x^2 + 24x + 9 = 4(4x^2 + 6x + 2) + 1 \\ = 4m + 1.$$

**908. Theorem III.**—Every square number is of the form  $5m$  or  $5m \pm 1$ .

For, every number is of the form  $5x$ ,  $5x \pm 1$ , or  $5x \pm 2$ .

$$\text{Now, } (5x)^2 = 25x^2 = 5(5x^2) = 5m;$$

$$(5x \pm 1)^2 = 25x^2 \pm 10x + 1 = 5(5x^2 \pm 2x) + 1 \\ = 5m + 1;$$

$$\text{and } (5x \pm 2)^2 = 25x^2 \pm 20x + 4 = 5(5x^2 \pm 4x + 1) - 1 \\ = 5m - 1.$$

**909. Theorem IV.**—If  $a^2 + b^2 = c^2$  when  $a$ ,  $b$ , and  $c$  are integers, then will  $abc$  be a multiple of 60.

For, 1.  $a^2$  and  $b^2$  can not both be of the form  $3m + 1$ , else would  $c^2$  be of the form  $3m + 2$ , which is not a square. Therefore, either  $a$  or  $b$  is a multiple of 3 [906].

2.  $a^2$  and  $b^2$  can not both be of the form of  $4m + 1$ , else would  $c^2$  be of the form  $4m + 2$ , which is not a square. Therefore, either  $a$  or  $b$  must be a multiple of 4, or each of them a multiple of 2 [907]. In either case,  $abc$  is a multiple of 4.

3.  $a^2$  and  $b^2$  can not both be of the form  $5m + 1$  or  $5m - 1$ , else would  $c^2$  be of the form  $5m \pm 2$ , which is not a square. Therefore, either  $a^2$  or  $b^2$  must be of the form  $5m$ , or one of the form  $5m + 1$  and the other of  $5m - 1$  [908]. In the former case, either  $a$  or  $b$  is a multiple of 5, and in the latter,  $c$  is a multiple of 5, and in either case,  $abc$  is a multiple of 5.

4. Since  $abc$  is a multiple of 3, 4, and 5, and these numbers are prime to each other,  $abc$  is a multiple of 60.

*Scholium.*—By means of this theorem and the formula  $a = \sqrt{(c+b)(c-b)}$ , rational values of  $a$ ,  $b$ , and  $c$  may be determined by inspection that will satisfy the equation  $a^2 + b^2 = c^2$ .

**910. Problem.** To determine the rational value of  $x$  that will render  $x^2 + px + q$  a perfect square.

**Solution:** Let  $x^2 + px + q = (x + m)^2$   
 then,  $x^2 + px + q = x^2 + 2mx + m^2$   
 whence,  $x = \frac{m^2 - q}{p - 2m}$ , in which  $m$  may have any rational value from  $-\infty$  to  $+\infty$ .

**Illustration.**—What value of  $x$  will render  $x^2 - 7x + 2$  a perfect square?

**Solution:** Here  $p = -7$ ,  $q = 2$ , and let  $m = 5$ ,

then, 
$$x = \frac{25 - 2}{-7 - 10} = \frac{23}{-17} = -1\frac{6}{17}$$

and, 
$$x^2 - 7x + 2 = \frac{529}{289} + \frac{161}{17} + 2 = \frac{3844}{289} = \left(\frac{62}{17}\right)^2.$$

**911. Cor. 1.**—For  $m > \sqrt{q}$  and  $2m < p$ , or  $m < \sqrt{q}$  and  $2m > p$ ,  $m$  being positive,  $x$  will be positive.

**912. Cor. 2.**—Put  $m^2 - q = n(p - 2m)$ ; then

$$q = m^2 - n(p - 2m);$$

$$x = n, \text{ an integer; and}$$

$$\begin{aligned} x^2 + px + q &= x^2 + px + m^2 - n(p - 2m) \\ &= (n + m)^2, \text{ an integer.} \end{aligned}$$

**913. Cor. 3.**—Put  $x = \frac{m^2 - q}{p - 2m} = -m$ ; or

$$m = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}, \text{ then,}$$

$$x^2 + px + q = 0, \text{ and } x = -\frac{p}{2} \mp \sqrt{\frac{p^2}{4} - q},$$

which conforms to Art. 337.

### Perfect Cubes.

**914. Theorem I.**—Every cube is of the form  $4m$  or  $4m \pm 1$ .

For, every number is of the form  $4x$ ,  $4x + 1$ ,  $4x + 2$ , or  $4x + 3$ .

Now,  $(4x)^3 = 64x^3 = 4(16x^3) = 4m$ ;

$$(4x + 1)^3 = 64x^3 + 48x^2 + 12x + 1$$

$$= 4(16x^3 + 12x^2 + 3x) + 1 = 4m + 1$$

$$(4x + 2)^3 = 64x^3 + 96x^2 + 48x + 8$$

$$= 4(16x^3 + 24x^2 + 12x + 2) = 4m$$

$$(4x + 3)^3 = 64x^3 + 144x^2 + 108x + 27$$

$$= 4(16x^3 + 36x^2 + 27x + 7) - 1 = 4m - 1$$

**915. Problem.** To determine rational values of  $x$  that will render  $x^3 + px^2 + qx + r$  a perfect cube.

**Solution:** Put  $x^3 + px^2 + qx + r = (x + m)^3$ ; then,  
 $(p - 3m)x^2 + (q - 3m^2)x + (r - m^3) = 0.$  (A)

1. Put  $3m = p$ , or  $m = \frac{1}{3}p$ ; then

$$x = \frac{m^3 - r}{q - 3m^2} = \frac{p^3 - 27r}{27q - 9p^2}; \text{ and}$$

$$\begin{aligned} x^3 + px^2 + qx + r &= (x + m)^3 = \left( \frac{p^3 - 27r}{27q - 9p^2} + \frac{p}{3} \right)^3 \\ &= \left( \frac{9pq - 27r - 2p^3}{27q - 9p^2} \right)^3. \end{aligned}$$

**Cor.**—If  $9pq - 27r - 2p^3 = 0$ , or  $r = \frac{9pq - 2p^3}{27}$ ,  
 $x^3 + px^2 + qx + r = 0$ , and  $x = \frac{p^3 - 3pq}{9q - 3p^2}.$

2. Put  $m^3 = r$ , and suppose  $r = r_1^3$ , then  $m = r_1$ ; and (A) will become  $(p - 3r_1)x^2 + (q - 3r_1^2)x = 0$ ; whence,

$$x = \frac{3r_1^2 - q}{p - 3r_1}; \text{ and } x^3 + px^2 + qx + r = x^3 + px^2 + qx + r_1^3;$$

and  $(x + m)^3 = \left( \frac{pr_1 - q}{p - 3r_1} \right)^3$ . Therefore,

$$x = \frac{3r_1^2 - q}{p - 3r_1} \text{ will render } x^3 + px^2 + qx + r_1^3 = \left( \frac{pr_1 - q}{p - 3r_1} \right)^3$$

**Cor.**—If  $q = pr_1$ ,  $x = -r_1$ , and  
 $x^3 + px^2 + qx + r_1^3 = 0.$

**Scholium.**—Other values under particular suppositions may be obtained by putting  $3m^2 = q = 3q_1^2$ .

#### EXERCISE 117.

1. Find which of the following numbers are prime : 197, 251, 313, 281, 461, 829, 957.

2. Find the least multiplier that will render 3174 a perfect square.

3. Find the least multiplier that will render 13168 a perfect cube.

4. Find which of the following numbers are divisible by 9, which by 11, and which by both 9 and 11 : 11205, 24530, 342738, 25916, 558657.

5. Show that, if  $p+q$  is an even number, then is  $p-q$  also an even number, provided  $p$  and  $q$  are integral.

6. Show that every cube number is of the form  $7n$  or  $7n \pm 1$ .

7. Find such a value of  $x$  as will render the  $\sqrt{ax}$  rational.

**Suggestion.**—Put  $\sqrt{ax} = p$ .

8. Find such values of  $x$  as will render  $\sqrt{ax+b}$  rational.

9. Prove that  $2^{4n} - 1$  is a multiple of 15.

10. Show that no square number is of the form  $3n - 1$ .

11. Show that  $n(n+1)(2n+1)$  is divisible by 6.

12. Show that  $(n^2+3)(n^2+7)$  is divisible by 32, when  $n$  is odd.

13. Show that  $n^5 - n$  is a multiple of 30.

14. Show that the fourth power of any number is of the form  $5m$  or  $5m+1$ .

15. Every even power of every odd number is of the form  $8n+1$ .

16. Show that  $2^{10} - 1$  is divisible by 11, and  $13^6 - 1$  is divisible by 7.

17. Show that  $a^n + a$  and  $a^n - a$  are even numbers.

18. Show that every number and its cube leave the same remainder when divided by 6.

19. If  $n > 2$ , show that  $n^5 - 5n^3 + 4n$  is divisible by 120.

20. If  $n$  is a prime number greater than 3, show that  $n^2 - 1$  is divisible by 24.

21. Find such a value of  $x$  as will render  $\sqrt{ax}$  rational.

**Suggestion.**—

Let  $\sqrt{ax} = p$ , any rational quantity, then will  $x = \frac{p^2}{a}$ .

22. Find such a value of  $x$  as will rationalize  $\sqrt{ax+b}$ .

**Suggestion.**—Put  $\sqrt{ax+b} = p$ , and prove  $x = \frac{p^2-b}{a}$ .

23. Find such a value of  $x$  as will rationalize

$$\sqrt{ax^2+bx}.$$

**Suggestion.**—Put  $\sqrt{ax^2+bx} = px$ , and prove  $x = \frac{b}{p^2-a}$ .

24. Find such a value of  $x$  as will rationalize

$$\sqrt{ax^2+bx+c^2}.$$

**Suggestion.**—

Put  $\sqrt{ax^2+bx+c^2} = px+c$ , and prove  $x = \frac{2pc-b}{a-p^2}$ .

25. Find such a value of  $x$  as will rationalize

$$\sqrt{a^2x^2+bx+c}.$$

**Suggestion.**—

Put  $\sqrt{a^2x^2+bx+c} = ax+p$ , and prove  $x = \frac{p^2-c}{b-2ap}$ .

26. Find such a value of  $x$  as will render  $\sqrt{ax^2+bx+c}$  rational when  $b^2-4ac$  is a perfect square.

**Suggestion.**—Put  $\sqrt{ax^2+bx+c} = 0$ , and  $b^2-4ac = q^2$ , and prove  $x = \frac{-b \pm q}{2a}$ .

27. Find such a value of  $x$  as will rationalize

$$\sqrt{ax^3+bx^2}.$$

**Suggestion.**—Put  $\sqrt{ax^3+bx^2} = px$ , and prove  $x = \frac{p^2-b}{a}$ .

28. Find such a value of  $x$  as will rationalize

$$\sqrt{ax^3+bx^2+cx+d^2}.$$

**Suggestion.**—

Put  $\sqrt{ax^3+bx^2+cx+d^2} = \frac{c}{2d}x+d$ , and prove  $x = \frac{c^2-4bd}{4ad^2}$ .

# ANSWERS.

## Exercise 1.

2.  $\frac{a n b}{m}$
3.  $\frac{200x-730b}{3}$
4.  $\frac{c(a-b)}{a+b}$
5.  $\frac{m n}{100 y}$
6.  $\frac{c(m+n)}{r}$
7.  $\frac{5280}{x} - \frac{5280}{y}$
8.  $\frac{m}{a}$
9.  $2ab+2ac+2bc$
10.  $\frac{a m n}{128}$
11.  $\frac{(100-x)a}{100}$
12.  $\frac{2(m+c)}{11}$
13.  $\frac{100 m}{y}$
14.  $\frac{(x-y)100}{x}$
15.  $\sqrt{a^2+b^2}$
16.  $\sqrt{x^2-y^2}$
17.  $\frac{r x y}{100}$
18.  $\frac{100 a}{100+t r}$
19.  $\frac{6000 p}{6000-d}$
20.  $\frac{a b c g h k}{d e f l m}$
21.  $x+y, x-y, x y, \frac{x}{y}$
22.  $x^2+y^2, (x+y)^2, (x-y)^2, x^2-y^2$
23.  $x^3+y^3, (x+y)^3, (x+y)(x-y), \frac{x+y}{x-y}$
24.  $x y(x+y), \frac{x}{y}(x-y), \sqrt{x} + \sqrt{y}, \sqrt{x+y}$
25.  $\frac{x y}{\sqrt{x} + \sqrt{y}}, \frac{x-y}{x^2-y^2}, \frac{x}{y}(\sqrt[3]{x} + \sqrt[3]{y})$

## Exercise 2.

1. 9
2.  $3\frac{3}{11}$
3.  $1\frac{7}{8}$
4.  $2\frac{1}{5}$
5.  $13\frac{3}{4}$
6.  $\frac{9}{11}$
7.  $\frac{49}{1600}$
8.  $\frac{2}{3}$
9.  $2\frac{2}{5}$
10.  $16\frac{3}{8}$
11. 125
12.  $3\frac{1}{24}$
13.  $3\frac{1}{3}$
14.  $\frac{3}{5}$
15.  $1\frac{2}{5}$
16.  $-\frac{3}{10}$
17. 0
18.  $\frac{49}{100}$
19. 0
20.  $2\frac{11}{14}$

**Exercise 3.**

1.  $2x^2y$
2.  $3x^2y^2z^2$
3.  $-9a^2b^2$
4.  $17x^2y^2$
5.  $a-c$
6.  $c-a$
7.  $(a-b+c-d)x$
8.  $-2(a-b)$
9.  $2ab-3cd+4x-mn$
10.  $(a+b)(x+y)$
11.  $3p^2q^2+4r^2s^2$
12.  $(x+y)^2$
13.  $-mp(p-q)$
14.  $-2a(x^2+y^2)$
15.  $5(a+b)-10(x+y)$
16.  $-11x^2+9x^2+13x+17$
17.  $2mnp-rst-mt$
18.  $-8(a+b)+10$
19.  $-3x^3+10x^2-10x-4$
20.  $-3a^3-3a^2b+6ab^2-5b^3$
21.  $x^3-3x^2-2x+7$
22.  $-2(a+b)^2-7(a+b)+5$
23.  $-a^2-2a^4$
24.  $bn+mn+bm-3m^2+2b^2-3n^2$

**Exercise 4.**

1.  $x^2$
2.  $3m^2n$
3.  $2yz$
4.  $2x^2y^2$
5.  $-2x^2y^2$
6. 0
7. 2
8. -1
9.  $-2y$
10.  $-5abc$
11.  $-2a+b-c$
12.  $-a$
13. 0
14.  $11x-8z$
15.  $3a$
16.  $-2x^2$
17. 5
18.  $2z$
19.  $-2ax^2$
20.  $2pq$

**Exercise 5.**

1.  $\begin{cases} (3x+4b)-(2c-4d)-(5x+3b) \\ (3x+4b-2c)+(4d-5x-3b) \end{cases}$
2.  $\begin{cases} (4y-2z)+(3x-5m)-(6n-2z) \\ (4y-2z+3x)-(5m+6n-2z) \end{cases}$
3.  $\begin{cases} (7a-3b)-(c+2d)-(e+3f) \\ (7a-3b-c)-(2d+e+3f) \end{cases}$
4.  $\begin{cases} -(5m-2n)-(3mn-4m^2)-(6n^2-3am) \\ -(5m-2n+3mn)+(4m^2-6n^2+3am) \end{cases}$
5.  $\begin{cases} (x^2-3xy)-(2y^2+7x^2y)+(7xy^2-y^3) \\ (x^2-3xy-2y^2)-(7x^2y-7xy^2+y^3) \end{cases}$
6.  $\begin{cases} -(3a^2+2ab)-(3b^2-4a)-(2b-3c) \\ -(3a^2+2ab+3b^2)+(4a-2b+3c) \end{cases}$
7.  $(a+b)-(c+d)+(e-f)+(g-h)-(k-l)-(m-n)$
8.  $(a+b-c)-(d-e+f)+(g-h-k)+(l-m+n)$
9.  $(a+b-c-d)+(e-f+g-h)-(k-l+m-n)$



## Exercise 9.

- |                     |                     |                       |                    |
|---------------------|---------------------|-----------------------|--------------------|
| 2. $\frac{1}{2}$    | 3. $\frac{1}{8}$    | 4. $-3$               | 5. $3\frac{1}{8}$  |
| 6. $\frac{13}{25}$  | 7. $\frac{1}{4}$    | 8. $\frac{3}{4}$      | 9. $-\frac{1}{12}$ |
| 10. $-30$           | 11. $-6$            | 12. $4$               | 13. $1$            |
| 14. $-\frac{1}{2}$  | 15. $0$             | 16. $-3\frac{13}{16}$ | 17. $0$            |
| 18. $1\frac{7}{18}$ | 19. $-1\frac{4}{5}$ | 20. $-\frac{3}{5}$    | 21. $\frac{5}{6}$  |
| 22. $1\frac{1}{8}$  | 23. $-20$           | 24. $5624$            | 25. $235$          |
|                     | 26. $-113$          | 27. $15628$           |                    |

## Exercise 10.

- |                                    |  |  |
|------------------------------------|--|--|
| 1. $27a^3b^6c^3$                   | 2. $-64a^6b^6c^4$                            | 3. $-128x^{14}y^{21}z^{14}$                      |
| 4. $a^{5m}b^{5n}$                  | 5. $64a^{6x}b^6$                             | 6. $a^{mn}b^{n^2}c^{2n}$                         |
| 7. $2^{2n}a^{10n}b^{3n}$           | 8. $6^{2n^2}a^{3n}b^{6n}$                    | 9. $(-1)^{5n} \cdot m^{10n}n^{20n}p^{5n}q^{15n}$ |
| 10. $-\frac{8}{27}a^{3n}b^{3m}c^6$ | 11. $27(a-b)^6$                              | 12. $a^8(x+y)^{12}$                              |
| 13. $(a+b)^{10}(x+y)^{15}$         | 14. $81(x^2+y^2)^8(x^2-y^2)^{12}$            |  |
| 15. $(a+b)^{mn}(a-b)^{n^2-n}$      | 16. $9^n x^{2n} y^{4n} (x^2-y^2)^{2mn+2n^2}$ |  |
| 17. $(a+b)^{5m}$                   | 18. $a^{mnpq}$                               | 19. $x^{m^2+n^2}$                                |
| 20. $x^3p^n$                       | 21. $-x^6n^2-1$                              | 22. $1972$                                       |
| 23. $443\frac{82}{179}$            | 24. $1000x^{15}$                             | 25. $(a+x)^{8n}$                                 |

## Exercise 11.

- |   |  |
|---|--|
| 1. $x^4+4x^2y^2+4y^4$   | 2. $\frac{4}{9}x^4-x^2y^2+\frac{9}{16}y^4$ |
| 3. $x^{2m-2n}-2+x^{2n-2m}$                                    | 4. $8x^3-36x^2y+54xy^2-27y^3$              |
| 5. $8x^6+48x^4y+96x^2y^2+64y^3$                               | 6. $x^6y^3-3x^5y^4+3x^4y^5-x^3y^6$         |
| 7. $x^4+4x^3y+6x^2y^2+4xy^3+y^4$                              | 8. $x^4-4x^3y+6x^2y^2-4xy^3+y^4$           |
| 9. $x^4+4x^3+6x^2+4x+1$                                       |  |
| 10. $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$                   |  |
| 11. $x^5-5x^4+10x^3-10x^2+5x-1$                               |  |
| 12. $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$          |  |
| 13. $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$ |  |
| 14. $x^7-7x^6+21x^5-35x^4+35x^3-21x^2+7x-1$                   |  |

$$15. x^n + nx^{n-1}y + \frac{n(n-1)}{1 \times 2} x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} x^{n-3}y^3 + \dots + y^n$$

16. The fourth power of the sum of two quantities equals the fourth power of the first + etc.

$$21. x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8$$

$$22. 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$$

$$23. 243x^{10} - 405x^8y^2 + 270x^6y^4 - 90x^4y^6 + 15x^2y^{12} - y^{15}$$

$$24. a^{10}x^5 + 5a^8bx^4y^2 + 10a^6b^2x^3y^4 + 10a^4b^3x^2y^6 + 5a^2b^4xy^8 + b^5y^{10}$$

$$25. 625x^{12} - 1500x^9y^2 + 1350x^6y^4 - 540x^3y^6 + 81y^8$$

$$26. a^6b^6 - 6a^5b^5cd + 15a^4b^4c^2d^2 - 20a^3b^3c^3d^3 + 15a^2b^2c^4d^4 - 6ab^5c^5d^5 + c^6d^6$$

$$27. a^{21} - 7a^{18} + 21a^{15} - 35a^{12} + 35a^9 - 21a^6 + 7a^3 - 1$$

$$28. 256a^{24} + 3072a^{21} + 16128a^{18} + 48384a^{15} + 90720a^{12} + 108864a^9 + 81648a^6 + 34992a^3 + 6561$$

$$29. a^{6n} + 12a^{5n} + 60a^{4n} + 160a^{3n} + 240a^{2n} + 192a^n + 64$$

$$30. 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$$

$$31. 81y^8 - 54y^6z^2 + \frac{27}{2}y^4z^4 - \frac{8}{2}y^2z^6 + \frac{1}{16}z^{12}$$

$$32. 64 + 960y + 6000y^2 + 20000y^3 + 37500y^4 + 37500y^5 + 15625y^6$$

$$33. -x^{10} - 5ax^9 - 10a^2x^8 - 10a^3x^7 - 5a^4x^6 - a^5x^5$$

$$34. -x^5 - 10x^4y - 40x^3y^2 - 80x^2y^3 - 80xy^4 - 32y^5$$

$$35. 729 - 2916x^2 + 4860x^4 - 4320x^6 + 2160x^8 - 576x^{10} + 64x^{12}$$

$$36. 243a^6c^6 - 810a^4c^4bd + 1080a^3c^3b^2d^2 - 720a^2c^2b^3d^3 + 240ac^4b^4d^4 - 32b^6d^5$$

$$37. a^{4m} + 4a^{3m}b^n + 6a^{2m}b^{2n} + 4a^mb^{3n} + b^{4n}$$

$$38. a^{5n+5} - 5a^{5n+3} + 10a^{5n+1} - 10a^{5n-1} + 5a^{5n-3} - a^{5n-5}$$

$$39. x^{6p} - 6x^{5p+q} + 15x^{4p+2q} - 20x^{3p+3q} + 15x^{2p+4q} - 6x^{p+5q} + x^{6q}$$

$$40. a^4x^{4n} + 4a^3bx^{3n}y^n + 6a^2b^2x^{2n}y^{2n} + 4ab^3x^ny^{3n} + b^4y^{4n}$$

### Exercise 12.

$$1. x^4 + 2x^3 + 3x^2 + 2x + 1$$

$$2. x^4 + 4x^3 + 2x^2 - 4x + 1$$

$$3. x^4 - 2x^3y + 8x^2y^2 - 2xy^3 + y^4$$

$$4. 4x^2 - 12xy + 9y^2 + 16xz - 24yz + 16z^2$$

$$5. x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3$$

$$6. x^4 - 3x^3 + 5x^2 - 3x - 1$$

$$7. x^6 - 6x^5 + 40x^3 - 96x - 64$$

$$8. 8x^6 + 36x^5 + 30x^4 - 45x^3 - 30x^2 + 36x - 8$$

$$9. x^8 + 4x^7 + 10x^6 + 16x^5 + 19x^4 + 16x^3 + 10x^2 + 4x + 1$$

$$10. x^{12} - 6x^{11} + 21x^{10} - 50x^9 + 90x^8 - 126x^7 + 141x^6 - 126x^5 + 90x^4 - 50x^3 + 21x^2 - 6x + 1$$

$$11. x^{10} + 5x^9y + 15x^8y^2 + 30x^7y^3 + 45x^6y^4 + 51x^5y^5 + 45x^4y^6 + 30x^3y^7 + 15x^2y^8 + 5xy^9 + y^{10}$$

12.  $x^{2m} + 2x^{2m-1} + 3x^{2m-2} + 2x^{2m-3} + x^{2m-4}$

13.  $a^3 + b^3 + c^3 + d^3 + 2ab - 2ac - 2ad - 2bc - 2bd + 2cd$

14.  $x^9 - 3x^8 + 6x^7 - 10x^6 + 12x^5 - 12x^4 + 10x^3 - 6x^2 + 3x - 1$

**Exercise 13.**

1.  $\pm 81$

2. 18

3.  $\pm 8$

4. 6

5.  $\pm 2x^3y^4z^5$

6.  $-3ab^2c^3$

7.  $-2ab^2c^4$

8.  $3x^4y^4z^6$

9.  $\pm 2ab^2c^3$

10.  $\pm 12a^n b^{2n} c$

11.  $\pm xy(x+y)^3$

12.  $-xy^2(x-y)^3$

13.  $\pm 3x^n(x+a)^3$

14.  $(a+b)(a-b)$

15.  $\pm 2(a+b)^2$

16.  $\pm x(xy+y^2)$

17.  $-3a^n b^{2n} c^3$

18.  $\pm 2a^{2n} b^n c$

19.  $a^2 b^3 c$

20.  $\pm x^3(x+y)^3$

21.  $xy^2z^3$

22.  $8^3 \times 7 \times 8^3$

23.  $3^3 \times 4 \times 5^3$

24.  $\pm x^2 y^3 (x-y)^n$

**Exercise 14.**

1.  $\pm(x+y)$

2.  $\pm(x+5)$

3.  $x+y$

4.  $x-3$

5.  $\pm(3x-2y)$

6.  $\pm(a-b)$

7.  $3x^2-2y$

8.  $\pm(x-2)$

9.  $x-y$

10.  $\pm(x-y-z)$

11.  $x-y-z$

12.  $\pm(a+b-c-d)$

**Exercise 15.**

1.  $\pm(2x^4-2x^2+3)$

2.  $\pm(2x^3-3xy+y^2)$

3.  $\pm x^2-2y^2+3z$

4.  $\pm(4x^{2n}-3x^n y^n-2y^{2n})$

5.  $\pm(2a^3-3b^2+4c^3)$

6.  $\pm(3x^3-4x^2y+6xy^2-y^3)$

7.  $\pm(x^4-4x^3y+6x^2y^2-4xy^3+y^4)$

8.  $\pm(a^{3n}+3a^{2n}b^n+3a^n b^{2n}+b^{3n})$

9.  $\pm(1-a^n+a^{2n}-a^{3n})$

**Exercise 16.**

1.  $2a-3$

2.  $x^2-3y^2$

3.  $10x^3-2$

4.  $x^m+2$

5.  $x^2+x+1$

6.  $x^2-2x-2$

7.  $2x^{2n}-3x^n+1$

8.  $x^4+3x^2y-y^2$

9.  $x-2y-z$

10.  $2x^2-2y^2+2z^2$

11.  $3a+2b+c$

12.  $a^{2m}-a^{2n}+a^m-1$

**Exercise 17.**

1.  $\pm 56$

2.  $\pm 81$

3.  $\pm 124$

4.  $\pm 206$

5.  $\pm 325$

6.  $\pm 508$

7.  $\pm 777$

8.  $\pm 989$

9.  $\pm 1024$

10.  $\pm 1345$

11.  $\pm 6006$

12.  $\pm 7801$

**Exercise 18.**

- |        |         |          |          |
|--------|---------|----------|----------|
| 1. 12  | 2. 34   | 3. 81    | 4. 95    |
| 5. 125 | 6. 202  | 7. 250   | 8. 365   |
| 9. 401 | 10. 525 | 11. 1234 | 12. 6006 |

**Exercise 19.**

- |               |                |               |      |
|---------------|----------------|---------------|------|
| 1. $\pm 16$   | 2. $\pm 12$    | 3. $\pm 5$    | 4. 7 |
| 5. $\pm 8$    | 6. $\pm 2ab^2$ | 7. $\pm(x+y)$ |      |
| 8. $\pm(x-2)$ | 9. $\pm(x-y)$  | 10. $x-y$     |      |

**Exercise 20.**

- |   |  |
|---|--|
| 1. $x^2 - xy + y^2$   | 2. $x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$ |
| 3. $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$                    | 4. $x^2 + xy + y^2$                            |
| 5. $a^3 + 2ab + 4b^2$   | 6. $x^3 + 2x^2y + 4xy^2 + 8y^3$                |
| 7. $x^3 - 2x^2y + 4xy^2 - 8y^3$                                   | 8. $4x^2 + 6xy + 9y^2$                         |
| 9. $4x^2 - 6xy + 9y^2$  | 10. Not divisible.                             |
| 11. $9m^2 + 12mn + 16n^2$   | 12. Not divisible.                             |
| 13. $x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4$                       |  |
| 14. $x^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 16y^4$                       |  |
| 15. $16a^4 + 24a^3b + 36a^2b^2 + 54ab^3 + 81b^4$                  |  |
| 16. Not divisible.  | 17. $64m^3 + 80m^2n + 100mn^2 + 125n^3$        |
| 18. $64m^3 - 80m^2n + 100mn^2 - 125n^3$                           | 19. Not divisible.                             |
| 20. $x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$                   |  |
| 21. $32a^5 + 48a^4b + 72a^3b^2 + 108a^2b^3 + 162ab^4 + 243b^5$    |  |
| 22. $x^6 - 2x^5y + 4x^4y^2 - 8x^3y^3 + 16x^2y^4 - 32xy^5 + 64y^6$ |  |
| 23. $x^{2n} - x^ny^n + y^{2n}$                                    | 24. $x^{2n} + x^{2n}y^n + x^n y^{2n} + y^{2n}$ |

**Exercise 21.**

- |  |                                       |                            |
|--|---------------------------------------|----------------------------|
| 1. $a^4 + a^2b + b^2$  | 2. $a^3 - ab^2 + b^4$                 | 3. $4x^6 + 6x^3y^2 + 9y^4$ |
| 4. $4x^4 - 6x^2y^2 + 9y^4$                                     | 5. Not divisible.                     | 6. Not divisible.          |
| 7. $a^8 - 2a^6b + 4a^4b^2 - 8a^2b^3 + 16b^4$                   |                                       |                            |
| 8. $8a^6 - 12a^4b + 18a^2b^2 - 27b^3$                          | 9. $8a^6 + 12a^4b + 18a^2b^2 + 27b^3$ |                            |
| 10. Not divisible.   | 11. Not divisible.                    |                            |
| 12. $a^4x^3 - a^3b^2x^2y + a^2b^4x^4y^2 - ab^6x^3y^3 + b^8y^4$ |                                       |                            |
| 13. $m^4p^8 - m^3n^2p^4q^3 + n^4q^6$                           |                                       |                            |
| 14. $16x^8y^4 - 24x^7y^5 + 36x^6y^6 - 54x^5y^7 + 81x^4y^8$     |                                       |                            |
| 15. $4x^{4m} - 6x^{2m}y^{2m} + 9y^{4m}$                        |                                       |                            |

16.  $(a+b)^4 - (a+b)^3 c + (a+b)^2 c^2 - (a+b) c^3 + c^4$   
 17.  $a-b$       18.  $a+b$       19.  $x-2y$       20.  $x+2y$   
 21.  $x^2-y^2$       22.  $a-b$       23.  $ax+by$       24.  $2x-3y$   
 25.  $x+y$       26.  $x-y$

**Exercise 22.**

1.  $(a+b^2)x$       2.  $3a(2a-x)$       3.  $2ab(a-2b+c)$   
 4.  $3xy(x^2+xy+2y^2)$       5.  $2xz(x-2y+3z)$   
 6.  $x^m(ax^n+b+cx)$       7.  $3am^2(4m-6mn+n^2)$   
 8.  $6x^m(x^n-2x+3)$       9.  $5abc(2a+3b-2c)$   
 10.  $4xy(2x-3y+5z)$       11.  $10a(2ab-3c+2b-a)$   
 12.  $5ac(6x-3y+2z+1)$       13.  $3c(7-2c-5a+m^2)$   
 14.  $5xy(2+5x-7c+a^2)$       15.  $5a^mb^m(2-3a^mb^m-b^m+4a^m)$   
 16.  $6a^m(a^2+2a^2-3a-4)$       17.  $a^mx^m(x^{2m}+a^mx^m-a^{2m}+a^{3m}x^m)$   
 18.  $a^2+y^2-4(a^2-ay+y^2)$

**Exercise 23.**

1.  $(2x+y)(2x-y)$       2.  $(3x+4y)(3x-4y)$   
 3.  $(4a^2+9b^2)(2a+3b)(2a-3b)$       4.  $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$   
 5.  $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$   
 6.  $(x-2y)(x^2+2xy+4y^2)$       7.  $(4x+1)(16x^2-4x+1)$   
 8.  $(x^2+1)(x^4-x^2+1)$       9.  $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$   
 10.  $(2x^2+2x+1)(2x^2-2x+1)$       11.  $(x^{2n}+y^{2n})(x^n+y^n)(x^n-y^n)$   
 12.  $(x^4+y^4)(x^8-x^4y^4+y^8)$   
 13.  $(x-y)(x+y)(x^2+y^2)(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$   
 14.  $(a^2x^2+1)(ax+1)(ax-1)$       15.  $(x^2+4x+8)(x^2-4x+8)$   
 16.  $(8a^2+4ab+b^2)(8a^2-4ab+b^2)$   
 17.  $(a+b+c)(a+b-c)$       18.  $(x-y+z)(x-y-z)$   
 19.  $(2x-3)(16x^4+24x^3+36x^2+54x+81)$   
 20.  $(x+2)(x^6-2x^5+4x^4-8x^3+16x^2-32x+64)$   
 21.  $(x^2+10xy+50y^2)(x^2-10xy+50y^2)$   
 22.  $(a^4+2a^2b+2b^2)(a^4-2a^2b+2b^2)$   
 23.  $(x^4+4x^2y+8y^2)(x^4-4x^2y+8y^2)$   
 24.  $(2x^n+3y^n)(4x^{2n}-6x^ny^n+9y^{2n})$   
 25.  $(a^2x^2+4abx+8b^2)(a^2x^2-4abx+8b^2)$   
 26.  $(ax^n+bn)(a^4x^{4n}-a^3bx^{3n}+a^2b^2x^{2n}-ab^3x^n+b^4n)$

**Exercise 24.**

1.  $(x+2)^2$
2.  $(3a-4b)^2$
3.  $(4x^2+5y^2)^2$
4.  $(x-11)(x+12)$
5.  $(x+3)(x-6)$
6.  $(2x+7)(x+3)$
7.  $(2x-3)(x+7)$
8.  $(x^2y^2+xy+1)(x^2y^2-xy+1)$
9.  $(4x^2+6x+9)(4x^2-6x+9)$
10.  $(5x^2-2)(2x^2+7)$
11.  $(4x-y)(4x+3y)$
12.  $(ax+1)(ax+99)$
13.  $(11x^2+12y^2)^2$
14.  $(x^n+y^n)^2$
15.  $(ay+bz)(ay+2bz)$
16.  $(3x+2a)(2x+a)$
17.  $(2x-3b)(5x-2b)$
18.  $(x^4-x^2y^2+y^4)(x^2+xy+y^2)(x^2-xy+y^2)$
19.  $(8x^4+4x^2y+9y^2)(8x^4-4x^2y+9y^2)$
20.  $(x^{2n}+x^n y^n+y^{2n})(x^{2n}-x^n y^n+y^{2n})$
21.  $(a+b+2)(a+b+3)$
22.  $(9a^2-1)(3a^2-7)$
23.  $(x+y+z)^2$
24.  $(2ax^2-3)(5ax^2+7)$

**Exercise 25.**

1.  $(a+b)(x-y)$
2.  $(c-d)(x^2+1)$
3.  $(x-y+z)(x-y-z)$
4.  $(x+y+z)(y-x-z)$
5.  $(5m+n-2)(5m-n+2)$
6.  $(x+1)^3$
7.  $(2x-3)^2$
8.  $(1+r+s)(1-r-s)$
9.  $x(x+y)(a-b)$
10.  $(x+y-z)(x^2+2xy+y^2+xz+yz+z^2)$
11.  $(x+1)^4$
12.  $(x^2+2)^4$
13.  $(1-x)^5$
14.  $(z+y+1)(z^2-yz-z+y^2+2y+1)$
15.  $(3ab+3a-2b)(3ab-3a+2b)$
16.  $(a+b+c+d)(a+b-c-d)$
17.  $(2z-3x)(2x+3y)$
18.  $(3a-2b)(a-5y)$
19.  $(3c-2x)(a-6b)$
20.  $(x-1)(x+2)(x^2+5x-2)$
21.  $(x+2y)(x+2y+6z)$
22.  $(2x+2a-1)(4x^2-4ax+2x+4a^2-4a+1)$
23.  $(x-y+3)(x-y+6)$
24.  $(2x-3y-4)(3x+2y+4)$
25.  $(2x+y-5)(x-3y+2)$

**Exercise 26.**

1.  $2ab(3a^2-3ab+c)$
2.  $5a(x+1)^2$
3.  $3ax(a^2+b^2)(a+b)(a-b)$
4.  $3(x^2+ax+a^2)(x^2-ax+a^2)$
5.  $5(x-3)(x+7)$
6.  $(a+x+z)(a-x-z)$
7.  $(2x+3y)(x-5y)$
8.  $(x+y+z^2)(x+y-z^2)$

9.  $(x^2+25)(x^2+1)$       10.  $(2x+3y)(3x-7y)$   
 11.  $(a+b-x+y)(a-b-x-y)$       12.  $8xy(x^2+y^2)$   
 13.  $(3+2x+y)(9-6x-3y+4x^2+4xy+y^2)$   
 14.  $(2a+3b+3c)(4a^2+12ab+9b^2-6ac-9bc+9c^2)$   
 15.  $(4x^4+3y^3)(x^4+3y^3)$       16.  $(a+c)(a-b)$   
 17.  $(a+b+c-d)(a-b-c+d)$       18.  $(x+2y)(x-2y+7)$   
 19.  $(x^4-x^2y^2+y^4)(x^2+xy+y^2)(x^2-xy+y^2)$       20.  $(3x+1)^2$   
 21.  $(x-3y)(x-3y)(x-3y)$       22.  $(1+2x-3y)(1-2x+3y)$   
 23.  $(x+y-2)^2$       24.  $(x+y)(x-y)(a+b)(a-b)$   
 25.  $(x^2-3x-1)^2$       26.  $(a+b)(a-b)(x+y)(x^2-xy+y^2)$   
 27.  $(x^2+3x-1)(x^2+3x-3)$       28.  $(a+3)(x^2+y^2)(x^4-x^2y^2+y^4)$   
 29.  $x^2y(a^2+1)(a^8-a^6+a^4-a^2+1)$   
 30.  $a(a^2x^4+1)(ax^2+1)(ax^2-1)$       31.  $a(a^4+b^4)(a^2+b^2)(a+b)(a-b)$   
 32.  $(x^2+z^2)(x^4-x^2z^2+z^4)(x+z)(x^2-xz+z^2)(x-z)(x^2+xz+z^2)$   
 33.  $(x^4+z^4)(x^8-x^4z^4+z^8)$       34.  $(2x^2+3y^2)(4x^4-6x^2y^2+9y^4)$   
 35.  $x(x+y)(x^6-x^3y+x^4y^2-x^2y^3+x^2y^4-xy^5+y^6)$   
 36.  $(x+y)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$   
 37.  $(2x^2-1)(16x^8+8x^6+4x^4+2x^2+1)$   
 38.  $(1-x+y)(1+x-y+x^2-2xy+y^2)$       39.  $(6a-1)(36a^2-30a+7)$   
 40.  $(x-y+4z^2)(x^2-2xy+y^2-4xz^2+4yz^2+16z^4)$   
 41.  $(x^2+2x+y)^2$       42.  $(2x+y-3)(3x-2y+5)$   
 43.  $(3x-2y)(a+b+c)$       44.  $(x+y+1)^5$   
 45.  $(5x+y+6)(x-7y-3)$       46.  $(3x-y+7)(x+5y+7)$   
 47.  $(2x-3y+12)(2x-3y+2)$       48.  $(16x^2+4x-30)^2$

**Exercise 27.**

1.  $a^4+a^3b-ab^3-b^4$       2.  $x^4-y^4$       3.  $x^2-y^2$   
 4.  $a^2+2ax+x^2$       5.  $a^2+2ab+b^2$       6.  $a^6+2a^3x^3+x^6$   
 7.  $x^2-y^2$       8.  $x^5+x^3y^2-x^2y^3-y^5$       9. 1  
 10.  $x-y+z$       11. 1      12.  $x$       13. 6      14. 1  
 15.  $x$       16.  $ax-by$       17.  $\frac{x^3-x^2y-xy^2+y^3}{x^2+y^2}$   
 18.  $-(a^2+b^2)$       19.  $a^2-b^2$       20. -4

**Exercise 28.**

1.  $6x^2y^2z$       2.  $4a^2b^2c$       3.  $15m^2(p+q)$   
 4.  $2(p+q)^2(p-q)^2$       5.  $a+b$       6.  $a-b$

- |                  |                        |                      |
|------------------|------------------------|----------------------|
| 7. $a+b$         | 8. $a^2-b^2$           | 9. $x-y$             |
| 10. $x^2-xy+y^2$ | 11. $x^2-xy+y^2$       | 12. $x-y$            |
| 13. $x+2$        | 14. $x^2-xy+y^2$       | 15. $3x-2$           |
| 16. $2x+7$       | 17. $x+y$              | 18. $3a-2b$          |
| 19. $x+y$        | 20. $a+b+c+d$          | 21. $x^3-x^4y^4+y^3$ |
| 22. $2x+3y+4$    | 23. $x^2+2xy+2y^2$     | 24. $x+3y+6$         |
| 25. $x+y+z$      | 26. $x^n+y^n-z^n$      | 27. $(p+q+r)^2$      |
|                  | 28. $x^4-4x^2y^2+8y^4$ |                      |

**Exercise 29.**

- |                      |                |                 |
|----------------------|----------------|-----------------|
| 1. $x^2+5x+6$        | 2. $x^2-6x+9$  | 3. $x+2$        |
| 4. $x^2+x+1$         | 5. $x^2-5x-8$  | 6. $2x-3y$      |
| 7. $x^2-2x+3$        | 8. $a-3b$      | 9. $6x^2+21x+9$ |
| 10. $3x^2-2z^2$      | 11. $3x+2$     | 12. $x^2+3x+5$  |
| 13. $7x^2+8x+1$      | 14. $x^2-3x+1$ | 15. $x^2+3x+2$  |
| 16. $x+2$            | 17. $x-6$      | 18. $x+6$       |
| 19. $x^4+x^2y^2+y^4$ | 20. $3a-b$     | 21. $x+8$       |

**Exercise 30.**

- |                                    |   |                             |
|------------------------------------|---|-----------------------------|
| 1. $36x^3y^2z^2$                   | 2. $48a^2b^3c^2$                        | 3. $720x^3y^2z^2$           |
| 4. $x^4-1$                         | 5. $a(b^2-c^2)$                         | 6. $a^2(b+1)^2$             |
| 7. $12(a-1)^2(a+1)$                | 8. $abc(x^2-y^2)$                       |                             |
| 9. $(a^2+b^2)(a+b)^2(a-b)^2$       | 10. $(x+y)^2(x^2+y^2)(x-y)(x^2-xy+y^2)$ |                             |
| 11. $x^4+x^2y^2+y^4$               | 12. $x^{12}-y^{12}$                     | 13. $144(x^6-y^6)(x^2+y^2)$ |
| 14. $(x^2-y^2)^2$                  | 15. $(a^2-b^2)xy$                       | 16. $(x+4)(x^2-9)(x-1)$     |
| 17. $(x-3)(x+4)(x-5)^2$            | 18. $(2x+3)^2(x+4)(x-5)$                |                             |
| 19. $2(4x+3)(x+3)(2x-5)(3x-4)$     |   |                             |
| 20. $(x-y)(x+y)(x^2+xy+y^2)$       | 21. $xy(x+y)^2(x^2-xy+y^2)$             |                             |
| 22. $(a^2-b^2)(c^2-d^2)$           | 23. $(4a^2-9b^2)(a^2-b^2)$              |                             |
| 24. $a^2c^2(a+b-c)(a+b+c)$         |   |                             |
| 25. $(x+y+z)(x+y-z)(x-y+z)(x-y-z)$ |   |                             |

**Exercise 31.**

- $(3x-5a)(x-3a)(x^2-3ax+a^2)$
- $(4a-3b)(3a^2-4a^2b+3ab^2-2b^3)$
- $(2y-5)(y^4-4y^2-16y^2+7y+24)$
- $(a^2+a-2)(a^4-a^2+2a^2+a+3)$

5.  $a b (2 a-b)(3 a-b)(a+b)$       6.  $(x+14)(x^2+9 x^2+27 x-98)$   
 7.  $(2 x^2+3 x-5)(2 x^2+x^2-8 x+5)$       8.  $(x-10)(x^2-39 x+70)$   
 9.  $(x^2+x+1)(x^2-2 x-1)$       10.  $(4 x-y)(3 x^2-3 x^2 y+x y^2-y^2)$   
 11.  $(2 x^2-6 x+2)(5 x^2+15 x^2+5 x+15)$   
 12.  $(x^2-y^2)(x^4-2 x^2 y+2 x^2 y^2-2 x y^3+y^4)$   
 13.  $(x^2-x+1)^2(x^2+x+1)(x+1)$   
 14.  $(2 x^2+x-1)(x+8)(4 x+1)(3 x+2)$   
 15.  $(x+2)(x+3)(x+4)(x-5)$       16.  $(3 x-4)(x+1)^2(x-1)^2$   
 17.  $x^4-8 x^4 y^2-8 x^2 y^4-9 y^2$       18.  $(2 x-1)(x^2-1)(x^2+1)(x^2-x+1)$   
 19.  $(5 a^5-10 a^3+7 a-14)(a^2-5 a+6)(3 a^2-8 a)$   
 20.  $(x^3-2 x+5)(x-3)(x+1)(2 x-3)$       21.  $x^4+x^2 y^2+y^4$   
 23.  $x^3+11 x^2+40 x+48, x^3+x^2-21 x-45$   
 24.  $x^4+14 x^2+75 x^2+4 x+20$       26.  $x^2+2 x^2 y+2 x y^2+y^2$   
 27.  $a^3+a x^2+a^2 x+x^2, a^3+a x^2-a^2 x-x^2$   
 28.  $x^2+5 x+6, x^2+7 x+10$

**Exercise 32.**

1.  $\frac{3 y^m}{5 x z}, \frac{2 a^{m-1} b^3}{8}$       2.  $\frac{x+2}{x+3}, \frac{x^2+x y+y^2}{x}$   
 3.  $\frac{3 x+2}{5 x-1}, 4 x^2-2 x+1$       4.  $\frac{8 x^2+31 x+26}{4 x+3}$   
 5.  $\frac{x^2+3 x+1}{x+5}$       6.  $4-\frac{5 x+14}{3 x^2+2}$       7.  $a+\frac{b^3}{a^2+a b+b^2}$   
 8.  $a-b+\frac{2 b^2}{a+b}$       9.  $\frac{x+2 x^2+2 x^3+x^4}{(1-x^2)(1+x+x^2)}, \frac{y+x y+x^2 y}{c \cdot d}, \frac{z+x z}{c \cdot d}$   
 10.  $\frac{x^4+x^3 y+x y^3+y^4}{x^6-y^6}, \frac{x^4-x^3 y-x y^3+y^4}{c \cdot d}, \frac{x^2+y^2}{c \cdot d}$   
 13.  $\frac{15}{6(1-x^2)}, \frac{6}{6(1-x^2)}, -\frac{4+4 x}{6(1-x^2)}$   
 14.  $\frac{x^2+2}{x^2-3}$       15.  $\frac{2 a^2-6 a-7}{6 a^2-17 a-20}$   
 16.  $-\frac{x y-x z-y z+y^2}{(x-y)(x-z)(y-z)}, -\frac{x y+y z-x z-z^2}{c \cdot d}, -\frac{x y-y z+x z-z^2}{c \cdot d}$   
 17.  $\frac{x^{2 m} y^{2 m}}{x^{2 m}-x^m y^m+y^{2 m}}$       18.  $\frac{c^m+d^m}{c^m-d^m}$

**Exercise 33.**

1.  $\frac{10 x-3}{8 x}$       2.  $\frac{1}{b}$       3.  $\frac{12 x-9}{x^2-x-6}$       4.  $\frac{5 x+7}{x^2-1}$

5.  $\frac{6x}{x^2-5x+6}$       6. 0      7.  $\frac{1}{x-1}$       8.  $\frac{7}{x-2}$
9.  $\frac{1}{a^4-1}$       10. 0      11.  $\frac{8n^2-2n^2-2n+2}{4n^4-5n^2+1}$
12.  $\frac{4x^3-x^2-3x-2}{x^2(x+1)^2}$       13.  $\frac{4xy(x^2-xy+y^2)}{x^4-y^4}$
14.  $\frac{69x-41}{30(x^2-1)}$       15. 0      16.  $-\frac{3(x^2+1)}{x^4+x^2+1}$
17.  $\frac{2x^2}{(a-c)(b-c)}$       18.  $\frac{b-c}{(a-x)(b-x)(c-x)}$
19. 0      20. 0      21.  $\frac{2x(a+bx)}{(a^2-b^2)(1-x^2)}$
22. 0      23.  $\frac{4p^3q^3}{p^6-q^6}$       24.  $\frac{2cx+c^2+a^2}{(a^2-c^2)(x^2-a^2)(x+c)}$

**Exercise 34.**

1.  $\frac{xy}{x-y}$       2.  $-(x+y)^2$       3.  $\frac{4xy(x^2+xy+y^2)}{x+y}$
4.  $\frac{2ab(a^2+ab+b^2)}{(a+b)(a^3+b^3)}$       5.  $\frac{a^2-b^2}{x-y}$       6.  $2 + \frac{ay}{bx} + \frac{bx}{ay}$
7.  $\frac{a^2}{x^2} + \frac{2ab}{xy} + \frac{b^2}{y^2}$       8.  $\frac{x+y}{x-y}$       9. 1
10.  $\frac{x^4}{y^4} + \frac{x^2}{y^2} + 1 + \frac{y^2}{x^2} + \frac{y^4}{x^4}$       11.  $\frac{x^{12}+y^{12}}{x^8+x^4y^4+y^8}$
12.  $\frac{x^3}{y} - 1 + \frac{y}{x^3}$       13.  $4ax$       14.  $\frac{(x-y)^2}{(x+y)^2}$
15.  $(a^2-3a+9)(a^2+2a+4)$       16.  $\frac{b(x+y-z)}{a(x-y+z)}$       17. 1
18.  $\frac{x+y}{x-y}$       19.  $\frac{(a+b)^2}{(a-b)^2}$       20.  $\frac{x}{z} - \frac{y}{a} + \frac{z}{b}$

**Exercise 35.**

1.  $\frac{adfh-bcfh}{bdeh-bdfh}$       2.  $\frac{x+2xy+y}{y-x}$       3.  $\frac{x^2+x+1}{x^2+1}$
4. 1      5.  $\frac{a^2+b^2}{2ab}$       6.  $x^3+y^3$
7.  $-\frac{xy(x-y)^2}{x^4+x^2y^2+y^4}$       8.  $a$       9.  $\frac{a-b}{8ab}$

**Exercise 36.**

1.  $\frac{a^2}{b^4}, \frac{a^4}{b^2}$
2.  $\frac{(a-x)^2}{(a+x)^4}, \frac{(a-x)^4}{(a+x)^2}$
3.  $\frac{x-a}{(x+y)^2(x^2-xy+y^2)}, \frac{(x+a)(x-a)^2(x^2+ax+a^2)}{x+y}$
4.  $\frac{a+b}{m(a-b)(c-d)(a^2+ab+b^2)}, \frac{(c+d)(a+b)^2(a^2-ab+b^2)x^2}{a-b}$
5.  $\frac{(a-b)^2}{(x-y)^2}$
6.  $\frac{acn}{bdm}$
7.  $\frac{a+cd}{2}, \frac{a-cd}{2}$

**Exercise 37.**

1.  $\frac{a^2 b^{12}}{c^8 d^{16}}$
2.  $\frac{81 a^8 b^{4n} c^{12}}{256 d^{4n} e^{20n} f^4}$
3.  $\frac{(a^2-2ab+b^2)x^2}{(a^2+2ab+b^2)y^2}$
4.  $\frac{a^2}{x^2} + 2 + \frac{x^2}{a^2}$
5.  $\frac{27x^4}{125y^3} - \frac{9x^2}{5y} + \frac{5y}{x^2} - \frac{125y^3}{27x^4}$
6.  $\frac{x^2}{y^3} + \frac{4x}{y} + 6 + \frac{4y}{x} + \frac{y^2}{x^2}$
7.  $1 + \frac{2}{x} + \frac{3}{x^2} + \frac{2}{x^3} + \frac{1}{x^4}$
8.  $x^6 - 3x^4 + 6x^2 - 7 + \frac{6}{x^2} - \frac{3}{x^4} + \frac{1}{x^6}$
9.  $\frac{ax+by}{ax-by}$
10.  $\frac{3}{4}$
11.  $\pm \frac{744}{2401}$
12.  $\frac{503}{739}$
13.  $\pm \frac{4}{5}$
14.  $-\frac{2a^2b^3}{3x^4y^5}$
15.  $\frac{(a+x)^3(a-x)}{(x+y)^3}$
16.  $\frac{a^2b^3c^4}{x^5y^4z^4}$
17.  $a+1-\frac{1}{a}$
18.  $\pm \frac{x^2+7x+10}{x^2-7x+10}$
19.  $\pm \left( \frac{m}{n} - \frac{n}{m} + m \right)$
20.  $\pm \left( \frac{a}{b} + \frac{b}{a} \right)$
21.  $1 - \frac{1}{3x} - \frac{4}{9x^2} - \frac{23}{81x^3} - \text{etc.}$

**Exercise 38.**

1.  $\pm 24, \pm 238, \pm 345$
2.  $\cdot 02, 6\cdot 7, 4\cdot 06$
3.  $\pm 447+, \pm 592-, \pm 2236-, \pm 632+, \pm 1029$
4.  $\cdot 928+, \cdot 431-, \cdot 693+, 1\cdot 709+, 1\cdot 913-$
5.  $\pm 1\cdot 19-, \pm 1\cdot 21-$
6.  $\cdot 687$
7.  $\pm 2\cdot 6457, \pm 3\cdot 1622, \pm 58603$
8.  $1, 1$
9.  $\pm 1732050, \pm 7453559, \pm 2847474$
10.  $1\cdot 25992, 2\cdot 15443, \cdot 272354$

**Exercise 39.**

1.  $1\frac{4}{9}$
2. 0
3. c
4. 2
5.  $\frac{a+b}{a+d}$

6.  $\frac{x^m}{a^m}$       7. 1      8.  $z^3$       9.  $\frac{(a+x)^2}{(a-x)^2} + 2 + \frac{(a-x)^2}{(a+x)^2}$
10.  $\frac{x^4}{y^2+2y+1} - 2 + \frac{y^2+2y+1}{x^4}$       11.  $\frac{2x^3}{y^3} + \frac{6y}{x}$
12.  $\frac{8xy(x^2+y^2)}{(x^2-y^2)^2}$       13.  $\frac{x^3+2x+1}{y^2} - 1$
14.  $\frac{x^4}{y^4} + 4\frac{x^2}{y^2} - 21$       15.  $\frac{(a+x)^2}{a^2x^2} + \frac{b(a+x)}{ax} + \frac{2b^2}{9}$
16.  $x^4+2x^2+3+\frac{2}{x^3}+\frac{1}{x^4}$ ,  $x^4-2x^2+3-\frac{2}{x^3}+\frac{1}{x^4}$
17.  $x^3+3x^2-5+\frac{3}{x^2}-\frac{1}{x^3}$ ,  $x^3-3x^2+6x-7+\frac{6}{x}-\frac{3}{x^2}+\frac{1}{x^3}$
18.  $\frac{2x^4}{y^4} + 12 + \frac{2y^4}{x^4}$       19.  $\frac{10x^6}{y^5} + \frac{20x^4}{y^4} + 2$
20.  $1 + \frac{2}{x} + \frac{3}{x^2} + \frac{4}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} + \frac{1}{x^6}$
21.  $x^4+2x^3+x^2+2x+4+\frac{2}{x}+\frac{1}{x^2}+\frac{2}{x^3}+\frac{1}{x^4}$
22.  $x^4+4x^3+2x^2-8x-5+\frac{8}{x}+\frac{2}{x^2}-\frac{4}{x^3}+\frac{1}{x^4}$
23.  $ab\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)$       24. 1
25.  $-\frac{4x^2+1}{x(1-4x^2)}$       26.  $\left(\frac{a}{b}+2\right)^2, \left(\frac{2a}{b}-3\right)^2$
27.  $\left(\frac{a}{x} + \frac{x}{a}\right)\left(\frac{a}{x} - \frac{x}{a}\right)\left(\frac{a+x}{a-x} + \frac{a-x}{a+x}\right)\left(\frac{a+x}{a-x} - \frac{a-x}{a+x}\right)$
28.  $\left(a + \frac{x}{a}\right)\left(a - \frac{x}{a}\right)\left(a^2 + \frac{x^2}{a^2}\right),$   
 $\left(\frac{x}{y} + a\right)\left(\frac{x}{y} - a\right)\left(\frac{x^2}{y^2} + a^2\right)\left(\frac{x^4}{y^4} + a^4\right)$
29.  $\left(\frac{a}{b} + 1\right)\left(\frac{a^2}{b^2} - \frac{a}{b} + 1\right), \left(\frac{a}{b} - 1\right)\left(\frac{a^2}{b^2} + \frac{a}{b} + 1\right)$
30.  $\left(\frac{x}{y} + z\right)\left(\frac{x^4}{y^4} - \frac{x^2z}{y^2} + \frac{x^2z^2}{y^2} - \frac{xz^3}{y} + z^4\right);$   
 $\left(\frac{x}{y} - z\right)\left(\frac{x^4}{y^4} + \frac{x^2z}{y^2} + \frac{x^2z^2}{y^2} + \frac{xz^3}{y} + z^4\right)$
31.  $\left(x+1+\frac{1}{x}\right)\left(x-1+\frac{1}{x}\right), \left(x-\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$
32.  $\left(\frac{x}{y} + 3\right)\left(\frac{x}{y} + 2\right), \left(\frac{2x}{y} - 5\right)\left(\frac{2x}{y} + 3\right)$

$$33. \frac{1}{4b^2} (a+b+c)(a+b-c)(a+c-b)(b+c-a)$$

$$35. \pm \left( x-1 + \frac{1}{x} \right)$$

$$36. \pm \left( \frac{a}{x} + b + \frac{x}{b} \right)$$

$$37. \pm \left( \frac{3a^2}{2} + 5 - \frac{3}{4ab} \right)$$

$$38. x+2 + \frac{1}{x}$$

$$39. \frac{2x}{y} + \frac{2y}{x} + xy$$

$$40. \frac{a}{b}, \frac{a+m}{b+m}$$

$$41. \frac{a-m}{b-m}, \frac{a}{b}$$

**Exercise 40.**

$$1. x = -\frac{5}{7}$$

$$2. x = -6$$

$$3. x = \frac{1}{2}$$

$$4. x = -11$$

$$5. x = -4$$

$$6. x = 20\frac{1}{2}$$

$$7. x = -\frac{2}{19}$$

$$8. x = -4\frac{14}{31}$$

$$9. x = -\frac{21}{43}$$

$$10. x = 1\frac{98}{481}$$

$$11. x = 1$$

$$12. x = 0$$

$$13. x = \frac{ab}{b+2bc+c}$$

$$14. x = -\frac{1}{3}a$$

$$15. x = \frac{a-3c}{2}$$

$$16. x = \frac{1-c}{d}$$

$$17. x = \frac{c(bc-a)}{bc(1-d)}$$

$$18. x = 0$$

$$19. x = 1\frac{1}{4}$$

$$20. x = -2$$

$$21. x = \frac{1}{4}$$

$$22. x = 2$$

$$23. x = -1\frac{5}{6}$$

$$24. x = \frac{a^2}{a-4}$$

$$25. x = 3b-4a-5$$

$$26. x = \frac{1}{5}$$

$$27. x = 5\frac{1}{4}$$

$$28. x = 1\frac{21}{37}$$

**Exercise 41.**

$$1. 32$$

$$2. 60, 80$$

$$3. 22\frac{10}{17}, 25\frac{7}{17}$$

$$4. 21, 54$$

$$5. 36, 26, 18, 12$$

$$6. 22, 31, 9, 54$$

$$7. 8000, 9000, 10,000$$

$$8. 30$$

$$9. 200$$

$$10. 12$$

$$11. 10$$

$$12. 9\frac{3}{8} \text{ mi.}$$

$$13. 53\frac{1}{3} \text{ mi.}$$

$$14. 2\frac{8}{11} \text{ da.}$$

$$15. \frac{abc}{ab+ac+bc} \text{ da.}$$

$$16. 13\frac{1}{3} \text{ da.}$$

$$17. 8\frac{4}{7} \text{ hr.}$$

$$18. \$480, \$280, \$800$$

$$19. 24$$

$$20. 4 \text{ hr. } 48 \text{ min. A. M.}$$

$$21. 275 \text{ bu., } 500 \text{ bu., } 975 \text{ bu.}$$

22. 250

23. 1 hr. 20 min. P. M.

24. 10 hr. 54  $\frac{6}{11}$  min., 10 hr. 38  $\frac{2}{11}$  min., 10 hr. 21  $\frac{9}{11}$  min.

25. 1500 rd., 900 rd.

26. 150  $\frac{6}{7}$  yd., 201  $\frac{1}{7}$  yd.

27. 20 mi., 40 mi.

28. 10  $\frac{1}{3}$  mi., 41  $\frac{1}{3}$  mi.29. 29  $\frac{2}{5}$  mi. from R.

30. 1 mi.

31. 60 and 40

32. 300

33. 25, 20

34. \$5000

35. 35%

36. \$5500, \$2500, \$3000

37. 2s. 8d.

38. 112 oz.

39. 15 min. past 10

40. 11, 22, 33

41. 6 mi.

## Exercise 42.

1.  $x = 2, y = 3$

2.  $x = 2\frac{1}{9}, y = \frac{13}{27}$

3.  $x = 2\frac{1}{5}, y = 5\frac{4}{5}$

4.  $x = 7\frac{1}{3}, y = 8\frac{2}{3}$

5.  $x = 2, y = 6\frac{1}{5}$

6.  $x = \frac{2}{3}, y = \frac{5}{6}$

7.  $x = m+n, y = m+n$

8.  $x = 3, y = 3$

9.  $x = 3, y = 2$

10.  $x = \frac{amnr - abcm}{an - bm}, y = \frac{abcn - bmnr}{an - bm}$

11.  $x = 2, y = 3$

12.  $x = \frac{bdmnr + bc nrs}{adnr + bcm s}, y = \frac{ad nrs - bd m^2 s}{adnr + bcm s}$

13.  $x = \frac{(a+b)c}{a^2+b^2}, y = \frac{(a-b)c}{a^2+b^2}$

14.  $x = \frac{1}{2}, y = \frac{1}{3}$

15.  $x = \frac{a^3 d + b^3 c}{ab(adm + bc n)}, y = \frac{b^3 c + a^3 d}{ab(b^2 m - a^2 n)}$

16.  $x = 13\frac{1}{2}, y = 5\frac{7}{10}$

17.  $x = 5, y = 3$

18.  $x = \frac{c(a-b)^2}{a^3 - 3a^2 b - ab^2 - b^3}, y = \frac{c(a+b)^2}{a^3 - 3a^2 b - ab^2 - b^3}$

19.  $x = \frac{rs - bc}{as - br}, y = \frac{r^2 - ac}{as - br}$

20.  $x = \frac{am^2 + bmn + cm}{am + bn}, y = \frac{amn + bn^2 + cn}{am + bn}$

21.  $x = \frac{m^2 n + bcdm + acm + b^2 m - dm - cn}{an + cm},$   
 $y = \frac{abcdn + a^2 cn + ab^2 n - cm^2 n + c^2 n + cdm}{abn + bcm}$

$$22. x = \frac{mnp + an + mb}{2n}, y = \frac{mnp - an - mb}{2m}$$

$$23. x = \frac{c}{a-2b}, y = \frac{2c}{2b-a}$$

$$24. x = \frac{ac + bc - ad + bd}{4ab}, y = \frac{ad + bd - ac + bc}{4ab}$$

$$25. x = \frac{4ab}{ad - ac + bc + bd}, y = \frac{4ab}{ad - ac - bc - bd}$$

$$26. x = \frac{2a(a^2 + b^2)}{a^2 - b^2}, y = \frac{2b(a^2 + b^2)}{a^2 - b^2}$$

$$27. x = \frac{a^2 + a^2b + ab^2 - b^2}{a^2 + b^2}, y = \frac{b^2 + ab^2 + a^2b - a^2}{a^2 + b^2}$$

**Exercise 43.**

1.  $x = 2, y = 3, z = 4$

2.  $x = 4, y = 5, z = 1$

3.  $x = 3, y = 6, z = 8$

4.  $x = -3, y = -2, z = -1$

5.  $x = 10, y = 5, z = 3$

6.  $x = a, y = b, z = c$

7.  $x = bc, y = ac, z = ab$

8.  $x = 3, y = 4, z = 5, u = 6$

9.  $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{4}$

10.  $x = 10, y = 12, z = 16$

11.  $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$

12.  $x = 10, y = 10, z = 10$

13.  $x = \frac{1}{2}(a+b-c), y = \frac{1}{2}(a-b+c), z = \frac{1}{2}(b+c-a)$

14.  $x = 6, y = 5, z = 4$

15.  $x = 10, y = 11, z = 12, u = 13, t = 14$

16.  $x = \frac{a^2b - ac + a - b - b^2 + c}{a(b^2 - 1)}, y = \frac{a - b - b^2 + c}{a(b + 1)},$

$$z = \frac{a^2 - b^2 + a - ab - c - b + c}{a(b^2 - 1)}$$

17.  $x = 3\frac{8}{7}, y = 2\frac{2}{3}, z = 2\frac{2}{11}$

18.  $x = 3, y = -4\frac{1}{2}, z = -1\frac{3}{4}$

19.  $x = 18, y = 5, z = 10$

20.  $x = \frac{44c - 138b - 168a}{239}, y = \frac{440c + 54b - 246a}{717},$

$$z = \frac{21a - 162b + 114c}{239}$$

**Exercise 44.**

1. \$1800, \$3200

2. \$200, \$240

3. 10 h., 60 s., 30 c.

4. \$1.10, \$1.20, \$1.40

5. 48

6. 234

7. \$1623  $\frac{9}{17}$ , \$2611  $\frac{13}{17}$ , \$2541  $\frac{3}{17}$       8. 41      9. 808
10.  $\frac{5}{9}$       11.  $34\frac{2}{7}$  d., 48 d., 240 d.
12. 20 hr., 30 hr., 60 hr.      13. 1200
14.  $17\frac{7}{9}$  hr.,  $22\frac{6}{7}$  hr., 32 hr.      15. 24 yd., 18 yd.
16. 60 rd., 40 rd.      17. 2652.57 ft., 2627.43 ft.
18. 4 ft., 25.13 ft.; 5 ft., 31.42 ft.      19. \$1500, 1 yr.
20. \$  $\frac{br-ar'}{r-r'}$ ,  $\frac{100(a-b)}{br-ar'}\%$       21. \$1800, 6%
22. \$  $\frac{bm-an}{m-n}$ ,  $\frac{100(a-b)}{bm-an}\%$       23. \$600, \$900, \$1500
24. \$2000, 5%; \$3500, 6%; \$4500, 7%
25. \$1140 in 5's, \$1470 in 6's      26. 556 ft., 500 ft.
27. 30 yd., 20 yd.      28. 30, 20, 20      29. 40 mi., 32 mi.
30. \$4800, \$300; \$5640, \$300; \$7360, \$320
31. 15 mi., 10 mi.      32. \$10,000, \$12,000, \$15,000
33. 200, 150, 120, 80      34. \$5000, \$4000, \$7000
35.  $\frac{a(c-b)}{a-b}$ ,  $\frac{b(a-c)}{a-b}$       36. 148 lb. of tin, 92 lb. of lead.

## Exercise 45.

1.  $a^{\frac{5}{6}}$ ,  $a^{\frac{1}{2}}$ ,  $a^{-2}$ ,  $a^{-\frac{3}{2}}$ ,  $a^{n-m}$
2. 1,  $x^{\frac{m+n}{mn}}$ ,  $x^{\frac{1-n}{n}}$ ,  $x^{-\frac{m+n}{mn}}$ ,  $x^{\frac{1-x^2}{x}}$
3.  $6a^{-3}b$ ,  $-12a^2b^{\frac{1}{2}}y$ ,  $-7^5a^{-5}b^{-4}c^{-\frac{3}{2}}$
4.  $x^4$ ,  $x^{\frac{1}{2}}$ ,  $x^{\frac{n-m}{mn}}$ ,  $x^{\frac{m^2-n^2}{mn}}$ ,  $x^{\frac{8}{5}}$
5.  $x^3y^{-2}z^4$ ,  $3a^{-8}b^{\frac{1}{2}}c^{-5}$ ,  $3a^{\frac{1}{2}}b^{\frac{1}{2}}c^{-\frac{1}{2}}$
6.  $a^{-6}$ ,  $a^{\frac{1}{3}}$ ,  $x^{-2}$ ,  $x^{\frac{1}{2}}$ ,  $x^{\frac{1}{2}}$ ,  $x^{-m^2}$ ,  $x^{-1}$
7.  $27a^{-6}b^9$ ,  $a^{10}b^{-\frac{1}{2}}c^{-\frac{3}{2}}$ ,  $4bc$ ,  $\frac{8}{27}x^{3m}y^{\frac{2}{m}}z^{\frac{3m}{n}}$ ,  $4a^2b^{-\frac{1}{2}}c^{\frac{3}{2}}$
8.  $12^2$ ,  $2^9$ ,  $3^{\frac{2}{3}}$ ,  $36$ ,  $\frac{1}{8}$
9.  $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ ,  $a^{\frac{1}{2}}-2a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{1}{3}}$ ,  $a^{-1}-2a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$ ,  $a^{-\frac{1}{2}}-b^{\frac{1}{2}}$
10.  $a^{\frac{1}{2}}-x^{\frac{1}{2}}$ ,  $x^4-x^{-4}$ ,  $a+3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}+b$ ,  
 $a^{-12}-3a^{-8}b^{-3}+3a^{-4}b^{-6}-b^{-9}$
11.  $x^{\frac{1}{2}}+12x^{\frac{3}{2}}+35$ ,  $x^{-6}-3x^{-3}-4$ ,  $x^{-2m}-ax^{-m}-6a^2$
12.  $x+x^{\frac{1}{2}}y^{\frac{1}{2}}-x^{\frac{2}{3}}y^{\frac{2}{3}}-y^{\frac{1}{2}}$ ,  $8+3x^{\frac{1}{2}}+5x^{-\frac{1}{2}}$ ,  $x^2+xy+y^2$

13.  $x^{-3} + 4x^{-2}y^{-1} + 6x^{-1}y^{-2} + 4x^{-2}y^{-3} + y^{-3}$ ,  
 $\frac{16}{81}x^3 - \frac{8}{9}x^2y^{\frac{1}{3}} + \frac{3}{2}x^{\frac{2}{3}}y^{\frac{1}{3}} - \frac{9}{8}x^{\frac{1}{3}}y^{\frac{1}{3}} + \frac{81}{256}y^{\frac{1}{3}}$ ,  
 $\frac{5}{x^m} + 5x^{\frac{4}{m}}y^{\frac{1}{m}} + 10x^{\frac{3}{m}}y^{\frac{2}{m}} + 10x^{\frac{2}{m}}y^{\frac{3}{m}} + 5x^{\frac{1}{m}}y^{\frac{4}{m}} + y^{\frac{5}{m}}$ ,  
 $32x^{-5} - 240x^{-2} + 720x - 1080x^4 + 810x^7 - 243x^{10}$
14.  $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ ,  $a^{\frac{3}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + b^{\frac{3}{2}}$ ,  $a^{\frac{5}{2}} - a^{\frac{3}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{3}{2}} - b^{\frac{5}{2}}$ ,  
 $a^{\frac{7}{2}} - a^{\frac{5}{2}}b^{\frac{1}{2}} + a^{\frac{3}{2}}b^{\frac{3}{2}} - a^{\frac{1}{2}}b^{\frac{5}{2}} + b^{\frac{7}{2}}$ ,  
 $a^{\frac{9}{2}} + a^{\frac{7}{2}}b^{\frac{1}{2}} + a^{\frac{5}{2}}b^{\frac{3}{2}} + a^{\frac{3}{2}}b^{\frac{5}{2}} + a^{\frac{1}{2}}b^{\frac{7}{2}} + b^{\frac{9}{2}}$
15.  $a^{\frac{1}{2}} - a^{\frac{1}{2}}b^{\frac{1}{2}} + b^{\frac{1}{2}}$ ,  $a^{\frac{3}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{3}{2}} + b^{\frac{3}{2}}$ ,  
 $a^{-\frac{3}{2}} + a^{-\frac{1}{2}}b^{-\frac{1}{2}} + b^{-\frac{1}{2}}$ ,  $4a^{\frac{2m}{n}} - 2a^{\frac{m}{n}} + 1$
16.  $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$ ,  $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$ ,  $(a^{-\frac{1}{2}} + b^{-\frac{1}{2}})(a^{-\frac{1}{2}} - b^{-\frac{1}{2}})$ ,  
 $(x^{-1} + y^{-1})(x^{-1} - y^{-1})$ ,  $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})$
17.  $(a^{\frac{1}{2}} - b^{\frac{1}{2}})(a^{\frac{3}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + b^{\frac{3}{2}})$ ,  $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{3}{2}} - a^{\frac{1}{2}}b^{\frac{1}{2}} + b^{\frac{3}{2}})$ ,  
 $(a^{\frac{3}{2}} - b^{\frac{3}{2}})(a^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + b^{\frac{1}{2}})$ ,  $(a^{\frac{3}{2}} + b^{\frac{3}{2}})(a^{\frac{1}{2}} - a^{\frac{1}{2}}b^{\frac{1}{2}} + b^{\frac{1}{2}})$ ,  
 $(a^{-\frac{3}{2}} - b^{-\frac{3}{2}})(a^{-\frac{1}{2}} + a^{-\frac{1}{2}}b^{-\frac{1}{2}} + b^{-\frac{1}{2}})$ ,  $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - a^{\frac{1}{2}}b^{\frac{1}{2}} + b^{\frac{1}{2}})$
18.  $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2$ ,  $(x^{\frac{1}{2}} - 2y^{\frac{1}{2}})^2$ ,  $(2x^{\frac{1}{2}} - 3y^{\frac{1}{2}})^2$
19.  $(x^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{1}{2}})$ ,  
 $(2x + \sqrt{2xy} + y)(2x - \sqrt{2xy} + y)$ ,  
 $(x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})(x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})$
20.  $(a + b)(x^{\frac{2}{3}} + y^{\frac{2}{3}})$ ,  
 $x(a^{-1} + b^{-2})(a^{-4} - a^{-3}b^{-2} + a^{-2}b^{-4} - a^{-1}b^{-6} + b^{-8})$
21.  $x^{\frac{1}{2}} - y^{\frac{1}{2}}$ ,  $\frac{x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}$ ,  $\frac{1}{x^{\frac{2}{3}} - y^{\frac{2}{3}}}$ ,  $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$
22.  $\pm 2a^{\frac{2}{3}}b^{-1}c^{\frac{1}{2}}$ ,  $2a^{-\frac{2}{3}}b^{\frac{2}{3}}c^{-\frac{1}{2}}$ ,  $2\frac{1}{3}a^{\frac{2}{3}}b^{\frac{1}{3}}c^{\frac{5}{6}}$ ,  $\pm x^{-\frac{1}{2}}y^{\frac{3}{6}}z^{\frac{1}{2}}$
23.  $\frac{b^3c^3x^5}{a^2y^2}$       24.  $\frac{b^2cx^{\frac{1}{2}}y^5}{a^3z^3}$ ,  $a^3b^3$ ,  $\frac{1}{x^1y^{\frac{1}{2}}z^{\frac{1}{2}}}$ ,  $\frac{y^2z^4}{x^2}$
25.  $\frac{x^2 + y^2}{x^2y^2z}$ ,  $\frac{1 + x^2y^2}{xy^2}$ ,  $\frac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}}$ ,  $-xy$ ,  $\frac{y-x}{xy}$
26.  $4a^{-4}b^3x^2y^{-3}$
27.  $a^3b^{-1}c^2x$ ,  $a^{\frac{1}{2}}b^2x^2y^{-4}$ ,  $m^{-\frac{1}{2}}n^{\frac{3}{2}}x^3y^{-\frac{1}{2}}$ ,  $54x^5yz^{-\frac{5}{2}}$ ,  $-x^2y$
29.  $\sqrt[12]{a^8b^3c^6}$ ,  $\sqrt[6]{a^3b^{-3}c^{-4}}$ ,  $\sqrt{a^2 - b^2}$ ,  $\sqrt{x^{-2}y^{-1}z^{-4}}$
30.  $\frac{xy}{y^3 - x^3}$ ,  $\frac{x^2y^{\frac{1}{2}}}{a^3}$ ,  $\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}$       31.  $\frac{1}{a^{\frac{1}{2}}}\sqrt{-1}$  or  $(-a^{-5})^{\frac{1}{2}}$
32.  $(a^3 - b^3)^2$ ,  $(4a^3 - 9b^3)^{\frac{1}{2}}$ ,  $(a^4 - b^4)^{-\frac{1}{2}}$ ,  $(a^4 - x^4)^{\frac{3}{2}}$
33.  $(x-y)^2$ ,  $(x^2 - xy + y^2)^2$ ,  $(a^2 + ab + b^2)^{\frac{1}{2}}$ ,  $(x^m - y^m)^{-\frac{2}{3}}$

$$34. (a^4 + a^2 b)^2, (2ab - 2b^2)^2, (8a^4 b + 8a^2 b^3)^2, (a^{12} x^2 + a^{12} y^2)^{-\frac{1}{2}}$$

$$35. 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \text{etc.}$$

$$36. \pm (x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}})$$

$$37. 1 - \frac{1}{3}x^{-2} - \frac{1}{9}x^{-4} - \text{etc.}$$

$$38. x^{\frac{2}{3}} - y^{\frac{2}{3}} + z^{\frac{2}{3}}$$

## Exercise 46.

$$1. 3\sqrt[3]{2}$$

$$2. \pm 2abc^2\sqrt[4]{3ab^3}$$

$$3. \frac{2}{5}\sqrt[3]{10}$$

$$4. \pm (a-b)\sqrt{a}$$

$$5. a(a-b)\sqrt[3]{a(a-b)^2}$$

$$6. a^2c\sqrt[3]{b^2c}$$

$$7. xyz^{n-1}\sqrt{xz^2}$$

$$8. \pm abc^2\sqrt[3]{a^2bc}$$

$$9. x^2y^2z^{n+1}\sqrt{z}$$

$$10. 2\sqrt[3]{2}, a\sqrt[3]{4a^2b^3c}, \sqrt[3]{a^2b^4c^2}$$

$$11. \pm \frac{1}{3}b\sqrt{6a}, \frac{1}{b}\sqrt[3]{a^2}, \pm \frac{1}{ca}\sqrt{ab}$$

$$12. \pm \frac{1}{a-b}\sqrt{a^2-b^2}, \frac{1}{a+b}\sqrt[3]{a^2-b^2}, \frac{1}{a-b}\sqrt[2]{a^3-b^3}$$

$$13. \sqrt[3]{ax^4}, \sqrt[3]{\frac{125}{576}}, \sqrt{(a+x)^2}, \sqrt{\frac{a+x}{a-x}}$$

$$14. \sqrt{\frac{x+1}{x-1}}, \sqrt[3]{\frac{1-ab}{ab}}, \sqrt[3]{\frac{1-x}{1+x}}$$

$$15. \sqrt[12]{\frac{64}{729}}, \sqrt[12]{\frac{81}{256}}, \sqrt[12]{\frac{125}{216}}$$

$$16. \sqrt[10]{a^5x^5}, \sqrt[10]{a^4x^2}, \sqrt[10]{ax^2}$$

$$17. \sqrt[2]{81a^{4n}}, \sqrt[4]{4a^{2(n+1)}}, \sqrt[8]{4a^{n-1}}$$

$$18. \sqrt[6]{64a^6}, \sqrt[6]{8a^2}, \sqrt[6]{4a^2}$$

$$19. 2a\sqrt{2a}, 3b\sqrt{2a}, 4a^2b^2\sqrt{2a}$$

$$20. -\frac{1}{2}\sqrt[3]{2}, \frac{2}{3}\sqrt[3]{2}, \frac{3}{4}\sqrt[3]{2}$$

$$21. \frac{1}{a+b}\sqrt{a+b}, \frac{1}{a-b}\sqrt{a+b}$$

$$22. \frac{x-1}{x+1}\sqrt{x+1}, \frac{x+1}{x-1}\sqrt{x+1}$$

$$23. \sqrt[3]{5}, 2\sqrt[3]{5}, -3\sqrt[3]{5}$$

$$24. \frac{1}{2}\sqrt[12]{64}, \frac{1}{3}\sqrt[12]{18}, \frac{1}{2}\sqrt[12]{1728}$$

$$25. \frac{31}{15}\sqrt{ab}$$

$$26. -a\sqrt[3]{3a}$$

$$27. 2(a+b-c)\sqrt[3]{3a}$$

$$28. 0$$

$$29. \frac{(a-b)^2}{ab^2}\sqrt{ab}$$

$$30. \frac{7}{12}\sqrt{2}$$

$$31. \frac{9}{4}\sqrt{3} + \sqrt{2}$$

$$32. \frac{5}{4}\sqrt{3} + \frac{11}{3}\sqrt{5} - 3\sqrt[4]{2}$$

$$33. 0$$

$$34. 6\sqrt[3]{90}$$

$$35. 16$$

$$36. \frac{6}{7}\sqrt{14}$$

$$37. 10\sqrt[3]{3}$$

$$38. \frac{1}{6}\sqrt[3]{9}$$

$$39. 2\sqrt[4]{32}$$

$$40. \frac{1}{5}\sqrt{15}$$

$$41. \frac{4}{9}\sqrt[12]{839808}$$

42.  $xy\sqrt{x+x^2y+x}\sqrt{y}$       43.  $\sqrt[13]{33} + \sqrt[4]{2} + \sqrt[6]{2}$   
 44.  $x-y+2\sqrt{xz+z}$       45.  $x-y$   
 46.  $\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}$       47.  $\sqrt{x}-\sqrt[4]{xy}+\sqrt{y}$   
 48.  $\sqrt[4]{x}-\sqrt[4]{y}$       49.  $\sqrt[3]{x^2}-\sqrt[3]{x}+1$       50.  $\frac{a^{10}}{b^{10}}, m^3n^3, \frac{1}{a^2x^2}$   
 51.  $9\sqrt{3ab}, x^4y^4\sqrt[3]{x^2y^2}, \sqrt[4]{27b^3}$   
 52.  $4y\sqrt[3]{4x^2y}, \sqrt[3]{ab^2}, \sqrt{3ab}$       53.  $2a-2\sqrt{a^2-b^2}$   
 54.  $a^4+6a^2b^2+b^4+(4a^4b+4ab^4)\sqrt{ab}$   
 55.  $2(2x+y)\sqrt{x-y}+2(2x-y)\sqrt{x+y}$       56.  $\frac{577}{144}-\frac{17}{6}\sqrt{2}$   
 57.  $\pm(\sqrt{x}-\sqrt[3]{x})$       58.  $\pm(2\sqrt{x}+3\sqrt{y})$   
 59.  $\sqrt[3]{x}-\sqrt[3]{y}$       60.  $\pm(x+\sqrt{xy}+y)$       61.  $x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}$   
 62.  $\frac{1}{x+2y}\sqrt{x^2-4y^2}, -(5+4\sqrt{2})$       63.  $\frac{2}{5}\sqrt[3]{25}, \frac{1}{3}\sqrt[4]{648}$   
 64.  $\frac{1}{19}(30-12\sqrt{6}), \frac{1}{7}(4\sqrt{15}-17)$   
 65.  $-\frac{1}{x^2}(2a^2+x^2-2a\sqrt{a^2+x^2})$       66.  $\frac{1}{x}(a-\sqrt{a^2-x^2})$   
 67.  $\frac{1}{14}(3\sqrt{5}+2\sqrt{6}+\sqrt{3}-\sqrt[3]{10})$   
 68.  $1+\frac{1}{6}(5\sqrt{6}-2\sqrt{15}-3\sqrt{10})$

**Exercise 47.**

1.  $2\sqrt{-1}$       2.  $-(2a^2-4b)\sqrt{-1}$   
 3.  $(2x-3y+10z)\sqrt{-1}$       4.  $-4, -2\sqrt{10}, -24$   
 5. 7, 1      6.  $\sqrt{-6}+3\sqrt{-1}-2\sqrt{-3}, 4\sqrt{8}-6\sqrt{2}-2\sqrt{15}$   
 7.  $16\sqrt{15}-12\sqrt{10}-\sqrt{6}+24$       8.  $2\frac{2}{3}, \frac{5}{2a}\sqrt{ab}$   
 9.  $\frac{7}{2}\sqrt{2}+3-2\sqrt{5}$       10.  $\frac{8}{3}\sqrt{-6}+\frac{2}{3}\sqrt{6}-3\sqrt{-1}$   
 11.  $2\sqrt{-1}-\sqrt{-5}$       12.  $-2\sqrt{-2}, 256, -27, x^3(1-x)^2$   
 13.  $2\sqrt{-1}, -2(2+5\sqrt{-2}), -5+2\sqrt{6}, -\frac{1}{4}$   
 14.  $-\frac{1}{2}\sqrt{-2}, -\frac{2}{3}\sqrt{-3}, \frac{1}{2}(1+\sqrt{-1}), 2\sqrt{6}-5$

**Exercise 48.**

1.  $\pm (2 + \sqrt{3})$
2.  $\pm (3 - \sqrt{2})$
3.  $\pm (1 - 2\sqrt{3})$
4.  $\pm (\sqrt{2} + \sqrt{3})$
5.  $\pm (1 - \sqrt{15})$
6.  $\pm (\sqrt{7} - \sqrt{5})$
7.  $\pm (3 + 2\sqrt{5})$
8.  $\pm (7 + 3\sqrt{6})$
9.  $\pm (6 - 3\sqrt{2})$
10.  $\pm (\sqrt{x} - \sqrt{y})$
11.  $\pm \left( \frac{1}{2}\sqrt{2} + \frac{1}{5}\sqrt{5} \right)$
12. Not a square.
13.  $\pm (3\sqrt{3} - 2\sqrt{2})$
14. Not a square.
15.  $\pm (x + \sqrt{a^2 - x^2})$
16.  $\pm (\sqrt{x} - \sqrt{x-y})$
17.  $\pm (\sqrt{x+a} + \sqrt{x-a})$
18.  $\pm (1 + \sqrt{-1})$
19.  $\pm (\sqrt{-3} + \sqrt{-2})$
20.  $\pm (\sqrt{x^2 + xy} + \sqrt{x^2 - xy})$
21.  $\pm (x + \sqrt{x})$
22.  $\pm (10 + \sqrt{-5})$
23.  $\pm \left( \frac{1}{2}\sqrt{3} - \frac{1}{3}\sqrt{6} \right)$
24.  $\pm \frac{1}{10} - \frac{1}{10}\sqrt{-1}$
25.  $\pm (\sqrt{x^2 + xy} + \sqrt{xy + y^2})$
26.  $\frac{1}{3}\sqrt{6}$

**Exercise 49.**

1.  $x = \frac{1}{4}$
2.  $x = a$
3.  $x = 36$
4.  $x = a \left( \sqrt{2a} - \frac{2}{3} \right)$
5.  $x = 26$
6.  $x = \frac{a^2}{(a+1)^2}$
7.  $x = \frac{a-2\sqrt{a}}{a-4}$
8.  $x = 1200$
9.  $x = b^3$
10.  $x = 342$
11.  $x = \frac{2am}{m^2+1}$
12.  $x = \left( \frac{1}{1-\sqrt{a}} \right)^2$
13.  $x = \frac{1}{a}$  or  $a$
14.  $x = 49$
15.  $x = 1$
16.  $x = a + 2\sqrt{b} + \frac{b}{a}$
17.  $x = c$
18.  $x = 18$
19.  $x = \frac{b^3}{a} \left( \frac{1+c}{1-c} \right)^3$
20.  $x = a^2 \left( \frac{b+1}{b-1} \right)^2 - a$
21.  $x = \sqrt{6} + \sqrt{10} - \sqrt{15} - 2$
22.  $x = 49 + 20\sqrt{6}$
23.  $x = 0$
24.  $x = 5$
25.  $x = \frac{1}{a} \left( \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} \right)^2$
26.  $x = \left( \frac{a+1}{a} \right)^3$

**Exercise 50.**

1.  $\pm \frac{3}{2}\sqrt{10}, \pm \frac{xy}{2ab}\sqrt{bx}, \pm 2x^2y\sqrt[4]{3y}, \frac{1}{2}\sqrt[5]{24}$

2.  $\pm 2\sqrt{a}, \pm x\sqrt{5y}, a^{s-1}$  3.  $\sqrt[3]{2x}$
4.  $\frac{y(9x-1)}{6b} \sqrt[3]{4b^2xy}$  5.  $\frac{4xy}{x^2-y^2} \sqrt{y}$
7.  $a\sqrt{a}+b\sqrt{b}$  8.  $\frac{1}{10}, \sqrt[5]{2}$  9.  $7\sqrt[3]{3}$
10.  $4\sqrt{5}, 5\sqrt{3}, 3\sqrt{6}$  11.  $\sqrt{3}, \sqrt[3]{5}, \sqrt[4]{6}$
12.  $\frac{2}{3}\sqrt{\frac{2}{3}}, \frac{2}{5}\sqrt{\frac{3}{4}}, \frac{1}{3}\sqrt{\frac{4}{5}}$  13.  $\frac{1}{20}\sqrt{6}$  14.  $\frac{x}{2z}\sqrt{xy}$
15.  $24\sqrt{10+6\sqrt{2}}, \sqrt{x^2-x}$  16.  $a-2\sqrt{ab}+b, \frac{1}{32}\sqrt{2}, \frac{1}{16}$
17.  $\sqrt{a+b}, \sqrt{3x^2-2y^2}, \sqrt[100]{3a^2b^2}$  18.  $\frac{\sqrt{x^2-4y^2}+x}{2y}$
19.  $\frac{2x^2+8x+8y}{x^2-4x-4y}$  20.  $2x$  21.  $0$
22.  $-4(1+\sqrt{-1}), \frac{81}{2}(\sqrt{-3}-1), -\sqrt{-1}(401+298\sqrt{2})$
23.  $\pm(2+\sqrt{3}), \pm(\sqrt{11}-2\sqrt{2}), \pm(\sqrt{-3}-\sqrt{-2})$
24.  $\sqrt[5]{x^4}+\sqrt[5]{x^3y}+\sqrt[5]{x^2y^2}+\sqrt[5]{xy^3}+\sqrt[5]{y^4},$   
 $\sqrt[5]{x^4}-\sqrt[5]{x^3y}+\sqrt[5]{x^2y^2}-\sqrt[5]{xy^3}+\sqrt[5]{y^4}$
25.  $x^{\frac{1}{2}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{1}{6}}$
26.  $(\sqrt{x}-\sqrt{y})^2, (\sqrt{x}+\sqrt{y})^2, (\sqrt[4]{x}+\sqrt[4]{y})(\sqrt[4]{x}-\sqrt[4]{y}),$   
 $(\sqrt{x}+5)(\sqrt{x}+3), (\sqrt{x}-5)(\sqrt{x}+3)$
27.  $\sqrt{x}-\sqrt{y}, \sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}, \frac{\sqrt{x}+3}{\sqrt{x}-3}$
28.  $\frac{1}{a-b}, \frac{\sqrt{a}+\sqrt{b}}{a-b}, \frac{\sqrt{a}-\sqrt{b}}{a-b}$
29.  $\frac{3+4\sqrt{2x}}{6x}$  30.  $270$
31.  $(ax^{\frac{2}{3}}-x^{\frac{7}{3}})^{\frac{3}{2}}, (x^{-2}+x^{-2})^{-\frac{3}{2}}, (ax^{\frac{1}{2}}+bx^{\frac{3}{2}})^{-\frac{2}{3}},$   
 $(ax^{-\frac{m}{p}}-x^{\frac{np-mq}{p}})^{-\frac{p}{q}}$
32.  $x^2(a-bx)^2, x^{\frac{2}{3}}(x+1)^{\frac{2}{3}}, x^{-\frac{2}{3}}(ax-b)^{-\frac{2}{3}}, x^{2m}(x^{2n}-1)^{\frac{m}{n}},$   
 $x^{2p}(1+x^{-2q})^{-\frac{p}{q}}$
33.  $x = \frac{b(a+4b+4\sqrt{ab})}{a^2}$  34.  $x = \frac{ab}{a+b}, y = \frac{a+b}{4}$
35.  $x = \frac{111}{25}, y = 11$  36.  $p, 2(p+1)$

## Exercise 51.

1.  $x = a, -3a$
2.  $x = 2, -3\frac{2}{3}$
3.  $x = \frac{1}{2}a(b \pm \sqrt{b^2 - 4ac})$
4.  $x = \pm 3$
5.  $x = 5, \frac{7}{9}$
6.  $x = \frac{1}{2}(1 \pm \sqrt{4a - 4a^2 + 1})$
7.  $x = \frac{1}{2}(a \pm 2b\sqrt{-2})$
8.  $x = 5, -2\frac{5}{6}$
9.  $x = -\frac{1}{40}(31 \pm \sqrt{-2399})$
10.  $x = \frac{1}{2m}(n \pm \sqrt{4mr + n^2})$
11.  $x = \frac{ab}{a-2b}, -b$
12.  $x = \pm 9\sqrt{3}$
13.  $x = 2 \pm \sqrt{a^2 + 4}$
14.  $x = \pm \frac{1}{5}\sqrt{415}$
15.  $x = \frac{3a-b}{2}, \frac{1}{2}(3b-a)$
16.  $x = a \pm \sqrt{a^2 - ab - ac + bc}$
17.  $x = -\frac{1}{2(p-q)}(p+q \pm \sqrt{p^2 - 2pq + 5q^2})$
18.  $x = a+b, \frac{a+b}{ab}$
19.  $x = c, b-a$
20.  $x = \pm \frac{1}{c-2}\sqrt{b(bc^2 - 2ac - 2bca + 4a)}$
21.  $x = 0, \pm c$
22.  $x = \frac{1}{2}, -4\frac{1}{2}$
23.  $x = 2, -26\frac{3}{4}$
24.  $x = \frac{a}{2} \left\{ -1 \pm \left( \frac{1}{b-2} \sqrt{b^2 - 4} \right) \right\}$
25.  $x = \frac{1}{60}(13 \pm \sqrt{649})$
26.  $x = \pm \frac{2}{3}a\sqrt{3}$
27.  $x = \frac{a^3}{a^4 - 4}\sqrt{a^4 - 4}$
28.  $x = \pm \frac{1}{2}\sqrt{5}$
29.  $x = \pm \frac{1}{2}\sqrt{2(a+1)}$
30.  $x = \pm \sqrt{a^2 - 2b - 1}$
31.  $x = -1, -2$
32.  $x = \frac{1}{2}(-1 \pm \sqrt{-3})$
33.  $x = 6, 4$
34.  $x = -m, -n$
35.  $x = \frac{2}{a+2b}, -\frac{6}{a+2b}$
36.  $x = \frac{1}{2}(a-b \pm \sqrt{(a+b)^2 - 4(a+b)})$
37.  $x = \frac{1}{4}(-9a \pm \sqrt{49a^2 - 16})$
38.  $x = \pm \frac{b}{a-2}\sqrt{a^2 - 4}$
39.  $x = -\frac{1}{2(a+b+c)}(a+b-c \mp \sqrt{5a^2 + 2ab - 3b^2 + 6ac - 2bc + 5c^2})$
40.  $x = \pm a$
41.  $-1 \pm \sqrt{-3}$
42.  $x = \frac{1}{2}(1 \pm \sqrt{89})$
43.  $x = \frac{1}{8}(17 \pm \sqrt{17})$

**Exercise 52.**

1.  $x = \pm 3, \pm \sqrt{-3}$
2.  $x = -2, 1 \pm \sqrt{-3}, \sqrt[3]{6}$
3.  $x = 4, 12\frac{1}{4}$
4.  $x = 4, \sqrt[3]{25}$
5.  $x = 8, -3\frac{3}{8}$
6.  $x = 1, -\frac{1}{5}$
7.  $x = 8, \frac{24}{289}\sqrt{51}$
8.  $x = 27, -216$
9.  $x = 16, -9\sqrt[3]{-9}$
10.  $x = \sqrt[3]{\frac{1}{2}(-p \pm \sqrt{4q+p^2})}$
11.  $x = 3, -11$
12.  $x = 12, 21$
13.  $x = 3, -4, \frac{1}{2}(-1 \pm \sqrt{35})$
14.  $x = 3, \frac{1}{3}, \frac{1}{3}(-8 \pm \sqrt{55})$
15.  $x = \pm \frac{1}{2}\sqrt{38}, \pm \sqrt{-1}$
16.  $x = 3, -4, \frac{1}{2}(-1 \pm \sqrt{-15})$
17.  $x = \frac{1}{3}(-1 \pm \sqrt{205 \pm 12\sqrt{15}})$
18.  $x = \frac{1}{2}(1 \pm \sqrt{-3})$
19.  $x = 2, \frac{1}{2}, \frac{1}{4}(-9 \pm \sqrt{65})$
20.  $x = \frac{1}{b}\left(-a + \sqrt[3]{\frac{1}{2}(-p \pm \sqrt{4q+p^2})}\right)$

**Exercise 53.**

1.  $x = 5, -8$
2.  $x = 3, -1\frac{1}{3}$
3.  $x = -2, -1$
4.  $x = 2, 1$
5.  $x = \frac{1}{7}(-3 \pm \sqrt{65})$
6.  $x = 2 \pm \sqrt{3}$
7.  $x = a, b$
8.  $x = 1 \pm \sqrt{a}$
9.  $x = \frac{1}{2}a, \frac{1}{3}a$
10.  $x = \frac{1}{2}, -\frac{5}{6}$
11.  $x = 1 \pm \sqrt{-1}$
12.  $x = a \pm \sqrt{a}$
13.  $x = \frac{1}{2}(1 \pm \sqrt{-2})$
14.  $x = -\frac{1}{2} \pm \sqrt{-3}$
15.  $x = \frac{1}{2}(a \pm b)$
16.  $x = \frac{1}{4}(-3 \pm \sqrt{-30})$

**Exercise 54.**

1.  $x^2 - 7x = -12$
2.  $x^2 + 7x = -12$
3.  $x^2 + x = 12$
4.  $x^2 - x = 12$
5.  $x^2 - 3x = -1\frac{1}{4}$
6.  $x^2 - \frac{3}{2}x = 7$
7.  $x^2 - 3ax = -2a^2$
8.  $x^2 - bx = 6b^2$
9.  $x^2 + \frac{17}{12}x = -\frac{1}{2}$
10.  $x^2 - ax = 0$
11.  $x^2 = 2$
12.  $x^2 = -1$

13.  $x^2 - 4x = -1$       14.  $x^2 - \frac{75}{4}x = -78\frac{1}{8}$       15.  $x^2 - 2x = -2$   
 16.  $x^2 - 2ax = b^2 - a^2$       17.  $x^2 - 6x = -11$       18.  $x^2 - \frac{4}{3}x = \frac{2}{9}$   
 19.  $x = 2, 2$       20.  $x = 6, -3$       21.  $x = 2, -7$   
 22.  $x = -2, -3$       23.  $x = -3, -4$       24.  $x = 3a, 3a$   
 25.  $x = a, -b$       26.  $x = a \pm b$       27.  $x = 2, \frac{1}{2}$   
 28.  $x = \frac{1}{2}a, \frac{1}{2}a$

**Exercise 55.**

1. Real, rational,  $- > +$       2. Real, rational,  $+ > -$   
 3. Real, rational,  $-$       4. Real, rational,  $+$   
 5. Real, surds,  $- > +$       6. Imaginary.  
 7. Imaginary.      8. Real, surds,  $+ > -$   
 9. Imaginary.      10. Real, rational,  $- > +$   
 11. Real, surds,  $- > +$       12. Real, rational,  $+ > -$   
 13. Real, surds,  $+ > -$       14. Imaginary.  
 15. Imaginary.      16. Real, rational,  $+ > -$   
 17. Real, rational,  $+, 0$       18. Real, rational,  $-$ , equal.  
 19. Imaginary.      20. Real, rational,  $+$ , equal.  
 21. Imaginary.

**Exercise 56.**

1.  $x = \pm\sqrt{2}, \pm\sqrt{-2}$       2.  $x = 2, -1 \pm \sqrt{-3}$   
 3.  $x = -2, 1 \pm \sqrt{-3}$       4.  $x = -3, -5$   
 5.  $x = 4, -7$       6.  $x = 5, 8$       7.  $x = \pm 1, \pm 2$   
 8.  $x = \pm 3, \pm 4$       9.  $x = \frac{2}{3}, -2\frac{1}{2}$       10.  $x = 0, 2, -1\frac{1}{3}$   
 11.  $x = \pm 1, \pm\sqrt{-1}$       12.  $x = 1 \pm \sqrt{-3}, -1 \pm \sqrt{-3}$   
 13.  $x = \pm(1 \pm \sqrt{-2})$       14.  $x = 1\frac{1}{2}, \frac{3}{4}(-1 \pm \sqrt{-3})$   
 15.  $x = \pm 2, \pm 2\sqrt{-1}$       16.  $x = 3, -3, -8$   
 17.  $x = -1, -2, \frac{1}{2}(1 \pm \sqrt{-3})$       18.  $x = -1, \pm 2, \frac{1}{2}(1 \pm \sqrt{-3})$   
 19.  $x = \pm a, \pm 2$       20.  $x = \pm 3, \frac{1}{2}$       21.  $x = -2, -2, -2$   
 22.  $x = \pm\sqrt{-1}$       23.  $x = 1, 1, -2, -2$

24.  $x = 3 \pm \sqrt{3}$ ,  $3 \pm \sqrt{-3}$       25.  $x = \pm \frac{1}{2}(\sqrt{2} \pm \sqrt{-2})$   
 26.  $x = 1$ ,  $-\frac{1}{2}(1 \pm \sqrt{-3})$       27.  $x = -1$ ,  $\frac{1}{2}(1 \pm \sqrt{-3})$   
 28.  $x = \pm 1$ ,  $\pm \sqrt{-1}$ ,  $\frac{1}{2}(\pm \sqrt{2} \pm \sqrt{-2})$   
 29.  $x = 1$ ,  $\frac{1}{4}(-1 + \sqrt{5} \pm \sqrt{-2\sqrt{5}-10})$ ,  $\frac{1}{4}(-1 - \sqrt{5} \pm \sqrt{2\sqrt{5}-10})$ ,  
 $-1$ ,  $\frac{1}{4}(1 + \sqrt{5} \pm \sqrt{2\sqrt{5}-10})$ ,  $\frac{1}{4}(1 - \sqrt{5} \pm \sqrt{-2\sqrt{5}-10})$   
 30.  $x = \pm 1$ ,  $\frac{1}{2}(1 \pm \sqrt{-3})$ ,  $\frac{1}{2}(-1 \pm \sqrt{-3})$   
 31.  $x = \frac{1}{2}\sqrt{2(1 \pm \sqrt{-3})}$ ,  $\pm \sqrt{-1}$       32.  $x = \pm 2$ ,  $\pm 2\sqrt{-1}$   
 33.  $x = \sqrt{2} \pm \sqrt{-2}$ ,  $-\sqrt{2} \pm \sqrt{-2}$   
 34.  $x = \pm a\sqrt{-1}$ ,  $\pm \frac{a}{2}\sqrt{2(1 \pm \sqrt{-3})}$   
 35.  $x = \frac{a}{2}(1 \pm \sqrt{-3})$ ,  $-\frac{a}{2}(1 \mp \sqrt{-3})$ ,  $\pm a$   
 36.  $x = \pm \frac{1}{3}$ ,  $\pm \frac{1}{3}\sqrt{-1}$       37.  $x = \pm \frac{1}{6}(\sqrt{2} \pm \sqrt{-2})$   
 38.  $x = \frac{1}{2}(-1 \pm \sqrt{5})$ ,  $\frac{1}{2}(-1 \pm \sqrt{-11})$   
 39.  $x = -5$ ,  $\frac{5}{2}(-5 \pm \sqrt{-3})$       40.  $x = -\frac{1}{2}\sqrt[3]{4}$ ,  $\frac{1}{2}\sqrt[3]{8 \pm 4\sqrt{-3}}$   
 41.  $x = -a$ ,  $-\frac{3}{2}a$ ,  $\frac{a}{4}(-5 \pm \sqrt{33})$       42.  $x = \frac{a}{3}(7 \pm 2\sqrt{-3})$ ,  $\frac{1}{3}a$

**Exercise 57.**

- |  |  |   |
|--|--|---|
| 1. $x = 5$ , 4<br>$y = 4$ , 5  | 2. $x = 6$ , -2<br>$y = 2$ , -6            | 3. $x = 10$ , 3<br>$y = 1\frac{1}{2}$ , 5                                   |
| 4. $x = 12$ , -9<br>$y = 3$ , -4                                     | 5. $x = 4$ , $2\frac{2}{3}$<br>$y = 4$ , 6 | 6. $x = \frac{1}{4}$ , $-\frac{1}{2}$<br>$y = \frac{2}{3}$ , $-\frac{1}{3}$ |
| 7. $x = \pm 4$ , $\pm 3$<br>$y = \pm 3$ , $\pm 4$                    | 8. $x = 9$ , 3<br>$y = 3$ , 9              | 9. $x = 6$ , -4<br>$y = 4$ , -6   |
| 10. $x = \pm 8$ , $\pm 2\sqrt{-1}$<br>$y = \pm 2$ , $\mp 8\sqrt{-1}$ | 11. $x = 8$<br>$y = 2$                     |   |
| 12. $x = 9$<br>$y = 6$   | 13. $x = 8$<br>$y = 4$                     | 14. $x = 4$ , 3<br>$y = 3$ , 4  |

15.  $x = 6, -3$   
 $y = 3, -6$
16.  $x = 10, 3$   
 $y = 3, 10$
17.  $x = 5, -4$   
 $y = 4, -5$
18.  $x = \pm 4, \pm 3$   
 $y = \pm 3, \pm 4$
19.  $x = \pm 3, \pm 1$   
 $y = \pm 1, \pm 3$
20.  $x = 5, -4\frac{1}{2}$   
 $y = 3, -3\frac{1}{2}$
21.  $x = 2, 3$   
 $y = 3, 2$
22.  $x = 4$   
 $y = 5$
23.  $x = \pm a, \pm b$   
 $y = \pm b, \pm a$
24.  $x = 3, 2, \frac{1}{2}(5 \pm \sqrt{53})$   
 $y = 2, 3, \frac{1}{2}(5 \mp \sqrt{53})$
25.  $x = \pm 5$   
 $y = \pm 2$
26.  $x = 27, 8$   
 $y = 8, 27$
27.  $x = 5, 4, -6 \pm \sqrt{-5}$   
 $y = 4, 5, -6 \mp \sqrt{-5}$
28.  $x = 3, -2, \pm \sqrt{7}$   
 $y = 2, -3, \pm \sqrt{7}$
29.  $x = 2, 3$   
 $y = 3, 2$
30.  $x = 5, -3$   
 $y = 3, -5$
31.  $x = 3, 2, -3 \pm \sqrt{3}$   
 $y = 2, 3, -3 \mp \sqrt{3}$
32.  $x = 5, -7\frac{3}{4}$   
 $y = 5, -3\frac{2}{4}$
33.  $x = \pm 5$   
 $y = \pm 2$
34.  $x = 6, 3$   
 $y = 3, 6$
35.  $x = \frac{1}{2}, \frac{1}{3}$   
 $y = \frac{1}{3}, \frac{1}{2}$
36.  $x = \frac{1}{5}, -\frac{1}{4}$   
 $y = \frac{1}{4}, -\frac{1}{5}$
37.  $x = \frac{1}{64}, -\frac{1}{27}$   
 $y = \frac{1}{27}, -\frac{1}{64}$
38.  $x = \pm 1, \pm 4$   
 $y = \pm 4, \pm 1$
39.  $x = \pm \sqrt{2}$   
 $y = \pm 2$
40.  $x = 4, 1$   
 $y = 1, 4$
41.  $x = 1, 32$   
 $y = 32, 1$

**Exercise 58.**

1.  $x = 3, \frac{1}{2}(-5 \pm \sqrt{33})$   
 $y = 3, \frac{1}{2}(-5 \mp \sqrt{33})$
2.  $x = 2, 4, 3 \pm \sqrt{-55}$   
 $y = 4, 2, 3 \mp \sqrt{-55}$
3.  $x = 2, 3, \frac{1}{2}(5 \pm \sqrt{-51})$   
 $y = 3, 2, \frac{1}{2}(5 \mp \sqrt{-51})$
4.  $x = 5, -2, \frac{1}{2}(3 \pm \sqrt{-67})$   
 $y = 2, -5, \frac{1}{2}(-3 \pm \sqrt{-67})$
5.  $x = 4, 3$   
 $y = 3, 4$
6.  $x = 4$   
 $y = 2$
7.  $x = 2, 1, \frac{1}{8}(3 \pm \sqrt{43})$   
 $y = 1, 2, \frac{1}{8}(3 \mp \sqrt{43})$
8.  $x = 4, 2, \frac{1}{2}(-7 \pm \frac{1}{2}\sqrt{-13251})$   
 $y = 2, 4, \frac{1}{2}(-7 \mp \frac{1}{2}\sqrt{-13251})$
9.  $x = 5, 4, \frac{1}{2}(-9 \pm \sqrt{-71})$   
 $y = 4, 5, \frac{1}{2}(-9 \mp \sqrt{-71})$
10.  $x = 5, 1, -2 \pm \sqrt{-19}$   
 $y = 1, 5, -2 \mp \sqrt{-19}$

**Exercise 59.**

1.  $x = \pm 3, \pm \frac{1}{2}\sqrt{34}$   
 $y = \pm 5, \pm \frac{5}{2}\sqrt{34}$
2.  $x = \pm 2, \pm \frac{4}{3}\sqrt{3}$   
 $y = \pm 3, \pm \frac{5}{3}\sqrt{3}$
3.  $x = \pm 6$   
 $y = \pm 3$
4.  $x = \pm \frac{1}{19}\sqrt{965}$   
 $y = \pm \frac{8}{38}\sqrt{965}$
5.  $x = \pm 2$   
 $y = \pm 1$
6.  $x = \pm 3, \pm \frac{4}{5}\sqrt{95}$   
 $y = \pm 5, \mp \frac{2}{5}\sqrt{95}$
7.  $x = \pm 1$   
 $y = \pm 10$
8.  $x = 0$   
 $y = 0$
9.  $x = \pm 7$   
 $y = \pm 5$
10.  $x = \pm 2, \pm \infty$   
 $y = \pm 5, \mp \infty$
11.  $x = 9, 13\frac{2}{3}$   
 $y = 1, 12\frac{1}{3}$
12.  $x = 2$   
 $y = 2$
13.  $x = \pm 2$   
 $y = \pm 2$
14.  $x = \pm 2$   
 $y = \pm 2$
15.  $x = \pm 3$   
 $y = \pm 1$
16.  $x = \pm 3, \pm \frac{5}{2}\sqrt{2}$   
 $y = \pm 2, \pm \frac{1}{2}\sqrt{2}$

**Exercise 60.**

1.  $x = \pm 4, \pm 3, \pm (\sqrt{22} \pm \sqrt{10})$   
 $y = \pm 3, \pm 4, \pm (\sqrt{22} \mp \sqrt{10})$
2.  $x = 5, 2, \frac{1}{2}(7 \pm \frac{1}{2}\sqrt{8337})$ , imaginary.  
 $y = 2, 5, \frac{1}{2}(7 \mp \frac{1}{2}\sqrt{8337})$ , imaginary.
3.  $x = \frac{1}{6}(3 \pm \sqrt{429}), \frac{1}{2}(1 \pm \sqrt{133})$   
 $y = \frac{1}{6}(-3 \pm \sqrt{429}), \frac{1}{2}(-1 \pm \sqrt{133})$
4.  $x = 2, 1$   
 $y = 1, 2$
5.  $x = \pm 4$   
 $y = \pm 4$
6.  $x = 5$   
 $y = 5$
7.  $x = \frac{1}{2}, \frac{1}{3}$   
 $y = \frac{1}{3}, \frac{1}{2}$
8.  $x = 136, 320\frac{1}{2}, 128\frac{1}{2}, 313$   
 $y = 120, 304\frac{1}{2}, 127\frac{1}{2}, 312$

**Exercise 61.**

1. 6, 9
2. 8, 24
3. 15 rd., 12 rd.
4. 30 yr., 20 yr.
5. 14 ft., 13 ft.
6. 20 ft., 15 ft.
7. 8
8. 6%
9. 5%
10. 5 ft.
11. 20, \$8
12. 12 mi.
13. 12 ft., 16 ft.
14. 9 ft., 16 in.
15. \$20 or \$80
16. 6%, 94

17. 5%, 105      18.  $6\frac{6}{19}\%$ , 95      19. 22.86 ct.,  $9\frac{3}{10}\%$  +  
 20. 5%      21. 10%      22. 25 ct.  
 23. A 28 mi.,  $2\frac{1}{3}$  mi.; B 21 mi.,  $1\frac{3}{4}$  mi.      24. 20 mi.  
 25. 12 ft., 10 ft.      26.  $15 \times 14 \times 13$  ft.,  $12 \times 11 \times 10$  ft.  
 27. 40 mi., 30 mi.      28. 4, 9      29. 4,  $\frac{1}{4}$   
 30. 20      31. 36 doz., 25 ct.      32. A \$700, B \$800  
 33. 52.86 yd.      34. 30s. per week.      35. 120  
 36. 15 da., 18 da.      37. \$80 @ 5%; \$120 @ 6%      38. 70 mi.  
 39. 36, 77 lb.; or 28, 45 lb.      40. 100 shares @ £15 each.  
 41. 2 d.,  $1\frac{1}{2}$  d.; or  $1\frac{1}{2}$  d.,  $1\frac{1}{5}$  d.      42. 40, 44, 50

## Exercise 62.

1.  $\frac{abcd}{abc-bc-ac-ab}$ , 400      2.  $\frac{a+b}{2}$ ,  $\frac{a-b}{2}$ ;  $18\frac{1}{2}$ ,  $6\frac{1}{2}$   
 3.  $\frac{abc}{bc-ac-ab}$ , 20 da.      4.  $\frac{bc}{a-b}$ , 28  
 5.  $\frac{and-bcn}{dn-bm}$ , 21 c.;  $\frac{nbc-adn}{cn-am}$ , 63 g.  
 6. A  $\frac{2abc}{bc+ac-ab}$ , 48 da.; B  $\frac{2abc}{bc+ab-ac}$ ,  $9\frac{3}{5}$  da.;  
 C  $\frac{2abc}{ac+ab-bc}$ ,  $6\frac{6}{7}$  da.  
 7. Number,  $-\frac{a}{2} + \frac{1}{2n} \sqrt{4am n + a^2 n^2}$ , 18;  
 price,  $\frac{m}{-\frac{a}{2} + \frac{1}{2n} \sqrt{4am n + a^2 n^2}}$ , \$4  
 8. Number,  
 $\frac{1}{2b} \{ab - 20(m-n) + \sqrt{400(m-n)^2 + 40ab(m+n) + a^2 b^2}\}$ , 75;  
 price,  $\frac{20m}{\text{n. of sheep}} = 16$  s.  
 9. Capital,  $\frac{1}{2(r+t)} \{mt + ar + mr - nt +$   
 $\sqrt{4mnt(r+t) + (tn - mt - ar - mr)^2}\}$ , \$100  
 10. Rate,  $\frac{1}{2c} \left\{ (a+c)t - (a-c)s + \sqrt{\{(a-c)s - (a+c)t\}^2 - 4c^2 st} \right\}$ , 30 s

$$11. B, \frac{a}{1 + \sqrt{\frac{n}{m}}}, 60; A, \sqrt{\frac{n}{m}} \left( \frac{a}{1 + \sqrt{\frac{n}{m}}} \right), 40$$

$$12. \text{Base, } b + \sqrt{2bd}, 9 \text{ ft.}; \text{hyp., } b + d + \sqrt{2bd}, 15 \text{ ft.}; \\ \text{per., } d + \sqrt{2bd}, 12 \text{ ft.}$$

**Exercise 63.**

- |   |  |  |
|---|--|--|
| 1. $x = 1, 3$<br>$y = 2, 1$   | 2. $x = 5$<br>$y = 1$                              | 3. $x = 3$<br>$y = 2$                        |
| 4. Impossible.  | 5. $x = 2$<br>$y = 1$                              | 6. Impossible.                               |
| 7. Impossible.  |  | 8. $x = 4, y = 4$                            |
| 9. $x = 1, 1, 1, 2, 2, 3$<br>$y = 1, 2, 3, 1, 2, 1$<br>$z = 3, 2, 1, 2, 1, 1$   |  | 10. $x = 1, 2$<br>$y = 1, 1$<br>$z = 3, 1$   |
| 11. $x = 1, 2, 3, 4, 5, 6$<br>$y = 6, 7, 8, 9, 10, 11$<br>$z = 6, 5, 4, 3, 2, 1$  | 12. $x = 22$<br>$y = 3$<br>$z = 3$                 | 13. $x = 6, y = 4$                           |
| 14. $x = 5, y = 2$  | 15. $x = 7, y = 4$                                 | 16. $x = 7, y = 7$                           |
| 17. Impossible.   | 18. $x = 36, y = 25$                               | 19. $x = 3, y = 6$                           |
| 20. Impossible.   | 21. 28   | 22. 21, 79                                   |
| 23. 21  | 24. 16, 12, 8, 4<br>2, 5, 8, 11                    | 25. In 9 ways.                               |
| 26. Pigs: 19, 26, 33, 40, 47, 54, 61, 68<br>Sheep: 80, 70, 60, 50, 40, 30, 20, 10<br>Heifers: 1, 4, 7, 10, 13, 16, 19, 22 |  |  |
| 27. A, 14, 12, 10, 8, 6, 4, 2<br>B, 5, 5, 5, 5, 5, 5, 5<br>C, 1, 2, 3, 4, 5, 6, 7   |  | 28. 1st. 2, 15, 28, 41<br>2d. 74, 55, 36, 17 |
| 30. 1 way, 8 $\frac{1}{2}$ G., 3 $\frac{1}{2}$ c.   |  | 29. \$1.80                                   |
| 32. 1st. 2, 4, 6, 8, 10<br>2d. 9, 8, 7, 6, 5<br>3d. 7, 6, 5, 4, 3<br>4th. 5, 4, 3, 2, 1                                   | 33. $\frac{3}{2}, \frac{4}{5}, \frac{5}{8}$ , etc. | 31. \$183                                    |
|   | 34. 164, 245, 326                                  |  |
|   | 35. $x = 5, y = 244, z = 1$                        |  |

**Exercise 64.**

1.  $a - c < b - d$       2.  $ad < bc$ ;  $-ac < -bd$ ,  $\frac{a}{c} < \frac{b}{d}$   
 4.  $ac < bd$ ; can not tell;  $-ac > -bd$



**Exercise 68.**

- |                           |                               |                             |
|---------------------------|-------------------------------|-----------------------------|
| 1. 20 ft., 16 ft.         | 2. 40, 16                     | 3. 1 : 4                    |
| 4. $\frac{bnt}{ar}$       | 5. corn, 20 bu., oats, 30 bu. |                             |
| 6. 300                    | 7. 8 : 7                      | 8. 11 : 9                   |
| 9. 5                      | 10. 248                       | 12. $\frac{bc}{a-b}$        |
| 13. $\frac{bcr}{at}$      | 14. 181 : 91                  | 15. $\frac{qr-ps}{p+s-q-r}$ |
| 16. $\frac{abdm p}{cenq}$ | 17. $\frac{ca^3}{n^3}$        | 18. $\sqrt[3]{a^2+b^2}$     |

**Exercise 69.**

- |                             |                                     |  |
|-----------------------------|-------------------------------------|--|
| 6. $y = 50$                 | 7. $50\sqrt{2}$                     | 8. $\frac{5}{2}$                                       |
| 9. $x = \frac{4y^2+14}{3y}$ | 10. $s = \frac{1}{2} \text{ ft.}^2$ | 11. $\frac{ac}{b}$                                     |
| 12. $\frac{ac}{d}$          | 15. .3183                           | 16. 128.64   |
| 18. 5 sec.                  | 19. 32.16                           | 20. $\sqrt{\left(\frac{890}{93}\right)^2} \text{ yr.}$ |

**Exercise 70.**

- |             |             |             |             |
|-------------|-------------|-------------|-------------|
| 1. 2.57403  | 2. 2.93247  | 3. 2.95617  | 4. 2.84510  |
| 5. 2.60314  | 6. 0.00000  | 7. 3.30103  | 8. 3.63347  |
| 9. 4.00000  | 10. 2.66558 | 11. 0.16732 | 12. 1.54158 |
| 13. 3.68682 | 14. 3.54108 | 15. 1.53453 | 16. 2.95467 |
|             | 17. 3.66582 | 18. 1.27333 |             |

**Exercise 71.**

- |             |               |                |                |
|-------------|---------------|----------------|----------------|
| 1. 353      | 2. 41.7       | 3. 5140        | 4. 3.18        |
| 5. .033     | 6. .01        | 7. .258        | 8. .000459     |
| 9. 3        | 10. .000274   | 11. .000000266 | 12. .0020137   |
| 13. 1000.28 | 14. .00000209 | 15. 1.2915     | 16. 1          |
| 17. .1      | 18. .00000022 | 19. .00000001  | 20. .000000299 |
| 21. .010013 | 22. 109760.1  | 23. .0000359   | 24. .0254701   |

**Exercise 72.**

- |               |                |               |
|---------------|----------------|---------------|
| 1. 8.37675-10 | 2. 7.45593-10  | 3. 7.37161-10 |
| 4. 9.17070-10 | 5. 11.36452-10 | 6. 6.36957-10 |

- |                 |                 |                 |
|-----------------|-----------------|-----------------|
| 7. 11·86967—10  | 8. 8·36442—10   | 9. 10·19552—10  |
| 10. 6·39794—10  | 11. 9·51428—10  | 12. 13·39794—10 |
| 13. 9·32495—10  | 14. 12·06550—10 | 15. 13·36653—10 |
| 16. 15·00000—10 | 17. 9·15478—10  | 18. 14·74473—10 |

**Exercise 73.**

- |              |             |              |             |
|--------------|-------------|--------------|-------------|
| 1. 155351·25 | 2. 15·173   | 3. ·19653    | 4. ·45936   |
| 5. 9·0467    | 6. 10857·61 | 7. ·007691   | 8. 1441·42  |
| 9. ·04670    | 10. ·82423  | 11. 102733·4 | 12. 1·72928 |

**Exercise 74.**

- |              |             |             |              |
|--------------|-------------|-------------|--------------|
| 1. 20·03     | 2. 165·837  | 3. ·003061  | 4. 2419·33   |
| 5. 24090·607 | 6. ·000039  | 7. 1·3478   | 8. 3468·88   |
| 9. ·0619186  | 10. 43·0465 | 11. 481·477 | 12. ·241317— |
|              | 13. 88·068  | 14. 1·2035  |              |

**Exercise 75.**

- |              |               |                |
|--------------|---------------|----------------|
| 1. 34·013    | 2. ·34362     | 3. 3293358·78— |
| 4. 2·15706   | 5. 21243137·2 | 6. ·91083      |
| 7. 139·65    | 8. 684190·47  | 9. 52·208      |
| 10. 53125·61 | 11. ·80148    | 12. 2·4761     |

**Exercise 76.**

- |   |   |               |
|---|---|---------------|
| 1. $x = 3$  | 2. $x = 2·93$   | 3. $x = 9·54$ |
| 4. $x = \frac{\log. b}{\log. a}$  | 5. $x = \left( \frac{3 \log. c}{\log. m} \right)^{\frac{1}{2}}$ |               |
| 6. $x = \frac{\log. l - \log. a}{\log. r} + 1$  | 7. $x = \frac{4 \log. p}{3 \log. (m-n)}$                        |               |
| 8. $x = \frac{m \log. r + \log. q}{\log. p + \log. q}$  |   |               |
| 9. $x = \frac{1}{2} \left\{ \frac{\log. b}{\log. a} + \frac{\log. d}{\log. c} \right\}$ , $y = \frac{1}{2} \left\{ \frac{\log. b}{\log. a} - \frac{\log. d}{\log. c} \right\}$          |   |               |
| 10. $x = \frac{1}{2} \left( \frac{2 \log. a}{\log. m} + \frac{3 \log. m}{2 \log. a} \right)$ , $y = \frac{1}{2} \left( \frac{3 \log. m}{2 \log. a} - \frac{2 \log. a}{\log. m} \right)$ |   |               |
| 11. $x = -\frac{3 \log. p + 5 \log. q}{6 \log. a}$ , $y = \frac{5 \log. q - 3 \log. p}{4 \log. a}$  |   |               |

$$12. x = \frac{\log. \{a+s(r-1)\} - \log. a}{\log. r}$$

$$13. x = \frac{\log. \{(4a^2-4b^2+1)^{\frac{1}{2}}-1\} - \log. 2}{\log. (a-b)}$$

$$14. x = \log. 2$$

$$15. x = \frac{\log. (a+b) + \log. (a-b)}{\log. a} - 2$$

$$16. x = \frac{r \log. s}{\log. p - \log. q}$$

$$17. x = 2\frac{1}{2}, y = \frac{1}{2}$$

$$18. x = \frac{1}{2 \log. a} \{ \log. n \pm \sqrt{\log.^2 n - 4 \log. m \log. a} \},$$

$$y = \frac{1}{2 \log. a} \{ \log. n \mp \sqrt{\log.^2 n - 4 \log. m \log. a} \}$$

$$19. x = \frac{\log. p + \log. a + \log. 3}{\log. a - \log. p + \log. 3} \sqrt{\log. a - \log. p + \log. 3},$$

$$y = \sqrt{\log. a - \log. p + \log. 3}$$

$$20. x = \frac{\log. \{1 \pm \sqrt{b+1}\}}{a(\log. a - \log. b)}$$

### Exercise 77.

$$1. 21$$

$$2. 16\frac{5}{6}$$

$$3. 3(6-n)$$

$$4. 3$$

$$5. 60$$

$$6. 16$$

$$7. 400, 8105, \frac{1}{4}(7n+3n^2), \frac{1}{2}(103n-3n^2)$$

$$8. l = 13\frac{2}{3}, d = \frac{13}{19}$$

$$9. l = 44\frac{1}{6}, a = -10\frac{5}{6}$$

$$10. l = 55\frac{2}{3}, S = 606\frac{2}{3}$$

$$11. n = 7, S = 490$$

$$12. d = -5\frac{11}{30}, S = 966\frac{2}{3}$$

$$13. a = -287\frac{1}{2}, S = -1750$$

$$14. a = 10\frac{1}{2}, n = 25$$

$$15. a = 133\frac{1}{3}, d = 1\frac{43}{57}$$

$$16. d = -8, n = 15 \quad 17. l = 32, n = 10 \quad 19. 21\frac{1}{2}, 17\frac{17}{24}; m^2$$

$$20. 34\frac{3}{5}, 37\frac{1}{5}, 39\frac{4}{5}, 42\frac{2}{5}; \frac{4x+y}{5}, \frac{3x+2y}{5}, \frac{2x+3y}{5}, \frac{x+4y}{5}$$

$$21. x-y, x, x+y; a+b, a+2b, a+3b$$

$$23. 3, 5, 7, 9, \text{etc.}$$

$$24. 5, 8, 11, 14, 17, 20$$

$$25. 1, 2, 3, \dots, n$$

$$26. x, 3x, 5x, 7x, \text{etc.}$$

$$27. d = \frac{b-a}{m+1}; \frac{am+b}{m+1}, \frac{am+2b-a}{m+1}, \frac{am+3b-2a}{m+1}, \text{etc.}$$

$$28. \frac{1}{5}(4a^2+a^{-2}), \frac{1}{5}(3a^2+2a^{-2}), \frac{1}{5}(2a^2+3a^{-2}), \frac{1}{5}(a^2+4a^{-2})$$

**Exercise 78.**

1. 1, 3, 5, 7, 9, etc.    2. 4, 8, 12    3. 3, 8, 13, 18  
 4.  $948\frac{11}{12}$  ft.    5. \$405    6. 4, 8, 12, 16, 20  
 7. 5    8. 15 days.    9. 6 days.  
 10. 9 rd., 12 rd., 15 rd.    11.  $d = 3$ ,  $l = 47$   
 12. 10 ft., 12 ft., 14 ft.    13. 10 ft., 15 ft., 20 ft.  
 14. 10 days.    15. 11 days.    16.  $n = 7$ ,  $d = 5$   
 17. 50    20. 8    21.  $(2n+1)^2$ , 6  
 23.  $b + (a-b + \frac{1}{2}rb)n + \frac{1}{2}rbn^2$

**Exercise 79.**

1. 6561,  $1953\frac{1}{8}$ ,  $\frac{25}{26244}$ ,  $-486a^5b$   
 2. 2912,  $26\frac{2729}{4096}$ ,  $1\frac{4}{9}$ ,  $\frac{1}{a^2}(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$   
 3.  $l = 98304$ ,  $S = 131070$     4.  $l = 972$ ,  $n = 6$   
 5.  $r = 3$ ,  $l = 135$     6.  $a = 20\frac{100}{133}$ ,  $l = -2\frac{7894}{10773}$   
 7.  $S = 155$ ,  $n = 5$     8.  $r = \frac{1}{2}$ ,  $S = 126$   
 9.  $a = 216$ ,  $S = 775\frac{1}{6}$     10.  $a = 10$ ,  $n = 3$   
 11.  $a = 4$ ,  $r = 5$     12.  $r = 5$ ,  $n = 3$   
 14. 36,  $10\frac{1}{9}$ ,  $\sqrt{ab}$ ,  $6ab$     15. 24, 72;  $a(a+b)$ ,  $a(a+b)^2$ ;  $a, b$   
 16. 1, 3, 9, 27, etc.    17. 0, 4, 20, 100, 500, etc.  
 18.  $l = (-1)^{n-1} \times \frac{1}{2^{n-1}}$     19.  $\frac{3}{5} \left\{ 1 - \left( -\frac{2}{3} \right)^n \right\}$   
 21. 8    22. 2, 6, 18, 30, 50, 70, etc.

**Exercise 80.**

1. 3, 9, 27    2. 1, 4, 16    3. 5, 15, 45, 135  
 6. \$1.34    7. 4, 12, 36, 108    8. 7, 21, 63  
 9. 1, 2, 4, 8, 16    11. 3, 6, 12, 24, 48, 96  
 12. 5 ft., 15 ft., 45 ft.    13.  $\frac{1024}{59049}$     14.  $\frac{1024}{8125}$   
 15. 60 yr., \$64000    16. 10, 20, 40  
 17. \$3000, \$9000, \$27000

**Exercise 81.**

1.  $1\frac{1}{3}$
2. 6
3.  $2\frac{2}{5}$
4.  $23\frac{1}{3}$
5.  $\frac{a^2}{a-1}$
6.  $\frac{ab}{a-1}$
7.  $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}+1}$
8.  $\frac{a^2}{a+b}$
9.  $\frac{34}{99}, \frac{1}{33}, \frac{124}{999}, \frac{2}{333}$
10.  $\frac{81}{90}, \frac{1}{330}, \frac{28}{225}, 1\frac{7}{33}$
11.  $43\frac{7}{11}$  min.
12. 225 rd.
13. \$100
14. 2400 bar.
16. 9, 3, 1,  $\frac{1}{3}$ , etc.

**Exercise 82.**

1.  $\frac{1-r^n}{(1-r)^2} - \frac{nr^n}{1-r}$
2. 14216
3.  $9\frac{107}{128}$
4.  $2\frac{2}{9}$
5.  $\frac{1+r}{(1-r)^2}$
6.  $2\frac{3}{16}$
7. 10

**Exercise 83.**

1.  $\frac{1}{19}$
2.  $\frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14}$
3.  $\frac{1}{98}$
4.  $\frac{3}{31}$
5.  $\frac{1}{a(2n-1)}$
6. 6,  $7\frac{1}{2}$ , 10
7.  $\frac{2ab}{a+b}$
8. H. m. =  $3\frac{1}{5}$ , g. m. = 4, a. m. = 5
13. 15

**Exercise 84.**

1. 1240
2. 816
3. 1196
4. 504
5. 26059
6. 18
7. 11940
9. 816, 1496
10. 190

**Exercise 85.**

1. \$560
2. \$800, 6%
3.  $\frac{100(n-1)}{q}\%$
4.  $\frac{n-1}{r}$  yr.
5. \$200
6.  $3\frac{1}{2}$  yr.
7. 9 yr.
8.  $21+\%$
9. \$9.64
10. 6089-58
14.  $\frac{100(n-m)}{m}\%$
15. 8 yr.

16. 15 yr. nearly.      17.  $\frac{100r}{100-r}$       18. \$2744.68  
 19. \$486.38      20.  $P(1-r)^n$       21.  $AR^n$   
 22. \$103.08      23. 90%      24. 4.85-  
 25. 25 yr. nearly.

**Exercise 86.**

1. 36      2. 56      3.  $10^5$       4. 4500  
 5. 9      9. 25      10. 3      11. 90720  
 12. 1058399      13. 924      14.  $n^2$       15. 64, 325  
 16. 3500      17.  $6^3-1$       18. 8      19. 120  
 20.  $\frac{\begin{array}{|c|} \hline 21 \\ \hline \end{array}}{\begin{array}{|c|} \hline 3 \\ \hline \end{array} \begin{array}{|c|} \hline 18 \\ \hline \end{array}} \times \frac{\begin{array}{|c|} \hline 5 \\ \hline \end{array}}{\begin{array}{|c|} \hline 3 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array}} \times \begin{array}{|c|} \hline 5 \\ \hline \end{array} ; 21 \times 20 \times 19 \times 5 \times 4$       21.  $\frac{\begin{array}{|c|} \hline 45 \\ \hline \end{array} \begin{array}{|c|} \hline 50 \\ \hline \end{array}}{\begin{array}{|c|} \hline 9 \\ \hline \end{array} \begin{array}{|c|} \hline 10 \\ \hline \end{array} \begin{array}{|c|} \hline 36 \\ \hline \end{array} \begin{array}{|c|} \hline 40 \\ \hline \end{array}}$   
 22.  $\frac{\begin{array}{|c|} \hline 6 \\ \hline \end{array} \begin{array}{|c|} \hline 5 \\ \hline \end{array}}{\begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline \end{array}}$       23. 40824      24. 293930, 24310, 45

**Exercise 87.**

1.  $1+x+x^2+x^3$       2.  $\frac{1}{3} + \frac{2}{9}x + \frac{4}{27}x^2 + \frac{8}{81}x^3$   
 3.  $1+x-x^2-x^4$       4.  $1+x-x^2-x^3$   
 5.  $1-x-x^2+2x^3$       6.  $-3+5x-2x^2-3x^3$   
 7.  $1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3}$       8.  $5x+27x^2+180x^3+623x^4$   
 9.  $1+x+x^2-x^3$

**Exercise 88.**

1.  $2 - \frac{1}{4}x - \frac{1}{64}x^2 - \frac{1}{512}x^3$       2.  $1 + \frac{1}{2}x - \frac{5}{8}x^2 + \frac{5}{16}x^3$   
 3.  $3 + \frac{1}{6}x - \frac{109}{216}x^2 + \frac{109}{888}x^3$       4.  $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$   
 5.  $3 + \frac{1}{27}x^2 - \frac{1}{3 \cdot 27^2}x^4 + \frac{15}{3 \cdot 27^4}x^6$   
 6.  $2 + \frac{1}{12}x + \frac{23}{2 \cdot 12^2}x^2 - \frac{278}{2 \cdot 12^4}x^3$       7.  $a^{\frac{1}{2}} + \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{8a^{\frac{3}{2}}} + \frac{x^3}{16a^{\frac{5}{2}}}$   
 8.  $a^{\frac{1}{3}} - \frac{x}{3a^{\frac{2}{3}}} - \frac{x^2}{9a^{\frac{4}{3}}} - \frac{5}{81a^{\frac{5}{3}}}$       9.  $a + \frac{x^2}{3a^3} - \frac{x^4}{9a^5} + \frac{5x^6}{81a^7}$

**Exercise 89.**

1. Divergent.      2. Convergent; divergent; divergent.  
 3. Convergent.      4. Convergent.      5. Divergent.

6. Convergent; divergent; divergent.

7. Divergent; convergent; divergent.

8. Convergent.

**Exercise 90.**

1.  $x^3 + 6x^2 + 11x + 6$
2.  $x^3 - 2x^2 - 9x + 18$
3.  $x^4 + 2x^3 - 7x^2 - 8x + 12$
4.  $x^4 - 37x^3 - 24x + 180$
5.  $x^4 + 8x^3 + 24x^2 + 32x + 16$
6.  $x^4 - 20x^3 + 150x^2 - 500x + 625$
7.  $16x^4 - 16x^3 - 64x^2 + 4x + 15$
8.  $(x+2)(x+3)(x+4)$
9.  $(x-3)(x-4)(x+5)$
10.  $(x+2)(x+3)(x+1)(x-1)$
11.  $(x+2)(x-3)(x+4)(x-5)$
12.  $(x+2)(x-2)(x+3)(x-3)(x+4)$

**Exercise 91.**

1.  $a^4 - 12a^3x^3 + 54a^2x^4 - 108ax^5 + 81x^6$
2.  $32 + 400x + 2000x^2 + 5000x^3 + 6250x^4 + 3125x^5$
3.  $x^6 - 18ax^5 + 135a^2x^4 - 540a^3x^3 + 1215a^4x^2 - 1458a^5x + 729a^6$
4.  $128x^{14} + 2240x^{12} + 16800x^{10} + 70000x^8 + 175000x^6 + 262500x^4 + 218750x^2 + 78125$
5.  $x^4 - 40x^{\frac{7}{2}} + 700x^3 - 7000x^{\frac{5}{2}} + 43750x^2 - 175000x^{\frac{3}{2}} + 437500x - 625000x^{\frac{1}{2}} + 390625$
6.  $2187x^{\frac{1}{2}} + 5103a^{\frac{2}{3}}x^{\frac{1}{2}} + 5103a^2x^{\frac{1}{2}} + 2835a^{\frac{2}{3}}x^{\frac{2}{3}} + 945a^2x^2 + 189a^{\frac{1}{2}}x^{\frac{2}{3}} + 21a^2x^{\frac{2}{3}} + a^{\frac{2}{3}}$
7.  $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$
8.  $a^{\frac{2}{3}} - \frac{2}{3}a^{-\frac{1}{3}}x - \frac{1}{9}a^{-\frac{2}{3}}x^2 - \frac{4}{81}a^{-\frac{1}{3}}x^3$
9.  $x^{\frac{3}{2}} - \frac{3}{4}x^{-\frac{1}{2}} - \frac{3}{32}x^{-\frac{3}{2}} - \frac{5}{128}x^{-\frac{5}{2}}$
10.  $x^{-\frac{1}{2}} - \frac{1}{2}ax^{-\frac{3}{2}} + \frac{3}{8}a^2x^{-\frac{5}{2}} - \frac{5}{16}a^3x^{-\frac{7}{2}}$
11.  $a^{-\frac{2}{3}}x^{-\frac{2}{3}} - \frac{5}{6}a^{-\frac{1}{3}}bx^{-\frac{1}{3}} + \frac{55}{72}a^{-\frac{1}{3}}b^2x^{-\frac{1}{3}} - \frac{935}{1296}a^{-\frac{2}{3}}b^3x^{-\frac{2}{3}}$
12.  $x^{-\frac{4}{3}} + \frac{2}{3}a^{\frac{2}{3}}x^{-\frac{10}{3}} + \frac{5}{9}a^{\frac{4}{3}}x^{-\frac{16}{3}} + \frac{40}{81}a^3x^{-\frac{22}{3}}$
14. 8.06225; 8.94427; 7.006797; 5.000960
15.  $2449440x^4$
16.  $-\frac{5}{128}x^4$
17.  $-\frac{63}{256}a^{-\frac{1}{2}}x^3$
18.  $\pm \frac{r(r+1)(r+2)(r+3)}{4} a^{-(r+4)} x^{r-1}$
19.  $2\frac{2}{3}$
20.  $\frac{110}{243}$
21. 0

**Exercise 92.**

1.  $n(n+1)$ ;  $\frac{1}{3}n(n+1)(n+2)$
2.  $(2n-1)^2$ ;  $\frac{1}{3}n(4n^2-1)$
3.  $\frac{1}{2}n(n+1)$ ;  $\frac{1}{6}n(n+1)(n+2)$
4.  $n(n+2)$ ;  $\frac{1}{6}n(n+1)(2n+7)$
5.  $n^2$ ;  $\frac{1}{6}n(n+1)(2n+1)$
6.  $2n(2n-1)$ ;  $\frac{1}{3}n(n+1)(4n-1)$
7.  $n(n+1)(n+2)$ ;  $\frac{1}{4}n(n+1)(n+2)(n+3)$
8.  $n(n+4)(n+8)$ ;  $\frac{1}{4}n(n+1)(n+8)(n+9)$
9.  $n(n+2)(n+1)^2$ ;  $\frac{1}{10}n(n+1)(n+2)(n+3)(2n+3)$
10.  $(2n+1)(2n+3)(2n+5)$ ;  $n(n+4)(2n^2+8n+11)$
11.  $\frac{n}{3n+1}$ ;  $\frac{1}{3}$
12.  $\frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$ ;  $\frac{11}{18}$
13.  $\frac{1}{8}\left(\frac{1}{4} - \frac{1}{2(n+1)(n+2)}\right)$ ;  $\frac{1}{32}$
14.  $\frac{11}{180} - \frac{6n+11}{12(2n+1)(2n+3)(2n+5)}$ ;  $\frac{11}{180}$
15.  $\frac{5}{4} - \frac{2n+5}{2(n+1)(n+2)}$ ;  $\frac{5}{4}$
16.  $\frac{n}{12}(9n^2+10n^2-3n-4)$
17.  $\frac{n(n+1)(n+2)(n+3)}{[4]}$
18.  $\frac{n}{n+1}$
19.  $2 - \frac{1}{n+1}$
20. 2.978809

**Exercise 93.**

1.  $x = y - y^2 + y^2 - y^4 + \dots$
2.  $x = y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{1}{24}y^4 + \dots$
3.  $x = y - 2y^2 + 3y^3 - 4y^4 + \dots$
4.  $x = y - y^2 + y^2 - y^7 + \dots$
5.  $x = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \frac{1}{4}(y-1)^4 + \dots$
6.  $x = (y-1) + 2(y-1)^2 + 7(y-1)^3 + 30(y-1)^4 + \dots$
7.  $x = \frac{1}{5} - \frac{4}{5^2} + \frac{27}{5^3} - \frac{58}{5^4} = .17590144$
8.  $x = .00999999$

**Exercise 94.**

1.  $p=3, q=-1$ ;  $\frac{1-x}{1-3x+x^2}$
2.  $p=2, q=2$ ;  $\frac{1+x}{1-2x-2x^2}$

3.  $p = 5, q = -3; \frac{2-8x}{1-5x+3x^2}$       4.  $p = 4, q = -5; \frac{3-10x}{1-4x+5x^2}$
5.  $p = 3, q = 2, r = -2; \frac{1-x-5x^2}{1-8x-2x^2+2x^3}$
6.  $p = 2, q = 3, r = 4; \frac{1-5x+8x^2}{1-2x-3x^2-4x^3}$
7.  $p = 3, q = 3; \frac{1-x-1809x^4-1431x^7}{1-3x-3x^2}$
8.  $p = 2, q = -3; \frac{1-4x-4x^7+159x^8}{1-2x+3x^2}$
9.  $p = -2, q = 2; \frac{2+3x-2208x^8+1616x^9}{1+2x-2x^2}$
10.  $p = 1, q = -1, r = 1; \frac{1-3x+6x^2+2x^9-5x^{10}-x^{11}}{1-x+x^2-x^3}$
11.  $(-1)^{n-1} \{4-3 \cdot 2^{n-1}\} x^{n-1}; \frac{4 \{1-(-1)^n x^n\}}{1+x} - \frac{3 \{1-(-2)^n x^n\}}{1+2x}$
12.  $(4n-3)x^{n-1}; \frac{4(1-x^n)}{(1-x)^2} - \frac{(4n-3)x^n+3}{1-x}$

**Exercise 95.**

1.  $\frac{2}{x+2} + \frac{3}{x-2}$       2.  $\frac{3}{2x+1} - \frac{2}{2x-1}$
3.  $\frac{1}{x} + \frac{2}{x+1} + \frac{3}{x+2}$       4.  $\frac{3}{x+1} - \frac{5}{(x+1)^2}$
5.  $\frac{1}{x+3} - \frac{7}{(x+3)^2} + \frac{13}{(x+3)^3}$       6.  $\frac{13}{5(x+2)} + \frac{2}{5(x-3)} + \frac{2}{(x-3)^2}$
7.  $\frac{a}{a-x} + \frac{b}{b+x}$       8.  $\frac{p}{px+q} + \frac{q}{(px+q)^2}$
9.  $\frac{2}{x^2+x+1} + \frac{3}{x^2-x+1}$       10.  $\frac{2}{x} - \frac{2}{1+2x} + \frac{2}{1-2x}$
11.  $\frac{5}{(x-1)^4} - \frac{7}{(x-1)^3} + \frac{1}{(x-1)^2} + \frac{3}{x-1}$
12.  $\frac{3}{2(x-1)} + \frac{1-3x}{2(1+x^2)}$       13.  $1 - \frac{3}{x-2} + \frac{8}{x-3}$
14.  $-\frac{1}{4(x-1)} + \frac{x+1}{4(x^2+1)} + \frac{x+5}{2(x^2+1)^2}$       15.  $\frac{1}{2(x+1)} + \frac{x+1}{2(x^2+1)}$
16.  $-\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{8(x-1)} + \frac{9}{8(x+1)} + \frac{1}{4(x+1)^2} - \frac{x+1}{4(x^2+1)}$
17.  $\frac{3x}{x^2+2x-5} - \frac{1}{x-3}$       18.  $\frac{1}{x-1} - \frac{1}{x+1} + \frac{3}{(x+1)^2} - \frac{3}{(x+1)^3} + \frac{2}{(x+1)^4}$
19.  $\frac{1}{x} - \frac{2}{x^2} + \frac{3}{x+1} - \frac{4}{(x+1)^2}$

$$20. \frac{1}{6} \left\{ \frac{1}{x-1} - \frac{1}{x+1} + \frac{x-2}{x^2-x+1} - \frac{x+2}{x^2+x+1} \right\}$$

$$22. \frac{17}{36(x+1)} - \frac{5}{6(x+1)^2} + \frac{8}{45(x-2)} - \frac{13}{20(x+3)}$$

**Exercise 97.**

$$1. dy = (15ax^2 - 6bx + 2c) dx$$

$$2. dy = 15z^2 x^2 dx + 10x^2 z dz + dz$$

$$3. dy = (3x^2 + 6x + 4) dx$$

$$4. dy = b m^2 (a + b x)^{m^2-1} dx$$

$$5. dy = (x + \frac{3}{2} - x^{-2}) dx$$

$$6. dy = \frac{a dx}{2c^{\frac{1}{2}}} \cdot \frac{1}{(a+x)b^{\frac{1}{2}}}$$

$$7. dy = \frac{5 dx}{3} \cdot \frac{1}{(5x+6)^{\frac{2}{3}}}$$

$$8. dy = \sqrt{\frac{2p}{x}} \cdot dx$$

$$9. dy = \frac{2}{3} (3a)^{\frac{1}{3}} \cdot x^{-\frac{1}{3}} dx$$

$$10. dy = -\frac{b}{a} \cdot \frac{x dx}{\sqrt{a^2 - x^2}}$$

$$11. dy = \frac{1}{2} (3ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}) dx$$

$$12. dy = \frac{2x^3(4a+x)}{(a+x)^4} \cdot dx$$

$$13. dy = \frac{\frac{1}{2}(x+2a) dx}{(x+a)^{\frac{3}{2}}}$$

$$14. dy = -\frac{6ax dx}{(b^2 + x^2)^4}$$

$$15. dy = (x+a)(x+b)^2(5x+3a+2b) dx$$

$$16. dy = (x+a)^{n-1}(x-b)^{p-1} \{ (p+n)x + pa - bn \} dx \quad 17. dy = \frac{2x+1}{x^2+x} \cdot dx$$

$$18. dy = (\log_e x)^2 \cdot \frac{3 dx}{x}$$

$$19. dy = 9x^3 \{ 1 + \log_e 3 + \log_e x \} dx$$

$$20. dy = 2a^{x^2} \cdot x \log_e a dx$$

$$21. dy = \frac{x^2 dx}{(a+x^2)^{\frac{3}{2}}} (4a+x^2)$$

$$22. dy = \frac{n dx}{a+x}$$

$$23. dy = \left( \frac{c}{d} \right)^n (\log_e c - \log_e d) dx$$

$$24. dy = \frac{dx}{(1-x)(1-x^2)^{\frac{1}{2}}}$$

$$25. dy = a^{\log_e x} \cdot \log_e a \cdot \frac{dx}{x}$$

**Exercise 98.**

$$2. 16 \text{ sq. in. per second.}$$

$$3. 4\pi r^2; 144\pi \text{ cu. in. per second.}$$

$$4. 2\sqrt{2} \text{ in.}$$

$$5. 1 \text{ in. per second.}$$

$$6. \text{About } 2.8 \text{ mi. an hr.}$$

$$7. 12\sqrt{3} \text{ in. per sec.}$$

$$8. .00517$$

$$9. 1.62842; .00003$$

**Exercise 99.**

$$1. 3x^2 - 8x + 7$$

$$2. (x+2)^2 (x-2)^2 (7x+2)$$

$$3. 3x^2 + 8x + 2$$

$$4. (a+x)^4 (a-x)^2 (2a-8x)$$

$$5. -8x^7 + 5ax^4 - 3ax^2$$

$$6. 10a \cdot \frac{(a+x)^4}{(a-x)^6}$$

**Exercise 100.**

1.  $(x+3)^2(x-2)^2$
2.  $(x-2)^2(x-1)^2(x+3)$
3.  $(x+3)^2(x-3)^2(x^2+x+1)$
4.  $(x-2)^2(x+2)^2(x-3)^2(x+3)^2$
5.  $(x-1)^4(x+1)^4(x+3)(x-3)$

**Exercise 101.**

1. Max.  $1\frac{3}{4}$ ; min.  $-5$
2. Max.  $-3$ ; min.  $-128$
3. Min.  $-4$
4. Min.  $-4$
5. Max.  $10$ ; min.  $-22$
6. Min.  $0$ ; max.  $18+$
7. Max.  $\left(\frac{a+b}{2}\right)^4$
8. No turning values.
9. Min.  $-14$
10. Min.  $-14$
11. Max.  $16$
12. Min.  $-16$
13. At the middle point.
15.  $\frac{1}{3}a$
16.  $\frac{32}{81}\pi r^3$ , or  $\frac{8}{27}$  of the vol. of the sphere.
17.  $\frac{1}{2}a^2$ , or an inscribed square.
18.  $\frac{1}{4}\pi a^2$
19.  $\frac{4}{81}\pi a^3$
20. Square  $= 2r^2$
21. An isosceles triangle.

**Exercise 102.**

1.  $x^4+2x^3-9x^2+63x-135=0$ ; roots of  $f_n(x)=3\times$  roots of  $F_n(x)$
2.  $x^5+4x^4-2x^3+4x^2-112=0$ ; roots of  $f_n(x)=2\times$  roots of  $F_n(x)$
3.  $x^5+12x^4-320x^3+1792x-1024=0$ ; roots of  $f_n(x)=4\times$  roots of  $F_n(x)$
4.  $x^5-2x^4+9x^3-18x^2+108x-162=0$ ; roots of  $f_n(x)=3\sqrt[5]{\text{roots of } (A)}$
5.  $x^6-2x^5+2x^4-4x^3+24x^2+32x+32=0$ ; roots  $=2\sqrt[7]{\text{roots of } (A)}$
6.  $x^{36}+8\cdot 9^2x^{33}+36\cdot 9^{11}x^{34}+10\cdot 9^{19}x^{16}-4\cdot 9^{25}=0$ ; roots  $=9\sqrt[12]{\text{roots of } (A)}$
7.  $x^7-x^6+18x^4-162x^2-2187=0$ ; roots  $=3\sqrt[4]{\text{roots of } (A)}$
8.  $x^6-10x^5-36x^4-12288=0$ ; roots  $=4\sqrt[6]{\text{roots of } (A)}$
9.  $x^4+6800x-9000=0$ ; roots  $=10\sqrt[3]{\text{roots of } (A)}$

**Exercise 103.**

1.  $x=1$
2.  $x=4$
3.  $x=5$
4.  $x=-2$
5.  $x=1, 2, 2, -2$
6.  $x=\frac{1}{2}, -\frac{1}{2}, -\frac{2}{3}$
7.  $x=1, 2, -2, -3$
8.  $x=5, -1, 2\pm\sqrt{5}$
9.  $x=+2, -2, -3$
10.  $x=2, 4$

11.  $x = 1\frac{1}{2}, \frac{1}{4}(1 \pm \sqrt{29})$       12.  $\frac{1}{2}, -\frac{1}{2}, 1\frac{1}{2}, 1\frac{1}{2}$   
 13.  $x = \pm\sqrt{3}, 2, -2, -\frac{2}{3}$       14.  $x = -3, -5, -1, \pm\frac{2}{3}$   
 15.  $x = -\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}$       16.  $x = 2, -3, -5, 2$   
 17.  $x = 2, 2, -3, -3$       18.  $x = 1, 1, -2, -2, -2$   
 19.  $x = -1, -1, -1, 2, -3, -\frac{2}{3}$

**Exercise 104.**

1.  $-1+, 4+, 9+$       2.  $-6+$       3.  $0.3+, 1$   
 4.  $0.8+, 3+, -3+, -0.1+, -0.6+$   
 5.  $2+, -2+$       6.  $2+, 0.6+, 0.4+, -3+$

**Exercise 105.**

1. 775      2. 248      3. 1.60168,  $-1.83987, -3.26180$   
 4. 3.85808, 1.60601, 1.44327,  $-2.90737$   
 5.  $-5, .0560, 11.15306, 12.7908$   
 6.  $-0.87421$       7.  $-6.3606, 2.34244$       8. 1.25992      9. 1.3798  
 10. 3.2131, 3.2295,  $-17.4426$       11.  $.80285, -5.4335$   
 12. 8.41445      13.  $x^3 + 32x^2 + 343x - 13087800 = 0$

**Exercise 106.**

1.  $x = 1, 1 \pm 2\sqrt{-1}$       2.  $x = 4, 1 \pm \sqrt{-1}$   
 3.  $x = 5, 3 \pm \sqrt{-2}$       4.  $x = -2 \pm \sqrt{-2}, -2, -2$   
 5.  $x = 3, \frac{1}{2}(1 \pm \sqrt{-7})$       6.  $x = -4, -\frac{1}{2}(1 \pm \sqrt{-7})$   
 7.  $x = -2, -2, -3$       8.  $x = 3, -2, -2$   
 9.  $x = 2, 2, -3$       10.  $x = 4, -2 \pm \sqrt{-1}$   
 11.  $x = -6, 3 \pm \sqrt{-1}$       12.  $x = 1, \frac{1}{2}(3 \pm \sqrt{21})$   
 13.  $x = 5, 1, -2$       14.  $x = -5, -1, 2$

**Exercise 107.**

1.  $x = 1, \frac{1}{2}(1 \pm \sqrt{-3})$       2.  $x = -1, 2 \pm \sqrt{3}$   
 3.  $x = 1, -1, \frac{1}{2}(3 \pm \sqrt{5})$       4.  $x = 1, -1, \frac{1}{2}(-3 \pm \sqrt{5})$   
 5.  $x = \frac{1}{2}(5 \pm \sqrt{21}, \pm \sqrt{-1})$       6.  $x = -1, -1, -1, -1, -1$   
 7.  $x = 1, 2, \frac{1}{2}, -3, -\frac{1}{3}$       8.  $x = -1, -5, -\frac{1}{5}, 2 \pm \sqrt{3}$

**Exercise 108.**

- |           |           |       |                  |
|-----------|-----------|-------|------------------|
| 1. -1     | 2. -59    | 3. 65 | 4. $abc - 2ab^2$ |
| 5. $2mnp$ | 6. $4abc$ | 7. 0  | 8. -96           |
| 9. -124   | 10. 0     | 11. 0 | 12. 0            |
| 13. 0     | 14. 0     | 15. 0 |                  |

**Exercise 109.**

- |           |                                |      |      |
|-----------|--------------------------------|------|------|
| 1. 0      | 2. 0                           | 3. 0 | 4. 0 |
| 5. $8xyz$ | 6. $6abc - 2a^2 - 2b^2 - 2c^2$ |      |      |

**Exercise 110.**

- |       |        |        |      |
|-------|--------|--------|------|
| 1. 10 | 2. -48 | 3. -86 | 4. 0 |
|-------|--------|--------|------|

**Exercise 111.**

- |   |                          |
|---|--------------------------|
| 1. $x = 4, y = 2$   | 2. $x = 3, y = 3$        |
| 3. $x = \frac{cn - bd}{an - bm}, y = \frac{ad - cm}{an - bm}$                           |                          |
| 4. $x = \frac{cm - dm + cn + dn}{2(ac - bd)}, y = \frac{an + bn - am + bm}{2(ac - bd)}$ |                          |
| 5. $x = 2, y = 3, z = 4$  | 6. $x = 5, y = 1, z = 2$ |

$$7. x = \frac{abd - acd - abe + ec^2 + abh - bce}{a^2b - a^2c - abc + c^2 + ab^2 - b^2c}$$

$$y = \frac{a^2e - a^2h - acd + c^2h + abd - bce}{a^2b - a^2c - abc + c^2 + ab^2 - b^2c}$$

$$z = \frac{abh - ace - bce + c^2d + b^2e - b^2d}{a^2b - a^2c - abc + c^2 + ab^2 - b^2c}$$

$$8. x = \frac{a^2(m+n-r) + ab(m-n-r) + b^2(m-n+r)}{2a^2 + 2b^2}$$

$$y = \frac{a^2(n-m+r) + ab(n-r-m) + b^2(n-r+m)}{2a^2 + 2b^2}$$

$$z = \frac{a^2(r-n+m) + ab(r-n-m) + b^2(r+n-m)}{2a^2 + 2b^2}$$

$$9. x = 2, y = 3, z = 4, u = 5$$

$$10. x = m \begin{vmatrix} c & 0 & a \\ 0 & a & b \\ a & b & c \end{vmatrix} - n \begin{vmatrix} b & c & 0 \\ 0 & a & b \\ a & b & c \end{vmatrix} + p \begin{vmatrix} b & c & 0 \\ c & 0 & a \\ a & b & c \end{vmatrix} - q \begin{vmatrix} b & c & 0 \\ c & 0 & a \\ 0 & a & b \end{vmatrix}$$


---


$$a \begin{vmatrix} c & 0 & a \\ 0 & a & b \\ a & b & c \end{vmatrix} - b \begin{vmatrix} b & c & 0 \\ 0 & a & b \\ a & b & c \end{vmatrix} + c \begin{vmatrix} b & c & 0 \\ c & 0 & a \\ a & b & c \end{vmatrix}$$

$$y = \left\{ a \begin{vmatrix} n & 0 & a \\ p & a & b \\ q & b & c \end{vmatrix} - b \begin{vmatrix} m & c & 0 \\ p & a & b \\ q & b & c \end{vmatrix} + c \begin{vmatrix} m & c & 0 \\ n & 0 & a \\ q & b & c \end{vmatrix} \right\} + c \cdot d$$

$$z = \left\{ a \begin{vmatrix} c & n & a \\ 0 & p & b \\ a & q & c \end{vmatrix} - b \begin{vmatrix} b & m & 0 \\ 0 & p & b \\ a & q & c \end{vmatrix} + c \begin{vmatrix} b & m & 0 \\ c & n & a \\ a & q & c \end{vmatrix} \right\} + c \cdot d$$

$$u = \left\{ a \begin{vmatrix} c & 0 & n \\ 0 & a & p \\ a & b & q \end{vmatrix} - b \begin{vmatrix} b & c & m \\ 0 & a & p \\ a & b & q \end{vmatrix} + c \begin{vmatrix} b & c & m \\ c & 0 & n \\ a & b & q \end{vmatrix} \right\} + c \cdot d$$

11.  $x = 2, y = 3, z = -2, u = -3$

12.  $x = \frac{p+q+r+s}{4a}, y = \frac{p-q+r-s}{4b}$

$z = \frac{p+q-r-s}{4c}, u = \frac{p-q-r+s}{4d}$

13.  $x = 1, y = 2, z = 3, u = 4, v = 5$

### Exercise 112.

1. Consistent.

2. Inconsistent.

### Exercise 115.

1.  $\frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{6} + \frac{1}{2} + \frac{1}{4}$

2.  $\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{11} + \frac{1}{6}$

3.  $\frac{1}{9} + \frac{1}{1} + \frac{1}{99}$

4.  $\frac{1}{6} + \frac{1}{1} + \frac{1}{7} + \frac{1}{6} + \frac{1}{2}$

5.  $3 + \frac{1}{6}$

6.  $3 + \frac{1}{2} + \frac{1}{6}$

7.  $5 + \frac{1}{2} + \frac{1}{10}$

8.  $7 + \frac{1}{1} + \frac{1}{4} + \frac{1}{1} + \frac{1}{1} + \frac{1}{14}$

9.  $\frac{1}{3} + \frac{1}{7} + \frac{1}{17} + \frac{1}{2} + \frac{1}{1} + \frac{1}{8}$

10.  $3 + \frac{1}{7} + \frac{1}{16} + \frac{1}{11}$

11.  $67 + \frac{1}{2} + \frac{1}{1} + \frac{1}{12} + \frac{1}{1} + \frac{1}{2} \text{ deg.}$

12.  $\frac{1}{2}, \frac{3}{7}, \frac{13}{80}, \frac{68}{157}$

13.  $\frac{1}{3}, \frac{1}{4}, \frac{10}{39}, \frac{11}{43}$

14.  $\frac{1}{2}, \frac{3}{7}, \frac{7}{16}, \frac{24}{55}, \frac{55}{126}, \frac{189}{433}, \text{ etc.}$

15.  $\frac{1}{1}, \frac{2}{3}, \frac{7}{10}, \frac{16}{23}, \frac{55}{79}, \text{ etc.}$

16.  $\frac{1}{2}, \frac{3}{7}, \frac{10}{23}, \frac{33}{76}, \frac{109}{251}, \text{ etc.}$

17.  $\sqrt{5} - 2$

18.  $\sqrt{6} - 2$

19.  $\frac{1}{5}\sqrt{5}$

20.  $\frac{1}{3}(\sqrt{10} - 1)$

21.  $\frac{1}{1}, \frac{1}{2}, \frac{3}{5}, \frac{4}{7}, \frac{11}{19}, \frac{15}{26}, \text{ etc.}$

22.  $\frac{6}{7}$

23.  $\frac{1}{2}, \frac{1}{3}, \frac{10}{29}, \frac{21}{61}$

24.  $\frac{95}{98}$

**Exercise 116.**

- |   |               |                   |
|---|---------------|-------------------|
| 1. 25542                                    | 2. 616        | 3. 434005, 327454 |
| 4. $58072\frac{8}{9}$ , $32853\frac{6}{11}$ | 5. 2950       | 6. 135344         |
| 7. $2e23$                                   | 8. 8202033030 | 9. 7692           |
| 10. 7640                                    |               |                   |
| 11. $r = 6$                                 | 12. $r = 6$   | 13. $r = 14$      |
|   |               | 14. 2             |
| 15. 1414, 1201                              | 16. $r = 7$   |                   |

**Exercise 117.**

- |                                     |      |            |
|-------------------------------------|------|------------|
| 1. 197, 251, 313, 281, 461, 829     | 2. 6 | 3. 2709316 |
| 4. By 9: 11205, 842738, 558657      |      |            |
| By 11: 24530, 842738, 25916, 558657 |      |            |
| By 9 and 11: 342738, 558657         |      |            |

7.  $x = \frac{p^2}{a}$

8.  $x = \frac{p^2 - b}{a}$

THE END.









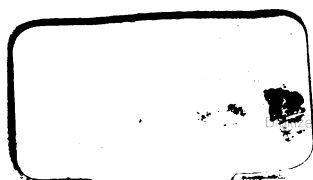
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